

CSC 411: Lecture 02: Linear Regression

Class based on Raquel Urtasun & Rich Zemel's lectures

Sanja Fidler

University of Toronto

Jan 13, 2016

(Most plots in this lecture are from Bishop's book)

Problems for Today

- What should I watch this Friday?

All

Movies, TV & Showtimes Celebs, Events & Photos News & Community Watchlist



BRING HIM HOME
THE MARTIAN
OCTOBER

The Martian (2015)

PG-13 | 144 min | Adventure, Comedy, Drama | 2 October 2015 (USA)

Your rating: ★★★★★★★★ -/10

8.1 Ratings: **8.1/10** from **271,829** users Metascore: **80/100**
Reviews: **750** user | **499** critic | **46** from **Metacritic.com**

During a manned mission to Mars, Astronaut Mark Watney is presumed dead after a fierce storm and left behind by his crew. But Watney has survived and finds himself stranded and alone on the hostile planet. With only meager supplies, he must draw upon his ingenuity, wit and spirit to subsist and find a way to signal to Earth that he is alive.

Director: [Ridley Scott](#)
Writers: [Drew Goddard](#) (screenplay), [Andy Weir](#) (book)
Stars: [Matt Damon](#), [Jessica Chastain](#), [Kristen Wiig](#) | [See full cast and crew »](#)

+ Watchlist Watch Trailer Share...

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Point Break (2015)

PG-13 | 114 min | Action, Crime, Sport | 25 December 2015 (USA)

Your rating: ★★★★★★★★ -/10
5.4 Ratings: **5.4/10** from **7,322** users Metascore: **34/100**
Reviews: **60** user | **84** critic | **19** from **Metacritic.com**

A young FBI agent infiltrates an extraordinary team of extreme sports athletes he suspects of masterminding a string of unprecedented, sophisticated corporate heists. "Point Break" is inspired by the classic 1991 hit.

Director: [Ericson Core](#)
Writers: [Kurt Wimmer](#) (screenplay), [Rick King](#) (story), [5 more credits](#) »
Stars: [Édgar Ramírez](#), [Luke Bracey](#), [Ray Winstone](#) | [See full cast and crew](#) »

[+ Watchlist ▾](#) [Watch Trailer](#) [Share...](#)

[See More on IMDb Pro](#) »

Problems for Today

- **Goal:** Predict movie rating automatically!



The image shows a screenshot of the IMDb website for the movie "Point Break (2015)". The page features a search bar at the top with the text "Find Movies, TV shows, Celebrities and more...". Below the search bar are navigation tabs for "Movies, TV & Showtimes", "Celebs, Events & Photos", "News & Community", and "Watchlist". The movie's poster is on the left, with the text "FIND YOUR BREAKING POINT" and "POINT BREAK" overlaid. To the right of the poster, the movie title "Point Break (2015)" is displayed, along with its rating "PG-13" and release date "25 December 2015". A callout box with a blue background and white text says "Predict this automatically!" with a dashed arrow pointing to a yellow star containing the number "5.4". Below the star, the text reads "Your rating: ★★★★★★ -/10" and "Ratings: 5.4/10 from 7,322 users Metascore: 34/100". Further down, there are sections for "Reviews: 60 user | 84 critic | 19 from Metacritic.com", a synopsis, and credits for "Director: Ericson Core", "Writers: Kurt Wimmer (screenplay), Rick King (story), 5 more credits »", and "Stars: Édgar Ramírez, Luke Bracey, Ray Winstone | See full cast and crew »". At the bottom, there are buttons for "+ Watchlist", "Watch Trailer", and "Share...".

Problems for Today

- **Goal:** How many followers will I get?

Red Leather Jacket

Updated on Jan 09, 2016



From This User

+1 282 VOTES

5 COMMENTS

67 FAVORITES

Like 0

Tweet

+1 0

...

Pin it 2

Tags

Chic
Everyday
Winter

SHARE

- **Goal:** Predict the price of the house

Why choose Nationwide? | Have your say | Corporate information | Media, Policy & Legal | **House Price Index** | Investor relations

Nationwide
House Price Index

Headlines | **House Price calculator** | Report archive | Download data | Methodology

House Price Calculator

Instructions

- Property Value: Enter the price paid for, or a more recent valuation of your property. Please ensure the value is entered without commas, for example 150000, rather than 150,000.
- Valuation Date 1: The date when your property was purchased, or revalued.
- Valuation Date 2: Date for which you would like a new estimate of your property's value.
- Region: Select region which the property is situated in. If you are not sure which region the property is in, click on the link below to find your region.

Please note: The Nationwide House Price Calculator is intended to illustrate general movement in prices only.

The calculator is based on the Nationwide House Price Index. Results are based on movements in prices in the regions of the UK rather than in specific towns and cities. The data is based on movements in the price of a typical property in the region, and cannot take account of differences in quality of fittings.

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(eg, a rating: a real number between 0-10, # of followers, house price)

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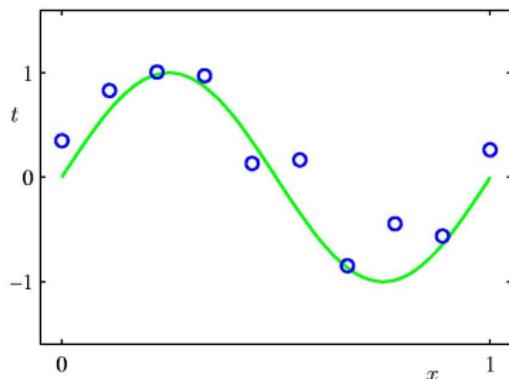
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 - ▶ **Optimization**, a way of finding the parameters of our model that minimizes the loss function

Today: Linear Regression

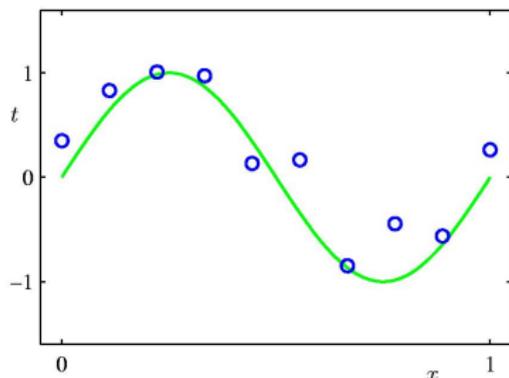
- Linear regression
 - ▶ continuous outputs
 - ▶ simple model (linear)
- Introduce **key concepts**:
 - ▶ loss functions
 - ▶ generalization
 - ▶ optimization
 - ▶ model complexity
 - ▶ regularization

Simple 1-D regression



- Circles are data points (i.e., training examples) that are given to us

Simple 1-D regression

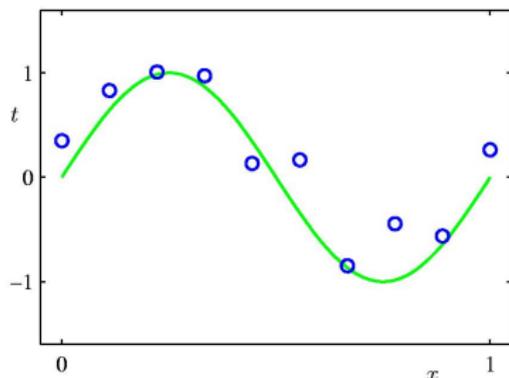


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- The data points are uniform in x , but may be displaced in y

$$t(x) = f(x) + \epsilon$$

with ϵ some noise

Simple 1-D regression



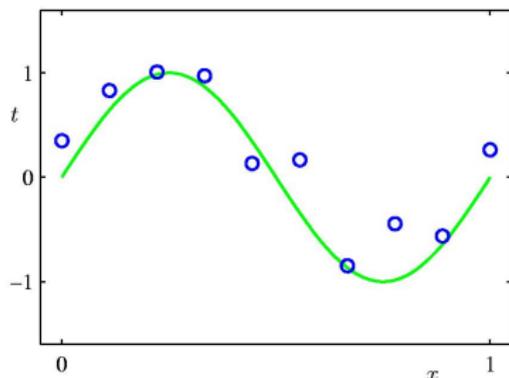
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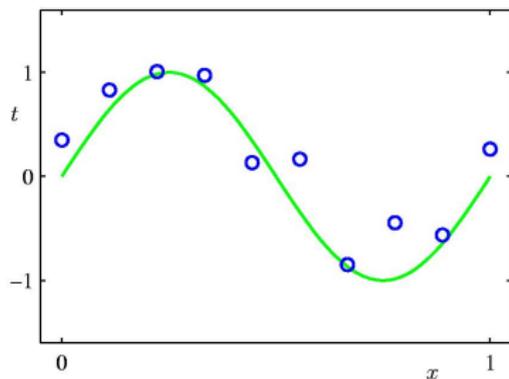
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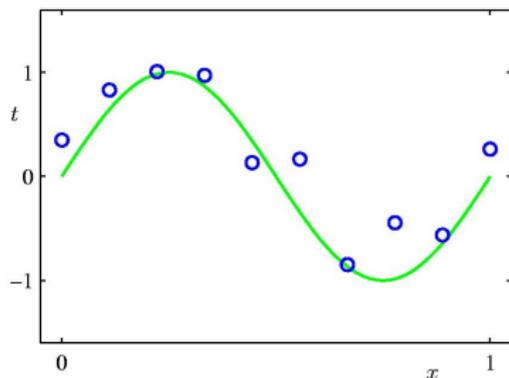
- In green is the "true" curve that we don't know
- **Goal:** We want to fit a curve to these points

Simple 1-D regression



- Key Questions:

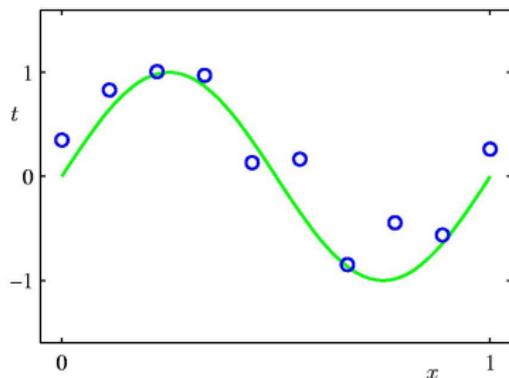
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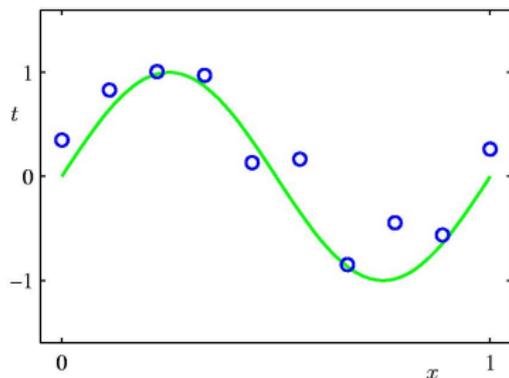
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Simple 1-D regression



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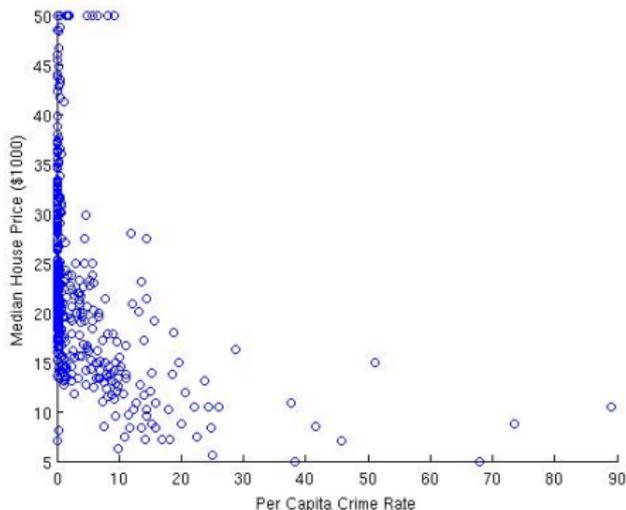
- ▶ How do we parametrize the **model**?
- ▶ What **loss (objective) function** should we use to judge the fit?
- ▶ How do we optimize fit to unseen test data (**generalization**)?

Example: Boston Housing data

- Estimate median house price in a neighborhood based on neighborhood statistics

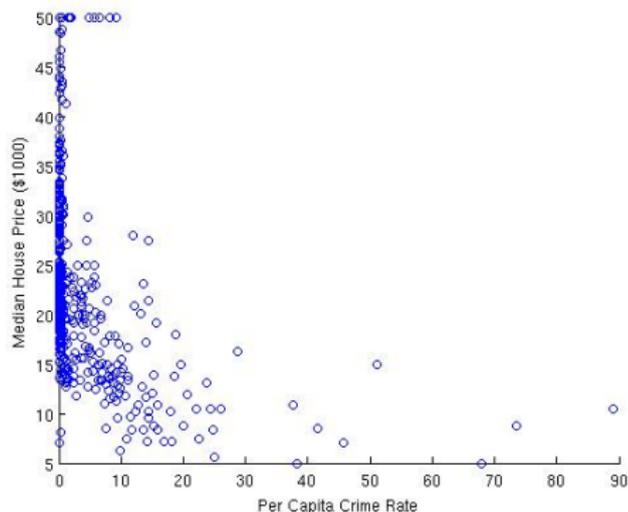
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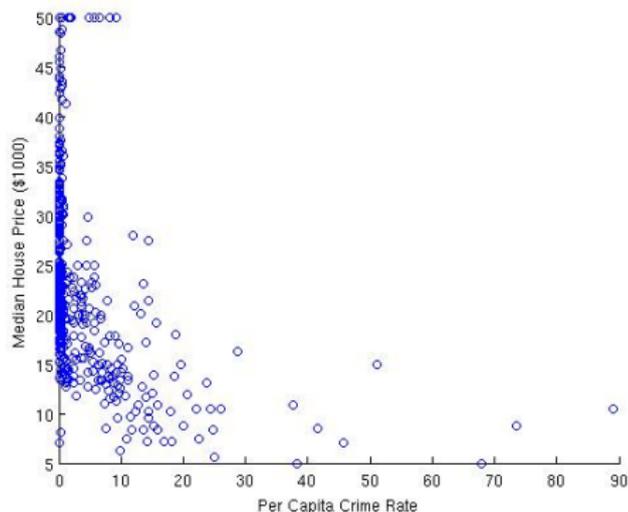
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- Use this to predict house prices in other neighborhoods
- Is this a **good input (attribute) to predict** house prices?

Represent the Data

- Data is described as pairs $\mathcal{D} = \{(x^{(1)}, t^{(1)}), \dots, (x^{(N)}, t^{(N)})\}$
 - ▶ $x \in \mathbb{R}$ is the **input feature** (per capita crime rate)
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 - ▶ Evaluate hypothesis on test set

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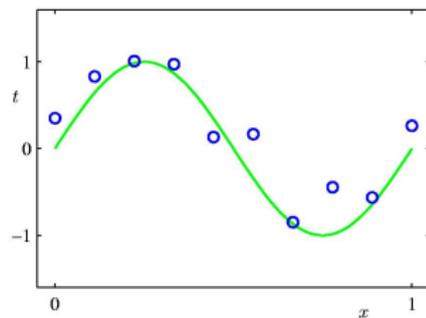
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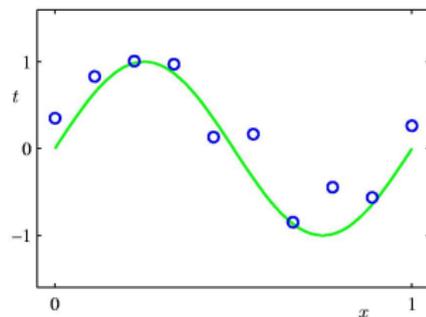
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 - ▶ Model may be too simple to account for data targets

Least-Squares Regression



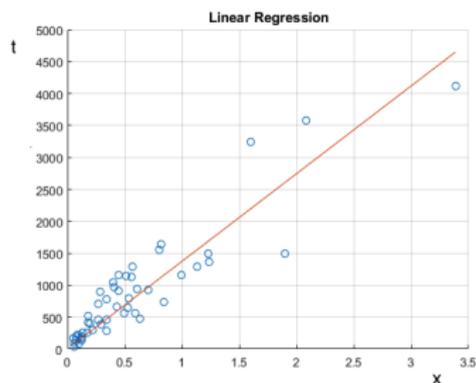
Least-Squares Regression



- Define a model

$$y(x) = \text{function}(x, \mathbf{w})$$

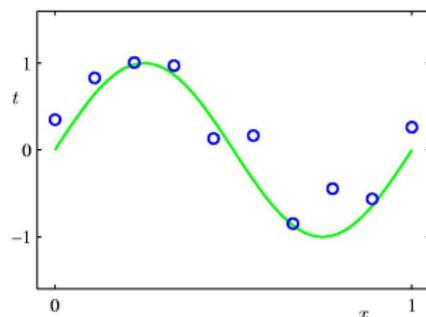
Least-Squares Regression



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Linear: $y(x) = w_0 + w_1x$

Least-Squares Regression



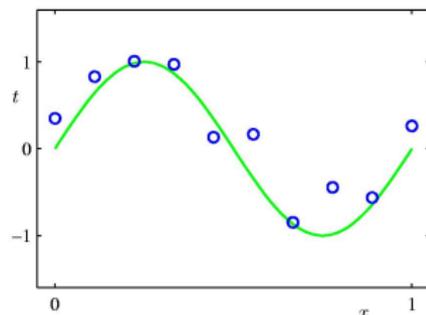
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Linear: $y(x) = w_0 + w_1x$

- Standard loss/cost/objective function measures the squared error between y and the true value t

$$\ell(\mathbf{w}) = \sum_{n=1}^N [t^{(n)} - y(x^{(n)})]^2$$

Least-Squares Regression



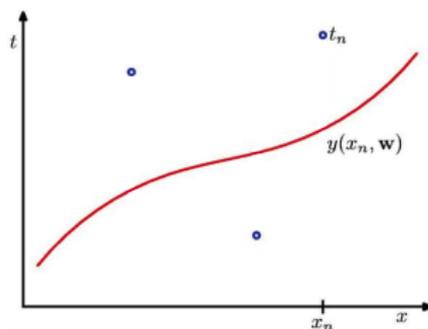
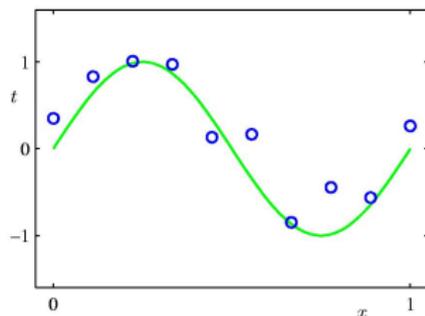
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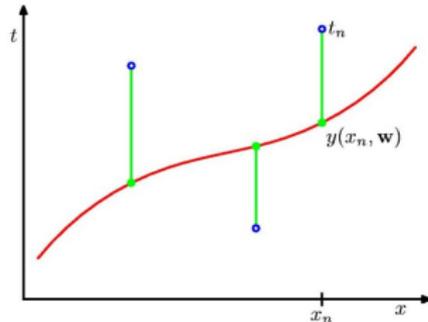
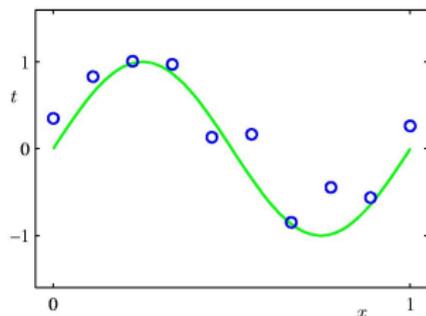
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- For a particular hypothesis ($y(x)$ defined by a choice of \mathbf{w} , drawn in red), what does the loss represent geometrically?

Least-Squares Regression



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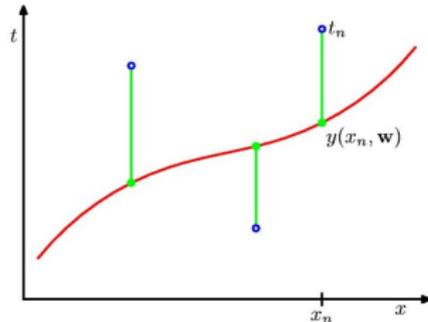
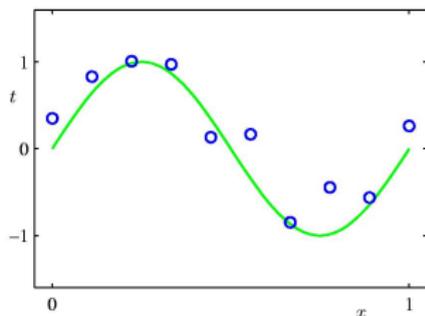
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- The loss for the red hypothesis is the **sum of the squared vertical errors** (squared lengths of green vertical lines)

Least-Squares Regression



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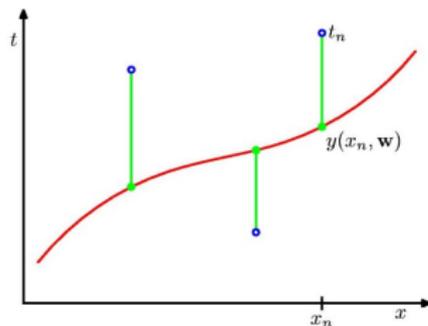
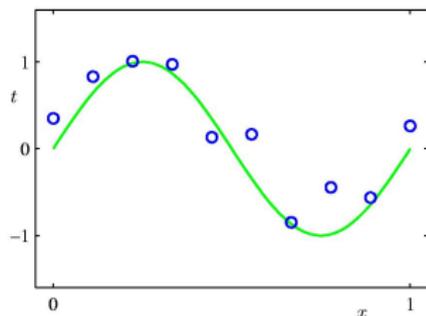
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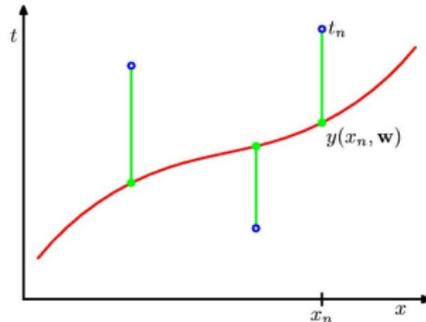
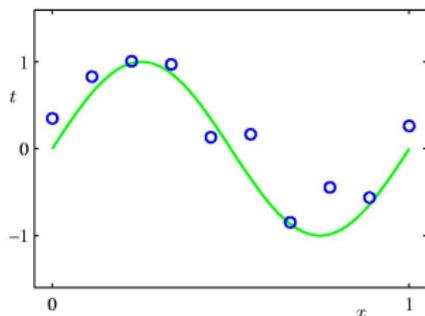
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- How do we obtain weights $\mathbf{w} = (w_0, w_1)$? Find \mathbf{w} that minimizes loss $\ell(\mathbf{w})$

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- How do we obtain weights $\mathbf{w} = (w_0, w_1)$?
- For the linear model, what kind of a function is $\ell(\mathbf{w})$?

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- One straightforward method: [gradient descent](#)

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Optimizing the Objective

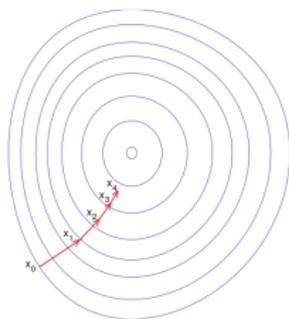
- One straightforward method: **gradient descent**
 - ▶ initialize \mathbf{w} (e.g., randomly)
 - ▶ repeatedly update \mathbf{w} based on the gradient

$$\mathbf{w} \leftarrow \mathbf{w} - \lambda \frac{\partial \ell}{\partial \mathbf{w}}$$

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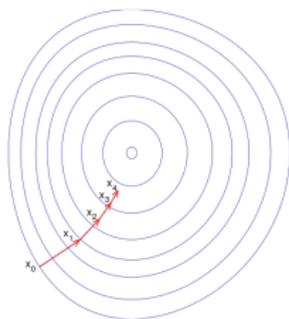


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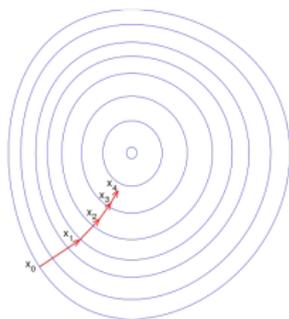
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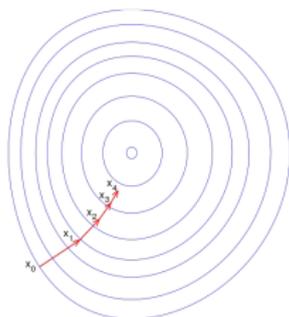
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$$\mathbf{w} \leftarrow \mathbf{w} + 2\lambda \underbrace{(t^{(n)} - y(x^{(n)}))}_{\text{error}} x^{(n)}$$

- Note: As error approaches zero, so does the update (\mathbf{w} stops changing)

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Algorithm 1 Stochastic gradient descent

- 1: Randomly shuffle examples in the training set
- 2: **for** $i = 1$ to N **do**
- 3: Update:

$$\mathbf{w} \leftarrow \mathbf{w} + 2\lambda(t^{(i)} - y(x^{(i)}))x^{(i)} \quad (\text{update for a linear model})$$

- 4: **end for**
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 - ▶ Underlying assumption: sample is independent and identically distributed (i.i.d.)

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- Compute the derivatives of the objective wrt \mathbf{w} and equate with 0
- Define:

$$\mathbf{t} = [t^{(1)}, t^{(2)}, \dots, t^{(N)}]^T$$
$$\mathbf{X} = \begin{bmatrix} 1, x^{(1)} \\ 1, x^{(2)} \\ \dots \\ 1, x^{(N)} \end{bmatrix}$$

- Then:

$$\mathbf{w} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{t}$$

(work it out!)

Multi-dimensional Inputs

- One method of extending the model is to consider other input dimensions

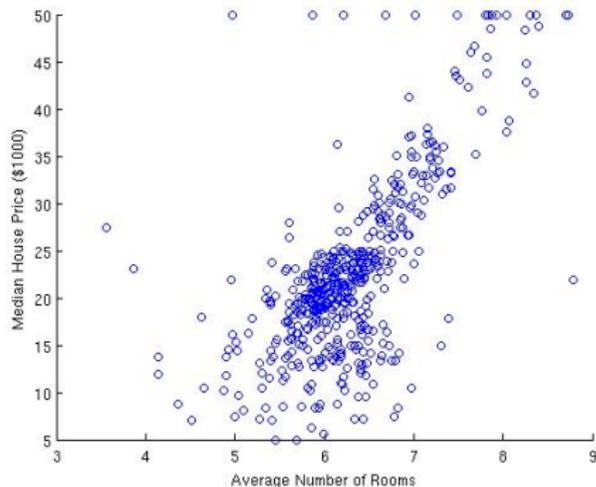
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- In the Boston housing example, we can look at the number of rooms



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- We can use gradient descent to solve for each coefficient, or compute \mathbf{w} analytically (how does the solution change?)

More Powerful Models?

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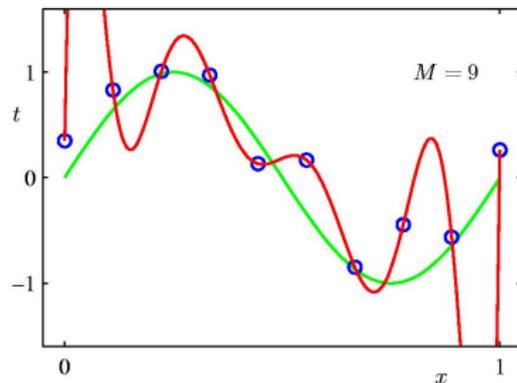
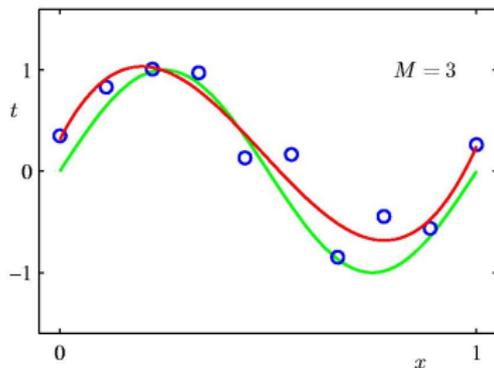
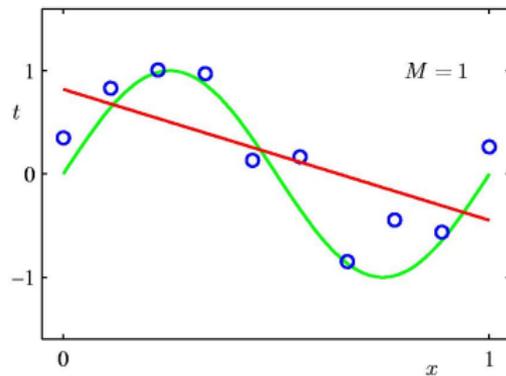
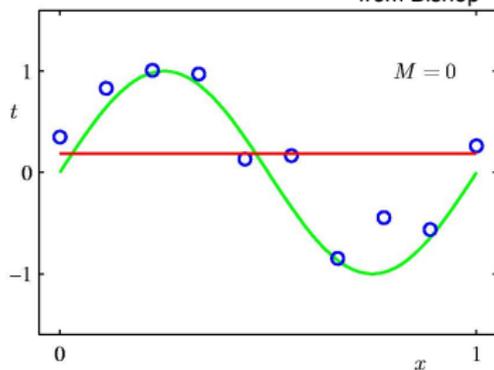
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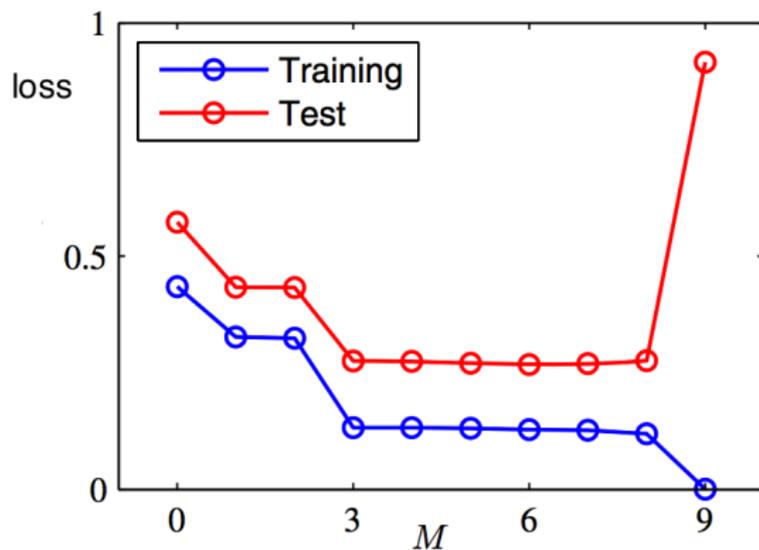
Which Fit is Best?

from Bishop



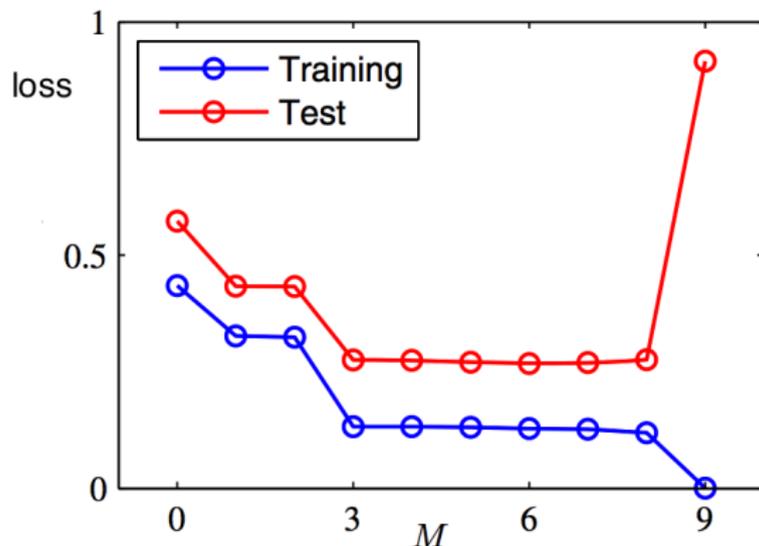
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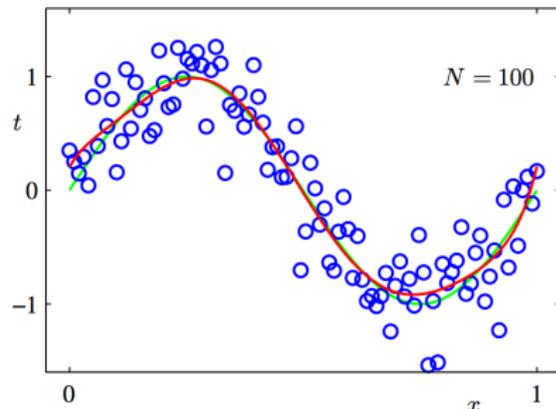
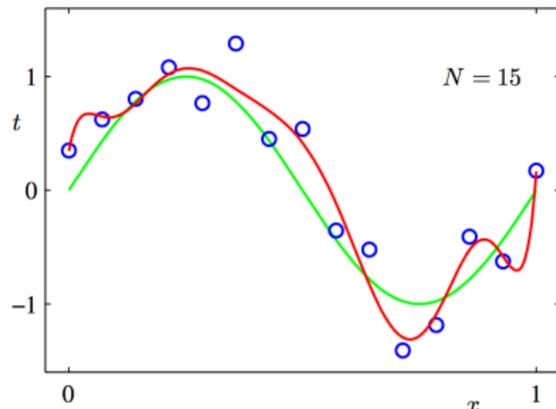
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	$M = 0$	$M = 1$	$M = 6$	$M = 9$
w_0^*	0.19	0.82	0.31	0.35
w_1^*		-1.27	7.99	232.37
w_2^*			-25.43	-5321.83
w_3^*			17.37	48568.31
w_4^*				-231639.30
w_5^*				640042.26
w_6^*				-1061800.52
w_7^*				1042400.18
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- One way of dealing with this is to encourage the weights to be small (this way no input dimension will have too much influence on prediction). This is called **regularization**.

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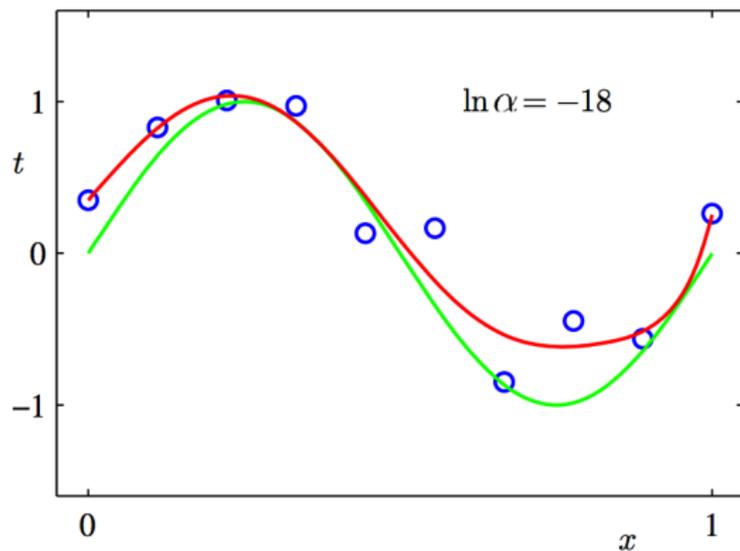
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- Also has an analytical solution: $\mathbf{w} = (\mathbf{X}^T \mathbf{X} + \alpha \mathbf{I})^{-1} \mathbf{X}^T \mathbf{t}$ (verify!)

Regularized least squares

- Better generalization
- Choose α carefully



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- One method of assessing fit: test [generalization](#) = model's ability to predict the held out data
- Optimization is essential: stochastic and batch iterative approaches; analytic when available

So...

- Which movie will you watch?



Now Playing

REFINE YOUR SEARCH



Abin And The Chipmunks: The Road Chip

1h 20m | Comedy, Family
View Ratings and Warnings

BUY TICKETS

TRAILER



Arosalia

1h 31m | Comedy, Animation, Fantasy
View Ratings and Warnings

BUY TICKETS

TRAILER



Bajrão Mestre (Hindi ver. s.1)

2h 30m | Foreign Language, Drama, Romance, History
View Ratings and Warnings

BUY TICKETS



Beauty And The Beast (Filipino ver. s.1)

1h 59m | Action, Foreign Language, Comedy
View Ratings and Warnings

BUY TICKETS



Brooklyn

1h 52m | Drama
View Ratings and Warnings

BUY TICKETS

TRAILER