CSC 2414H (metric embeddings) - Assignment 1

General rules : In solving this you may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the people you consulted. This list will not affect your grade. **Due Jan 29, 2006**.

Remember that a norm on $\mathbb{R}^n$ is a function $\|\cdot\|$ from $\mathbb{R}^n$ to $\mathbb{R}^+$ which satisfies the following properties:

- $\|v\| = 0$ iff $v = 0$.
- $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.
- $\|v + u\| \leq \|v\| + \|u\|$ for all $u, v \in V$.

Every norm $\|\cdot\|$ on $\mathbb{R}^n$ induces a metric defined as $d(u, v) = \|u - v\|$.

1. (a) Show that $\|v\|_1 = \sum_{i=1}^n |v_i|$, $\|v\|_2 = \sqrt{\sum_{i=1}^n v_i^2}$, $\|v\|_\infty = \max_{i=1}^n |v_i|$ are norms.

   (b) Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is in the ith coordinate. Show that for every norm $\|\cdot\|$ and every vector $v \in \mathbb{R}^n$ we have
   $$\|v\| \leq (\max_{1 \leq i \leq n} \|e_i\|)\|v\|_1.$$
   (It is also possible to show that there exists $m > 0$ (depending on $\|\cdot\|$) such that $m\|v\|_1 \leq \|v\|$.)

   (c) Two norms $\|\cdot\|$ and $\|\cdot\|'$ are called equivalent if there exists constants $m, M > 0$ such that
   $$m\|v\| \leq \|v\|' \leq M\|v\|,$$
   for every $v \in \mathbb{R}^n$. From the previous part conclude that every two norms in $\mathbb{R}^n$ are equivalent.

2. (a) Suppose that $\|\cdot\|$ is a norm, and $r > 0$. Show that $B_r = \{v : \|v\| \leq r\}$ is a bounded symmetric convex body, i.e. it satisfies

- (boundedness) There is an $M > 0$ such that $B_r \subseteq \{v : \|v\|_2 \leq M\}$.
- (symmetry) $v \in B_r$ iff $-v \in B_r$.
- (convexity)If $v, w \in B_r$, then for every $0 \leq \alpha \leq 1$, $\alpha v + (1-\alpha)w \in B_r$.

(b) Let B be a bounded symmetric convex body, that is also closed and that contains a Euclidean ball $\{x | \|x\|_2 \leq \rho\}$ for some $\rho > 0$. Show that there exists a unique norm that satisfies $B_1 = B$.

(c) Given two norms $\|\cdot\|$ and $\|\cdot\|'$, is $\|v\|'' = \min\{\|v\|, \|v\|'\}$ necessarily a norm? What about $\|v\|_{\max} = \max\{\|v\|, \|v\|'\}$?

(d) Show that ℓ_∞^2 is isometrically isomorphic to ℓ_1^2 , i.e. there exists a bijection T from $\mathbb{R}^2$ to itself, so that $\|Tx - Ty\|_1 = \|x - y\|_\infty$ for every $x, y \in \mathbb{R}^2$.

3. (a) Consider the graph distance d in C_4, the cycle of size 4. Show that (C_4, d) is not an ℓ_2 metric.

   (b) Conclude that not all ℓ_1 metrics are ℓ_2.

   (c) (Bonus) Show that all finite ℓ_2 metrics are also ℓ_1 metrics.

4. Let S be a set of n points in $\mathbb{R}^2$. Define the distance between two points x and y, $d(x, y)$, as the area of $B_1(x)\Delta B_1(y) = (B_1(x) \setminus B_1(y)) \cup (B_1(y) \setminus B_1(x))$ (where $B_1(z)$ is the disk of radius 1 centered at z). By defining a proper function $f : S \to \mathbb{R}^m$ for some m, show that (S, d) is an ℓ_1 metric.

5. Consider the proof of Bourgain's theorem.

   (a) For any one of the follwoing cases find a (semi) metric space (X, d) containing two specific points x and y so that the corresponding Δ_j (as in the proof of Bourgain's theorem; recall we have $\Delta_1, \ldots, \Delta_t$ for some $t = \Theta(\log n)$) we have the following behaviour:

   - $\Delta_j \geq d(x, y)/4$ for $j = t$ and 0 otherwise.
   - $\Delta_j \geq d(x, y)/4$ for $j = 1$ and 0 otherwise.
   - $\Delta_j \geq \Omega(\frac{d(x,y)}{\log n})$ for all $j = 1 \ldots t$.

   (b) Supopose that instead of choosing sets with different parameters as in Bourgain's construction (i.e. an element is in a set with probability 2^{-j} for varying values of j) we pick sets with just one value of j. Show that such an embedding is not good in the following sense: with probability $o(1)$ we get distortion $O(\log n)$.