CSC2414 - Metric Embeddings*
Lecture 3: Diamond graph, and embdedding planar graphs into ℓ_2.

Notes taken by Hamed Hatami

Summary: In this tutorial we prove that there exists a planar graph which requires distortion $\Omega(\sqrt{\log(n)})$ to be embedded into the Euclidian space.

1 Introduction

In Tutorial 1 we mentioned a result of Bourgain that embedding the complete binary tree into ℓ_2 requires the distortion $\sqrt{\log\log(n)}$. This gives a lower bound for the required distortion for embedding the planar graphs into the Euclidian space. But this bound is not sharp, as Newman and Rabinovich [NR03] showed that there exists a planar graph which requires a distortion of at least $\sqrt{\log n}$ to be embedded into ℓ_2.

In Lecture 5 we saw that it is not possible to embed C_4 into the Euclidian space with distortion better than $\sqrt{2}$. The idea is to amplify this: Define G_0 to be a single edge; For $i \geq 1$, obtain G_i by replacing every edge of G_{i-1} by two parallel paths each containing 2 edges (i.e. by a copy of C_4). Note that the *diamond graph* G_k contains 4^k edges and $\frac{2\times 4^k+4}{3}$ vertices.

To simplify the presentation, we need to define the notion of anti-edge in the diamond graph. Assume that the edge (a,b) of G_{i-1} was replaced in G_i by edges $(a,x),(x,b)$ and $(a,y),(y,b)$. The pair $\{x,y\}$ will be called the *anti-edge* of (a,b) at *level* $i-1$. We denote by $\mathcal{A}^i$ the set of anti-edges at level i of G_n, and by $\mathcal{A}$ the set of all anti-edges of G_n, i.e. $\mathcal{A} = \bigcup_{i=0}^{n-1} \mathcal{A}^i$.

In order to show the lower-bound we need to prove a Poincaré inequality. First recall the short diagonal inequality from Lecture note 6:

Lemma 1.1. *Let x_1, x_2, x_3, x_4 be arbitrary points in a Euclidian space. Then*

$$\|x_1 - x_3\|^2 + \|x_2 - x_4\|^2 \leq \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_4\|^2 + \|x_4 - x_1\|^2.$$

The Poincaré inequality that we wish to prove is a simple application of this lemma:

Lemma 1.2. *Label the vertices of G_0 as s and t, and let $f : G_n \to \ell_2$. Then*

$$\|f(s) - f(t)\|_2^2 + \sum_{xy\in\mathcal{A}} \|f(x) - f(y)\|_2^2 \leq \sum_{xy\in G_n} \|f(x) - f(y)\|_2^2. \quad (1)$$

* Lecture Notes for a course given by Avner Magen, Dept. of Computer Sciecne, University of Toronto.

Proof. Apply Lemma 1.1 to all copies of C_4 that substituted edges of G_{n-1}, and get

$$\sum_{xy\in G_{n-1}} \|f(x)-f(y)\|_2^2 + \sum_{xy\in \mathcal{A}^{n-1}} \|f(x)-f(y)\|_2^2 \leq \sum_{xy\in G_n} \|f(x)-f(y)\|_2^2,$$

Now apply the same argument to the term $\sum_{xy\in G_{n-1}} \|f(x)-f(y)\|_2^2$ in the left hand side to obtain:

$$\sum_{xy\in G_{n-2}} \|f(x)-f(y)\|_2^2 + \sum_{xy\in \mathcal{A}^{n-1}\cup A^{n-2}} \|f(x)-f(y)\|_2^2 \leq \sum_{xy\in G_n} \|f(x)-f(y)\|_2^2.$$

Repeating this will eventually lead to (1). □

Now we are ready to prove the lower-bound:

Theorem 1.3. *[NR03] The required distortion to embed the shortest path metric of G_n into the Euclidian space is at least $\sqrt{n+1}$.*

Proof. Let d denote the shortest path metric in G_n, and s and t be as in Lemma 1.2. Trivially

$$d(s,t) = 2^n,$$

and

$$\sum_{xy\in\mathcal{A}^i} d(x,y)^2 = |E(G_i)| \times 2^{2(n-i)} = 4^i \times 2^{2n-2i} = 4^n,$$

which shows

$$d(s,t)^2 + \sum_{xy\in\mathcal{A}} d(x,y)^2 = 4^n + \sum_{i=0}^{n-1} 4^n = (n+1)4^n.$$

On the other hand

$$\sum_{xy\in G_n} d(x,y)^2 = |E(G_n)| = 4^n.$$

So

$$d(s,t)^2 + \sum_{xy\in\mathcal{A}} d(x,y)^2 \geq (n+1) \sum_{xy\in G_n} d(x,y)^2.$$

Combining this with Lemma 1.2 completes the proof. □

Next we want to prove a similar result for ℓ_p. The following analogue of short diagonal lemma is valid in ℓ_p.

Lemma 1.4. *[LN04] Let x_1, x_2, x_3, x_4 be arbitrary points in ℓ_p where $1 \leq p \leq 2$. Then*

$$\|x_1-x_3\|_p^2+(p-1)\|x_2-x_4\|_p^2 \leq \|x_1-x_2\|_p^2+\|x_2-x_3\|_p^2+\|x_3-x_4\|_p^2+\|x_4-x_1\|_p^2.$$

Replacing Lemma 1.1 with Lemma 1.2 in the proof of Lemma 1.4 proves the following Pincaré inequality.

Lemma 1.5. *Label the vertices of G_0 as s and t, and let $f : G_n \to \ell_p$ where $1 \le p \le 2$. Then*

$$\|f(s) - f(t)\|_p^2 + (p-1) \sum_{xy \in \mathcal{A}} \|f(x) - f(y)\|_p^2 \le \sum_{xy \in G_n} \|f(x) - f(y)\|_p^2. \tag{2}$$

Now repeating the proof of Theorem 1.3 proves the following theorem.

Theorem 1.6. *[LN04] The required distortion to embed the shortest path metric of G_n into ℓ_p for $1 \le p \le 2$ is at least $\sqrt{(p-1)n + 1}$.*

Exercise 1.7. Prove Lemma 1.5 and Theorem 1.6.

References

[LN04] J. R. Lee and A. Naor. Embedding the diamond graph in L_p and dimension reduction in L_1. *Geom. Funct. Anal.*, 14(4):745–747, 2004.

[NR03] Ilan Newman and Yuri Rabinovich. A lower bound on the distortion of embedding planar metrics into Euclidean space. *Discrete Comput. Geom.*, 29(1):77–81, 2003.