Hereditary Dominating Pair Graphs

Nataša Pržulj* Derek G. Corneil† Ekkehard Köhler‡

November 1, 2002

Abstract

An *asteroidal triple* (AT) is a set of vertices such that each pair of vertices is joined by a path that avoids the neighborhood of the third. Every AT-free graph contains a *dominating pair*, a pair of vertices such that for every path between them, every vertex of the graph is within distance one of the path. We say that a graph is a *hereditary dominating pair* (HDP) graph if each of its connected induced subgraphs contains a dominating pair. In this paper we introduce the notion of *frame HDP graphs* in order to capture the structure of HDP graphs that contain asteroidal triples. We also determine the maximum diameter of frame HDP graphs.

Keywords: hereditary dominating pair graphs, AT-free graphs, dominating pairs

1 Introduction

An *asteroidal triple* (AT) is an independent set of vertices such that each pair of vertices is joined by a path that avoids the neighborhood of the third. Lekkerkerker and Boland[8] introduced the class of *AT-free* graphs, graphs without asteroidal triples, in their study of interval graphs and showed that a graph is interval if and only if it is chordal and AT-free. Thus, the AT-free property seems to impose

*Dept. of Computer Science, 10 King's College Rd., University of Toronto, Toronto, ON, M5S 3G4, Canada. E-mail: natasha@cs.toronto.edu

†Dept. of Computer Science, 10 King's College Rd., University of Toronto, Toronto, ON, M5S 3G4, Canada. E-mail: dgc@cs.toronto.edu

‡Sekr. MA 6-1, Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany. E-mail: ekoehler@math.tu-berlin.de

a "linear" structure that a chordal graph must have in order to be interval. AT-free graphs contain such families as cocomparability, trapezoid, permutation, interval, families that all exhibit some form of linearity. The hope of finding structural properties that capture such linearity led Corneil, Olariu, and Stewart [1] to study the structure of AT-free graphs. One of their most interesting results is that every connected AT-free graph has a dominating pair, a pair of vertices such that for every path between them, every vertex of the graph is either on the path, or is at distance one from it. We say that a graph is *hereditary dominating pair* (HDP) if all of its connected induced subgraphs have dominating pairs. Clearly, AT-free graphs are HDP. However, there are graphs that are HDP, but have asteroidal triples, such as, for example, C_6.

In this paper we study the structure of HDP graphs and determine whether various properties of AT-free generalize to HDP graphs. We assume that all graphs are finite with no loops or multiple edges and use the standard graph-theoretic terminology compatible with [11]. A path is not necessarily induced and standard definitions of the path *length* and the path *size* are used to denote the number of edges and the number of vertices respectively. In addition, we say that a vertex v *intercepts* a path P if v is adjacent to at least one vertex of P; otherwise, v *misses* P. For a graph G and a pair of vertices x, y of G, $D(x, y)$ represents the set of vertices that intercept all x, y-paths. Note that (x, y) is a dominating pair of $G = (V, E)$ if and only if $D(x, y) = V$.

As mentioned above, Corneil, Olariu, and Stewart [1] provided a common generalization of interval, permutation, trapezoid, and cocomparability graphs in the sense that the linearity of their structure is demonstrated by the existence of a dominating pair in every connected AT-free graph. They also showed the following interesting Polar Theorem for AT-free graphs.

Theorem 1 [1] *Let G be a connected AT-free graph with diameter at least four. There exist nonempty, disjoint sets X and Y of vertices of G such that (x, y) is a dominating pair if and only if $x \in X$ and $y \in Y$.*

Later, Deogun and Kratsch [3] studied *weak dominating pair graphs*, graphs that contain dominating pairs, but their subgraphs do not necessarily have dominating pairs, and proved the following Polar Theorem.

Theorem 2 [3] *Let $G = (V, E)$ be any weak dominating pair graph with diameter at least five.*

Then there are disjoint sets $X \subseteq V$ and $Y \subseteq V$ such that for all $x, y \in V$: (x, y) is a dominating pair of G if and only if $x \in X$ and $y \in Y$.

Later we will present a similar type of polarity result for HDP graphs.

Consider the following operations on graphs. A graph G is a *join* of graphs G_1 and G_2, if it consists of the disjoint union of graphs G_1 and G_2 plus the edges $\{uv | u \in V(G_1), v \in V(G_2)\}$. If K_j is a complete graph on j vertices, a graph G is obtained by *K_j-bonding* of graphs G_1 and G_2 if vertices of a K_j of G_1 are identified with the vertices of a K_j of G_2. We say that a graph G' is obtained by *substituting a vertex* u of a graph G by a graph U, if the neighborhood in $G' \setminus U$ of each vertex of U is equal to the neighborhood of u in G. A graph G' is obtained from graph G by *contracting an edge* $uv \in E(G)$, if $|V(G')| = |V(G)| - 1$ and the vertex w of G' that is obtained by identifying vertices u and v in G has $N(w) = N(u) \cup N(v) \setminus \{u, v\}$.

It is easy to see that HDP graphs are closed under vertex substitution, join, and edge contraction. They are not closed under complement and K_j-bonding, $j \geq 2$. An example of a non-HDP graph obtained by K_2-bonding of two C_6s is presented in Figure 1.

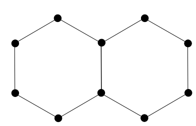

Figure 1: A non-HDP graph obtained by K_2-bonding of two HDP graphs.

In this paper we first explore the structure of HDP graphs. Section 2 describes some important structural properties of HDP graphs that contain asteroidal triples. In Section 3 we study frame HDP graphs, a family of graphs that capture the structure of the paths that establish an AT in an HDP graph. We determine the position of DP vertices in all frame HDP graphs, prove that the diameter of a frame HDP graph is always less than or equal to 5, and give a Polar Lemma for frame HDP graphs. Finally, in Section 4 we examine the complexity of HDP and chordal HDP graph recognition, determine whether various properties of AT-free graphs generalize to HDP graphs, and suggest directions for future work in this area.

2 Structure of HDP graphs

Since AT-free graphs are HDP and they have already been extensively studied, we restrict our attention to the structure of HDP graphs that contain asteroidal triples. We call these graphs HDP∩AT graphs.

Let $\mathcal{P}^c_{a,b}$ be defined as the set of all induced paths between vertices a and b in a graph G that avoid the neighborhood of a vertex c in G. Henceforth, in an HDP∩AT graph H with an AT $\{x, y, z\}$, we will use $\mathcal{P}_{x,y}$, $\mathcal{P}_{x,z}$, and $\mathcal{P}_{y,z}$ to denote $\mathcal{P}^z_{x,y}$, $\mathcal{P}^y_{x,z}$, and $\mathcal{P}^x_{y,z}$ respectively. In all claims in this section, H denotes an HDP∩AT graph with an AT $\{x, y, z\}$, and $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$ denote arbitrary induced paths that establish the AT $\{x, y, z\}$ in H.

Definition 1 *Let $\{x, y, z\}$ be an AT of an HDP graph H, and let $\mathcal{P}_{x,y}$, and $\mathcal{P}_{x,z}$ be defined as above. An AT vertex x is called* path-disjoint with respect to y, z *if for all paths $P \in \mathcal{P}_{x,y}$ and for all paths $Q \in \mathcal{P}_{x,z}$, $P \cap Q = \{x\}$. An AT vertex x is called* non-path-disjoint *if there exist paths $P \in \mathcal{P}_{x,y}$ and $Q \in \mathcal{P}_{x,z}$ such that $P \cap Q \supseteq \{x, x'\}$, where $x \neq x'$.*

Note that if x is a non-path-disjoint vertex of an AT $\{x, y, z\}$ of an HDP graph G, there may exist vertices $x'' \neq x'$ with the corresponding paths $P'' \in \mathcal{P}_{x,y}$ and $Q'' \in \mathcal{P}_{x,z}$, such that $P'' \cap Q'' = \{x, x''\}$. Also, there may exist paths $P' \in \mathcal{P}_{x,y}$ and $Q' \in \mathcal{P}_{x,z}$ such that $P' \cap Q' = \{x\}$.

Claim 1 *Let x be a non-path-disjoint vertex of H with respect to y, z, and let $P_{x,y}$, $P_{x,z}$, $P_{y,z}$ be induced paths establishing the AT. If any $x' \in P_{x,y} \cap P_{x,z}$, where $x' \neq x$, then xx' is an edge in H.*

Proof: Assume xx' is not an edge in H, where x' is the first vertex on $P_{x,y}$ in the x to y direction such that $x' \in P_{x,z}$. Let H' be the subgraph of H induced on $P_{x,y} \cup P_{x',z}$, where $P_{x',z}$ is the subpath of $P_{x,z}$ between z and x'. Denote by $P_{x,x'}$ the subpath of $P_{x,y}$ between x and x', and by $P_{x',y}$ the subpath of $P_{x,y}$ between x' and y. Note that $len(P_{x,x'}) \geq 2$ by assumption, $len(P_{x',y}) \geq 2$, since $P_{x,z}$ avoids the neighborhood of y, and $len(P_{x',z}) \geq 2$, since $P_{x,y}$ avoids the neighborhood of z.

Since H is HDP, H' has a DP. Denote by (α, β) a DP of H'. α and β cannot both be in $P_{x,y}$, since the path between them induced on $P_{x,y}$ misses z, by definition of $P_{x,y}$. They cannot both be in $P_{x',z}$, since the path between them induced on $P_{x',z}$ misses y, by definition of $P_{x',z}$. If one of α, β is in $P_{x,x'}$ and the other one is in $P_{x',z}$, then any path between them induced on $P_{x,x'} \cup P_{x',z}$

misses y; this is because y cannot be adjacent to a vertex in $P_{x,x'}$, since $P_{x,x'}$ and $P_{x',y}$ are subpaths of an induced path $P_{x,y}$ and $len(P_{x',y}) \geq 2$, and also y cannot be adjacent to a vertex in $P_{x',z}$, since $P_{x',z}$ is a subpath of $P_{x,z}$ which avoids the neighborhood of y. Similarly, if one of α, β is in $P_{x',y}$ and the other one is in $P_{x',z}$, then any path between them induced on $P_{x',y} \cup P_{x',z}$ misses x, since x cannot be adjacent to a vertex in $P_{x',y}$ because $P_{x,y}$ is an induced path, $P_{x,x'}$ and $P_{x',y}$ are subpaths of $P_{x,y}$, and $len(P_{x,x'}) \geq 2$, and similarly x cannot be adjacent to a vertex in $P_{x',z}$ because $P_{x,z}$ is an induced path, $P_{x,x'}$ and $P_{x',z}$ are subpaths of $P_{x,z}$, and $len(P_{x,x'}) \geq 2$. Thus, H' does not have a DP contradicting H being HDP. □

The following follows immediately from Claim 1.

Corollary 1 *If G is an HDP graph, then there does not exist an AT $\{x, y, z\}$ in G with paths $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$ establishing the AT $\{x, y, z\}$, such that all three paths have a common vertex w.*

The following theorem is the main result of this section.

Theorem 3 *If G is an $AT \cap HDP$ graph, then for any AT in G at least two of its AT vertices are path-disjoint.*

To prove this theorem, we first prove the following two claims.

We say that H is a *frame HDP graph* if for a fixed AT $\{x, y, z\}$ of H, all vertices of H belong to a path in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z} \cup \mathcal{P}_{y,z}$. We say that this fixed AT $\{x, y, z\}$ is the *defining AT* of H. Note that a frame HDP graph may have other ATs as well. We study this subclass of HDP graphs because they capture the difficulty of HDP graphs without containing all the tedious, easier to handle cases as well.

Claim 2 *Let $P_{x,y}$, $P_{x,z}$, $P_{y,z}$ be induced paths of H establishing the AT, such that $x' \in P_{x,y} \cap P_{x,z}$, and $x' \neq x$. Let $H \setminus \{x'\}$ be disconnected, and let C denote the connected component of $H \setminus \{x'\}$ that contains y and z. Then, for every DP (α, β) of H, one of α, β is in $H \setminus C$ and the other is in C. Furthermore, if H is a frame HDP graph, then $H \setminus C = \{x, x'\}$.*

Proof: Let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z} \cup \{\alpha, \beta\} \cup S$, where S is the set of vertices of H that induces a path between α and $P_{x,y} \cup P_{x,z} \cup P_{y,z}$ and a path between β and $P_{x,y} \cup P_{x,z} \cup P_{y,z}$ (S can be empty). Clearly, (α, β) is a DP of $\tilde{H}$ as well.

Suppose both $\alpha, \beta \in C \cap \tilde{H}$. Since $\tilde{H}$ is an induced subgraph of H, any path L from α to β in $C \cap \tilde{H}$ is also in H. Every vertex v of L is non-adjacent to a vertex in $\{H \setminus \{x'\}\} \setminus C$, for the following reasons. Let $x \in \{H \setminus \{x'\}\} \setminus C$. Since $v \in L \subset C \cap \tilde{H}$, we know that $v \in P_{x,y} \cup P_{x,z} \cup P_{y,z} \cup \{S \cap C\} \setminus \{x', x\}$. If $v \in \{y, z\}$, then $vx \notin E$ in H, since $\{x, y, z\}$ is an AT of H. If $v \in P_{x,y} \setminus \{x, x', y\}$, or $v \in P_{x,z} \setminus \{x, x', z\}$, then $vx \notin E$ in H, since $P_{x,y}$ and $P_{x,z}$ are induced paths. If $v \in P_{y,z} \setminus \{y, z\}$, then $vx \notin E$ in H, since all vertices on the path $P_{y,z}$ must miss x because paths $P_{x,y}$, $P_{x,z}$, $P_{y,z}$ establish the AT $\{x, y, z\}$. v cannot be equal to x, since $v \in C$ and $x \in \{H \setminus \{x'\}\} \setminus C$. If $v \in S \cap C$ and $vx \in E$ then the connected component of $H \setminus \{x'\}$ that contains y and z also contains x contradicting our assumption. Thus, every vertex of L misses x. Assume now that x does not belong to $\{H \setminus \{x'\}\} \setminus C$, i.e., $x \in C$. Since $H \setminus \{x'\}$ is disconnected, there must exist a vertex x_1 such that $x_1 \neq x$, $x'x_1 \in E$, and x_1 does not belong to the connected component of $H \setminus \{x'\}$ that contains y and z. If this is the case, then $\{x_1, y, z\}$ is also an AT, so the above discussion for x holds for x_1 as well, and therefore $vx_1 \notin E$ for every $v \in L$. Therefore, every vertex of L misses a vertex in $\{H \setminus \{x'\}\} \setminus C$, contradicting (α, β) being a DP of H. Therefore, at least one of α, β must be in $H \setminus C$.

Now, suppose $\alpha, \beta \in H \setminus C$. There has to exist a path P between α and β such that $P \in H \setminus C$ for the following reason. Assume to the contrary, i.e., let α and β belong to different connected components C_1 and C_2 of $H \setminus C$. Therefore, any path P between α and β must contain a vertex $v \in C$. But this means that at least one of α, β belongs to C contradicting our assumption. Now, any path from α to β in $H \setminus C$ misses both y and z contradicting (α, β) being a DP of H. Therefore, not both α and β are in $H \setminus C$.

It is proven that at least one DP vertex must be in $H \setminus C$, but not both of them are in $H \setminus C$. Thus, one of α, β is in $H \setminus C$ and the other is in C. An example is presented in Figure 2.

We now prove that if H is a frame HDP graph, then $H \setminus C = \{x, x'\}$. Let H be a frame HDP graph. Assume that there exists a vertex v in $H \setminus C$, such that $x \neq v \neq x'$. Let $vx' \notin E$. Since H is connected, $vx \in E$. But now $\{v, y, z\}$ is an AT of H with $vx' \notin E$ contradicting Claim 1. Therefore, $vx' \in E$. Now, either $xv \in E$, or $xv \notin E$. If $xv \in E$, then v belongs to a path in $\mathcal{P}_{x,y}$ that is not an induced path, which contradicts the definition of $\mathcal{P}_{x,y}$. If $xv \notin E$, then v does not belong to any path in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z} \cup \mathcal{P}_{y,z}$, which again contradicts the definition of a frame HDP graph. $\Box$

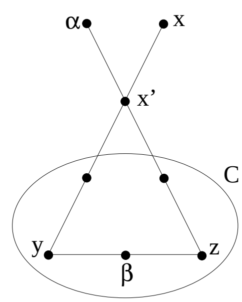

Figure 2: An illustration of Claim 2.

Claim 3 *If x is non-path-disjoint with respect to y, z, and y is non-path-disjoint with respect to x, z in a graph H with an AT $\{x, y, z\}$, then H is not an HDP graph.*

Proof: Let x be non-path-disjoint with respect to y, z, and y be non-path-disjoint with respect to x, z in H, and let $x' \in P_{x,y} \cap P_{x,z}$ and $y' \in Q_{x,y} \cap P_{y,z}$, $x \neq x'$ and $y \neq y'$, for some paths $P_{x,y}, Q_{x,y} \in \mathcal{P}_{x,y}$, $P_{x,z} \in \mathcal{P}_{x,z}$, and $P_{y,z} \in \mathcal{P}_{y,z}$. Assume that $P_{x,y} \neq Q_{x,y}$. Consider the subgraph H' of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Note that the x', y-path induced on $P_{x,y}$ is of length at least 2, since $P_{x,y}$ is induced and y misses $P_{x,z}$. Similarly, the x', z-path induced on $P_{x,z}$ and the y', z-path induced on $P_{y,z}$ are both of length at least 2. Since H is HDP, H' has a DP. Denote by (α, β) a DP of H'. Where are α and β positioned in H'? As we have seen in the second paragraph of the proof of Claim 1, α and β cannot both be on $P_{x,y}$, cannot both be on $P_{x,z}$, and cannot both be on $P_{y,z}$. If one of α and β is an internal vertex of $P_{x,y}$ and the other one is an internal vertex of $P_{x,z}$, then we have the following. Without loss of generality let α be an internal vertex of $P_{x,y}$, and let β be an internal vertex of $P_{x,z}$. Note that both α and β are different from x', since otherwise they would both be on $P_{x,y}$, or they would both be on $P_{x,z}$. If $\alpha y \in E$, then the α, β-path induced on $\{\alpha\} \cup P_{y,z} \cup P_{x,z}$ misses x contradicting (α, β) being a DP of H'. If $\alpha y \notin E$, then the α, β-path induced on $P_{x,y} \cup P_{x,z}$ misses y contradicting (α, β) being a DP of H'. Thus, one of α and β cannot be an internal vertex of $P_{x,y}$ while the other one is an internal vertex of $P_{x,z}$. Similarly, one of α and β cannot be an internal vertex of $P_{x,y}$ while the other one is an internal vertex of $P_{y,z}$, and also one of α and β cannot be an internal vertex of $P_{x,z}$ while the other one is an internal vertex of $P_{y,z}$. Therefore, H' does not have a DP contradicting H being HDP. Thus, we must have $P_{x,y} = Q_{x,y}$. An illustration is presented in Figure 3.

We now prove that if there exists $x' \in P_{x,y} \cap P_{x,z}$, $x' \neq x$, and there exists $y' \in P_{x,y} \cap P_{y,z}$,

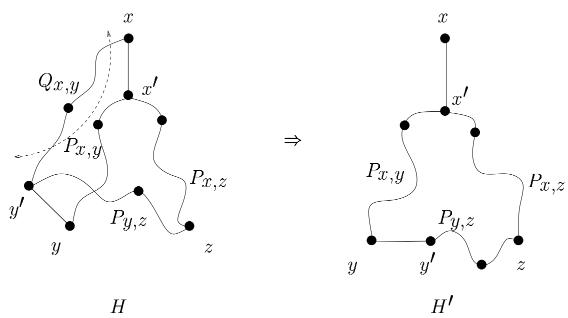

Figure 3: An illustration for Claim 3.

$y' \neq y$, for some paths $P_{x,y} \in \mathcal{P}_{x,y}$, $P_{x,z} \in \mathcal{P}_{x,z}$, and $P_{y,z} \in \mathcal{P}_{y,z}$, then H is not an HDP graph. Assume to the contrary. Thus, H is HDP. By Claim 1, xx' and yy' are edges in H. First note that $x' \neq y'$. This is because if $x' = y'$, then y intercepts path $P_{x,z}$ in H contradicting the fact that $\{x, y, z\}$ is an AT in H.

Let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. $\tilde{H}$ is a frame HDP graph. Let (α, β) be a DP of $\tilde{H}$. By Claim 2, $\tilde{H} \setminus C_x = \{x, x'\}$ contains a DP vertex, without loss of generality say $\alpha \in \tilde{H} \setminus C_x$, where C_x is the connected component of $\tilde{H} \setminus \{x'\}$ that contains y and z. By the same reasoning, $\beta \in \tilde{H} \setminus C_y = \{y, y'\}$, where C_y is the connected component of $\tilde{H} \setminus \{y'\}$ that contains x and z. Consider the path joining α and β that is induced on $\{\alpha, \beta\} \cup P_{x',y'}$, where $P_{x',y'}$ is a subpath of $P_{x,y}$ between x' and y'. This path misses z, since $\{x, y, z\}$ is an AT, contradicting (α, β) being a DP of $\tilde{H}$. $\square$

Since Claim 3 shows that an AT of an HDP graph cannot have two non-path-disjoint vertices, Theorem 3 follows directly from it. From the definition of frame HDP graphs and this theorem, we can conclude that there exist only two types of frame HDP graphs: those with no non-path-disjoint vertices, and those with exactly one non-path-disjoint vertex in their defining ATs. This motivates the following definitions.

Definition 2 *A frame HDP graph is called a* $\{$2,2,2$\}$ graph *if its defining AT* $\{x, y, z\}$ *has no non-path-disjoint AT vertices. A frame HDP graph is called a* $\{$1,2,2$\}$ graph *if its defining AT* $\{x, y, z\}$ *has exactly one non-path-disjoint AT vertex.*

Henceforth we will assume that in a $\{1, 2, 2\}$ graph with the defining AT $\{x, y, z\}$, vertex x is non-path-disjoint. The following section examines the structure of frame HDP graphs. We now state the last result of this section which is used to prove some structural properties of frame HDP graphs.

Claim 4 *Let (α, β) be a DP of an AT$\cap$HDP graph H. α and β cannot both belong to paths in $\mathcal{P}_{x,y}$, cannot both belong to paths in $\mathcal{P}_{x,z}$, and cannot both belong to paths in $\mathcal{P}_{y,z}$.*

Proof: Without loss of generality assume that α and β both belong to $\mathcal{P}_{x,y}$. Since the subgraph induced on the vertices in $\mathcal{P}_{x,y}$ is connected, there is a path between them that misses z, contradicting (α, β) being a DP. $\square$

3 Frame HDP graphs

In this section we first establish results common to $\{2,2,2\}$ and $\{1,2,2\}$ graphs. Then we describe some structure of $\{2,2,2\}$ and $\{1,2,2\}$ graphs which leads to the description of the position of DP vertices in frame HDP graphs. Finally, we determine the diameter of frame HDP graphs and give the Polar Lemma for frame HDP graphs.

The following definitions and notation are used throughout this section to describe frame HDP graphs more easily. We call a path of size bigger than 3 a *long path*. Also, in a $\{1,2,2\}$ graph with an AT $\{x,y,z\}$ and a non-path-disjoint AT vertex x, we define $\mathcal{R}_{x,y} \subseteq \mathcal{P}_{x,y}$ to be the set of paths $P \in \mathcal{P}_{x,y}$ such that there exists some path $Q \in \mathcal{P}_{x,z}$ and $P \cap Q \supsetneq \{x\}$. For a fixed $x' \in N(x)$, we let $\mathcal{R}^{x'}_{x,y} \subseteq \mathcal{R}_{x,y}$ denote the set of paths $P \in \mathcal{P}_{x,y}$ such that there exists some path $Q \in \mathcal{P}_{x,z}$ and $P \cap Q = \{x, x'\}$. $\mathcal{R}_{x,z}$ and $\mathcal{R}^{x'}_{x,z}$ are defined similarly. We denote by $\mathcal{R}_{x',y}$ the set of subpaths of paths in $\mathcal{R}^{x'}_{x,y}$ between x' and y, and we define $\mathcal{R}_{x',z}$ similarly.

A $\{2,2,2\}$ graph is called *two-long-sided* if it has long paths $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$; a $\{1,2,2\}$ graph is called *two-long-sided* if it has long paths $R_{x',y} \in \mathcal{R}_{x',y}$ and $R_{x',z} \in \mathcal{R}_{x',z}$. Similarly, a $\{2,2,2\}$ graph is called *one-long-sided* if it has a long path $P_{x,y} \in \mathcal{P}_{x,y}$ and both $\mathcal{P}_{x,z}$ and $\mathcal{P}_{y,z}$ consist of P_3s only; a $\{1,2,2\}$ graph is called *one-long-sided* if it has a long path $R_{x',y} \in \mathcal{R}_{x',y}$ and both $\mathcal{R}_{x',z}$ and $\mathcal{P}_{y,z}$ consist of P_3s only. A frame HDP graph is called *long sided* if it is either one-long-sided, or two-long-sided. Finally, a $\{2,2,2\}$ graph is called *no-long-sided* if $\mathcal{P}_{x,y}, \mathcal{P}_{x,z}$ and $\mathcal{P}_{y,z}$ consist of P_3s only; a $\{1,2,2\}$ graph is called *no-long-sided* if $\mathcal{R}_{x',y}, \mathcal{R}_{x',z}$ and $\mathcal{P}_{y,z}$ consist of P_3s only.

Lemma 1 *Let (α, β) be a DP of a frame HDP graph H.*

(1) If H is a two-long-sided $\{2,2,2\}$ graph with long paths $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$, then it is not the case that one DP vertex of H is an internal vertex of $P_{x,y}$ and the other one is equal to z. By symmetry, it is not the case that one DP vertex of H is an internal vertex of $P_{x,z}$ and the other one is equal to y.

(2) If H is a two-long-sided $\{2,2,2\}$ graph, then α and β cannot both belong to the union of the internal vertices of $P_{x,y}$ and $P_{x,z}$, where $P_{x,y}$ and $P_{x,z}$ are long paths in $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$ respectively.

(3) If H is a $\{1,2,2\}$ graph, then α and β cannot both belong to the union of the vertices of $R_{x',y}$ and $R_{x',z}$, where $R_{x',y}$ and $R_{x',z}$ are paths in $\mathcal{R}_{x',y}$ and $\mathcal{R}_{x',z}$ respectively.

Proof: Assume to the contrary.

(1) Without loss of generality assume that $\alpha = z$ and $\beta \in P_{x,y} \setminus \{x, y\}$. Since $|P_{x,y}| > 3$, β cannot be adjacent to both x and y. If $\beta x \notin E$, then the path from α to β induced on $P_{y,z} \cup P_{y,\beta}$ does not hit x contradicting (α, β) being a DP, where $P_{y,\beta}$ is the subpath of $P_{x,y}$ between y and β. If $\beta y \notin E$, then the path from α to β induced on $P_{x,z} \cup P_{x,\beta}$ does not hit y contradicting (α, β) being a DP, where $P_{x,\beta}$ is the subpath of $P_{x,y}$ between x and β.

(2) Without loss of generality let α be an internal vertex of $P_{x,y}$ and let β be an internal vertex of $P_{x,z}$. Note that α cannot be adjacent to both x and y, and that β cannot be adjacent to both x and z, since $|P_{x,y}| > 3$ and $|P_{x,z}| > 3$. If $\alpha y \in E$ and $\beta z \in E$, then the path from α to β induced on $\{\alpha y\} \cup P_{y,z} \cup \{\beta z\}$ does not hit x contradicting (α, β) being a DP of H, where $P_{y,z}$ is any path in $\mathcal{P}_{y,z}$. If one of these two edges αy and βz does not exist, i.e. without loss of generality if $\alpha y \notin E$, then the path from α to β induced on $P_{\alpha,x} \cup P_{x,\beta}$ does not hit y contradicting (α, β) being a DP, where $P_{\alpha,x}$ is the subpath of $P_{x,y}$ between α and x, and $P_{x,\beta}$ is the subpath of $P_{x,z}$ between x and β. Therefore, α and β do not both belong to the union of the internal vertices of $P_{x,y}$ and $P_{x,z}$.

(3) Without loss of generality let $\alpha \in R_{x',y}$ and $\beta \in R_{x',z}$. Clearly, either $\beta = x'$, or $\beta \in R_{x',z} \setminus \{x'\}$. If $\beta = x'$ (and $\alpha \in R_{x',y}$), then both α and β belong to $\mathcal{P}_{x,y}$ contradicting Claim 4. If $\beta \in R_{x',z} \setminus \{x'\}$, then we have the following cases:

(i) if $\alpha = x'$, then both α and β belong to $\mathcal{P}_{x,z}$ contradicting Claim 4.

(ii) if $\alpha \in R_{x',y} \setminus \{x'\}$, then the path between α and β induced on $P_{\alpha,y} \cup P_{y,z} \cup P_{z,\beta}$ does not hit x, where $P_{\alpha,y}$ is the path between α and y induced on $R_{x',y}$, $P_{z,\beta}$ is the path between z and β induced on $R_{x',z}$, and $P_{y,z}$ is any path in $\mathcal{P}_{y,z}$; note that no vertex on $P_{\alpha,y}$ and no vertex of $P_{z,\beta}$ is adjacent to x, since $R_{x',y}$ and $R_{x',z}$ are induced paths. $\Box$

Claim 5 *All paths in $\mathcal{P}_{y,z}$ of a two-long-sided $\{2,2,2\}$ graph are P_3s.*

Proof: Assume to the contrary. Thus, there exists $P_{y,z} \in \mathcal{P}_{y,z}$ such that $|P_{y,z}| > 3$. So, $|P_{x,y}| > 3$, $|P_{x,z}| > 3$, $|P_{y,z}| > 3$ for some $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$. Let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Let (α, β) be any DP of $\tilde{H}$. Where could α and β be positioned?

By Lemma 1 (2), α and β do not both belong to the union of the internal vertices of $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$. Therefore, one of α, β must be in $\{x, y, z\}$. Without loss of generality let $\alpha = x$. Then by Claim 4, $\beta \notin P_{x,y} \cup P_{x,z}$. Therefore, $\beta \in P_{y,z} \setminus \{y, z\}$.

β cannot be adjacent to both y and z, since $|P_{y,z}| > 3$. Without loss of generality assume that $\beta z \notin E$. Now the path between α and β induced on $P_{y,\beta} \cup P_{x,y}$ does not hit z contradicting (α, β) being a DP, where $P_{y,\beta}$ is the subpath of $P_{y,z}$ between y and β. Thus, $\tilde{H}$ does not have a DP contradicting H being HDP. $\Box$

In the following claim, let H be a $\{1,2,2\}$ graph with an AT $\{x, y, z\}$, a non-path-disjoint vertex x, and a vertex $x' \in P_{x,y} \cap P_{x,z}$, $x \neq x'$, for some $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$. Let $P_{x',y}$ be the subpath of $P_{x,y}$ between x' and y, $P_{x',z}$ the subpath of $P_{x,z}$ between x' and z, and let $P_{y,z}$ be any path in $\mathcal{P}_{y,z}$.

Claim 6 *In a $\{1,2,2\}$ graph H, for any $a \in P_{x',y} \setminus \{x'\}$ and any $b \in P_{x',z} \setminus \{x'\}$, $ab \notin E$.*

Proof: Assume to the contrary. Take a subgraph $\hat{H}$ of H induced on $P_{x,y} \cup P_{x,z}$. Note that $\hat{H}$ also has an AT because the path $P_{x,y}$ avoids the neighborhood of z, $P_{x,z}$ avoids the neighborhood of y, and the path from y to z induced on $P_{y,a} \cup \{ab\} \cup P_{b,z}$, where $P_{y,a}$ is the subpath of $P_{x,y}$ between y and a, and $P_{b,z}$ is the subpath of $P_{x,z}$ between b and z, avoids the neighborhood of x. (Note that no vertex in $P_{x',y} \cup P_{x',z} \setminus \{x'\}$ is adjacent to x since all paths in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z}$ are induced.) But now, vertices x, y, and z in $\hat{H}$ are all non-path-disjoint contradicting Theorem 3. $\Box$

The following claim is used in the proof of Claim 8, the analogue of Claim 5 for $\{1,2,2\}$ graphs. Let H be an HDP$\cap$AT graph with an AT $\{x, y, z\}$ and a non-path-disjoint vertex x. Let $x' \in P_{x,y} \cap P_{x,z}$, $x \neq x'$ for some induced paths $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$. If $H \setminus \{x'\}$ is disconnected into a connected component containing y and z and not containing x, and some other connected components, then we say that H is *1-disjoint with respect to* x. Let C denote the connected component of $H \setminus \{x'\}$ containing y and z. Then it is easy to see that the following holds for H.

Claim 7 *Let (α, β) be a DP of H. If $\alpha \notin C$, then $\beta \notin P_{x,y} \cup P_{x,z}$ for any path $P_{x,y} \in \mathcal{P}_{x,y}$, and any path $P_{x,z} \in \mathcal{P}_{x,z}$.*

Claim 8 *All paths in $\mathcal{P}_{y,z}$ of a $\{1, 2, 2\}$ graph H are P_3s.*

Proof: Assume to the contrary. Let $P_{y,z}$ be a path in $\mathcal{P}_{y,z}$ that is of length bigger than 2. Let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$, for some $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$ such that $P_{x,y} \cap P_{x,z} = \{x, x'\}$. Clearly, $\tilde{H}$ is 1-disjoint with respect to x. Let (α, β) be a DP of $\tilde{H}$. One of α, β must be in $\{x, x'\}$, since otherwise the path between them induced on $V(\tilde{H}) \setminus \{x, x'\}$ would miss x. (Note that no vertices in $V(\tilde{H}) \setminus \{x, x'\}$ are adjacent to x, since $\tilde{H}$ is 1-disjoint with respect to x.) By Claim 7, $\beta \in P_{y,z} \setminus \{y, z\}$ in $\tilde{H}$. Since by assumption $P_{y,z}$ is not a P_3, β is not adjacent to at least one of y, z. Without loss of generality let $\beta y \notin E$. But now, the path induced on $P_{\beta,z} \cup P_{x,z}$, where $P_{\beta,z}$ is the subpath of $P_{y,z}$ between β and z, is an α, β-path missing y contradicting (α, β) being a DP of $\tilde{H}$. $\square$

The following claim will be used to prove Claim 10 below, which explains the relationship between long paths in a two-long-sided $\{2, 2, 2\}$ graph.

Claim 9 *In a two-long-sided $\{2, 2, 2\}$ graph H, a vertex of distance $i \geq 2$ from x on $P_{x,y} \in \mathcal{P}_{x,y}$, cannot be adjacent to a vertex of distance $j \geq 2$ from x on $P_{x,z} \in \mathcal{P}_{x,z}$.*

Proof: Assume to the contrary. Denote by u a vertex of distance i from x on $P_{x,y}$, for $i \geq 2$, and by v a vertex of distance j from x on $P_{x,z}$, for $j \geq 2$, with $uv \in E$. Note that $u \neq y$, because otherwise u could not be adjacent to any vertex on $P_{x,z}$, since y is an AT vertex. Similarly, $v \neq z$. Consider the subgraph $\tilde{H}$ of H induced on $P_{x,y} \cup P_{x,z}$. Since $uv \in E$, $\tilde{H}$ has an AT $\{x, y, z\}$ with AT vertices y and z non-path-disjoint, contradicting Theorem 3. $\square$

Claim 10 *For all paths $P, Q \in \mathcal{P}_{x,y}$ that are long paths of a two-long-sided $\{2, 2, 2\}$ graph H, and for all $v_1 \in P \setminus \{x, x_p\}$ and for all $v_2 \in Q \setminus \{x, x_q\}$, where $x_p = N(x) \cap P$ and $x_q = N(x) \cap Q$, $v_1 \in D(x, v_2)$, or $v_2 \in D(x, v_1)$.*

Proof: Assume to the contrary. Let $P_{x,z}$ be any long path in $\mathcal{P}_{x,z}$, and let x_z be the neighbor of x on $P_{x,z}$. Let $v_1 \in P \setminus \{x, x_p\}$, $v_2 \in Q \setminus \{x, x_q\}$ be such that $v_1 \notin D(x, v_2)$ and $v_2 \notin D(x, v_1)$. Note that $v_1x_z \notin E$ and $v_2x_z \notin E$ since H is a $\{2,2,2\}$ graph; that is, if $v_1x_z \in E$, then x would not be path-disjoint. Let $\check{H}$ be the subgraph of H induced on $P \cup Q \cup P_{x,z}$.

Consider the graph $\check{H}$. $\{v_1, v_2, z\}$ is an AT of $\check{H}$ for the following reasons. The path from v_1 to v_2 induced on $P_{v_1,y} \cup P_{v_2,y}$ misses z, where $P_{v_1,y}$ is the path between v_1 and y induced on P, and $P_{v_2,y}$ is the path between v_2 and y induced on Q; this is true by the definition of $\mathcal{P}_{x,y}$. The path between z and v_1 induced on $P_{x,z} \cup P_{x,v_1}$ misses v_2, where P_{x,v_1} is the path between v_1 and x induced on P; this is true because $v_2 \notin D(x, v_1)$, $v_2x_z \notin E$ (since H is $\{2,2,2\}$), and also, by Claim 9, v_2 is not adjacent to any non-neighbor of x on $P_{x,z}$. Similarly, the path between z and v_2 induced on $P_{x,z} \cup P_{x,v_2}$ misses v_1, where P_{x,v_2} is the path between v_2 and x induced on Q. Therefore, $\{v_1, v_2, z\}$ is an AT of $\check{H}$ and $xz \notin E$, contradicting Claim 1. □

Other claims similar to the above claim hold for $\{1,2,2\}$ graphs as well [10].

From the above we can see that frame HDP graphs have a rich and interesting structure. This structure enables us to determine the position of dominating pair vertices in all frame HDP graphs, as described in the following two theorems.

Theorem 4 *Let M_1, M_2, M_3 be the sets of mid-points of P_3s of $\mathcal{P}_{x,z}, \mathcal{P}_{y,z}$ and $\mathcal{P}_{x,y}$ respectively of a $\{2,2,2\}$ graph H. Note that either or both of M_1 and M_3 could be empty. DP vertices of H satisfy the following. Either:*
(a) one DP vertex is in $N[x]$ and the other one is in M_2, or
(b) one DP vertex is in $N[y]$ and the other one is in M_1, or
(c) one DP vertex is in $N[z]$ and the other one is in M_3.
Each of these three types of DPs can occur.

Proof: Let (α, β) be a DP of H. First assume that H is a two-long-sided $\{2,2,2\}$ graph with all paths in $\mathcal{P}_{x,y}$ and all paths in $\mathcal{P}_{x,z}$ being long. Let $P_{x,y}$ be any path in $\mathcal{P}_{x,y}$, let $P_{x,z}$ be any path in $\mathcal{P}_{x,z}$, and let $P_{y,z}$ be any path in $\mathcal{P}_{y,z}$ of H.

By Claim 4, α and β cannot both belong to $P_{x,y}$, cannot both belong to $P_{x,z}$, and cannot both belong to $P_{y,z}$. By Lemma 1 (2), α and β cannot belong to the union of the internal vertices of $P_{x,y}$

and $P_{x,z}$. Let $x' = P_{x,y} \cap N(x)$. It is not the case that one of α, β (say α) is the internal vertex of $P_{y,z}$ and the other one (namely β) belongs to $P_{x,y} \setminus \{x, x', y\}$ since the path from α to β induced on $\{\alpha y\} \cup P_{y,\beta}$ does not hit x contradicting (α, β) being a DP, where $P_{y,\beta}$ is the subpath of $P_{x,y}$ between β and y. Similarly, it is not the case that one of α, β is in $P_{y,z} \setminus \{y, z\}$ and the other one belongs to $P_{x,z} \setminus \{x, x'', z\}$, where $x'' = P_{x,z} \cap N(x)$. By Lemma 1 (1), it is not the case that one of α, β is equal to z and the other one belongs to $P_{x,y} \setminus \{x, y\}$. Similarly, it is not the case that one of α and β is equal to y and the other one belongs to $P_{x,z} \setminus \{x, z\}$. Therefore, the only possible position for (α, β) is that one of them is in $N[x]$ and the other one is in M_2.

Next we assume that H is two-long-sided with a short path in a long side. Without loss of generality, let $\mathcal{P}_{x,y}$ contain a short path. Following the above argument, we conclude that the only two options for positions of α and β are that one of them is in $N[x]$ and the other one is in M_2, or that one of them is in $N[z]$ and the other one is in M_3. Similar arguments prove the theorem for one-long-sided and no-long-sided $\{2, 2, 2\}$ graphs.

Examples showing that each of these three types of DPs can occur are given in Figure 4. □

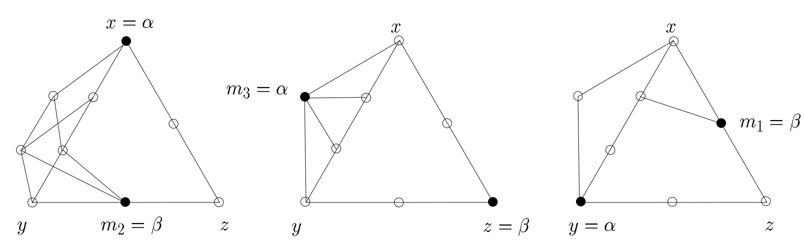

Figure 4: Examples of positions of dominating pairs of $\{2, 2, 2\}$ graphs. α and β are DP vertices.

In a $\{1, 2, 2\}$ graph, denote by $\tilde{\mathcal{R}}_{x,y}$ the set of all induced paths between x and y that avoid $N(z)$ and do not share vertices with paths in $\mathcal{R}_{x,y}$. Define $\tilde{\mathcal{R}}_{x,z}$ similarly. We use the following simple observation in the proof the next theorem.

Observation 1 *For any HDP graph G and any of its DPs (α, β), if H is an induced connected subgraph of G containing α and β, then (α, β) is a DP of H.*

Proof: Assume that (α, β) is not a DP of H. Then, since H is connected, there exists a path P from α to β in H that misses a vertex $w \in H$. However, P also belongs to G and misses w in G, contradicting (α, β) being a DP of G. □

Theorem 5 *Let M_1, M_2, M_3 be the sets of mid-points of P_3s of $\tilde{\mathcal{R}}_{x,z}, \mathcal{P}_{y,z}$ and $\tilde{\mathcal{R}}_{x,y}$ respectively of a $\{1,2,2\}$ graph H. Note that either or both of M_1 and M_3 could be empty. DP vertices of H satisfy the following. Either:*
(a) one DP vertex is in $N[x]$ and the other one is in M_2, or
(b) one DP vertex is in $N[y]$ and the other one is in M_1, or
(c) one DP vertex is in $N[z]$ and the other one is in M_3.
Each of these three types of DPs can occur.

Proof: Denote by (α, β) a DP of H. By Claim 4, α and β cannot both belong to $\mathcal{P}_{x,y}$, cannot both belong to $\mathcal{P}_{x,z}$, and cannot both belong to $\mathcal{P}_{y,z}$. By Lemma 1 (3), α and β cannot both belong to the union of vertices of $R_{x',y}$ and $R_{x',z}$, where $R_{x',y}$ and $R_{x',z}$ are any paths in $\mathcal{R}_{x',y}$ and $\mathcal{R}_{x',z}$ respectively.

If $\tilde{\mathcal{R}}_{x,y} \neq \emptyset$ and $\tilde{\mathcal{R}}_{x,y}$ has a long path, then it is not the case that one of α, β (say α) is equal to z and the other one (namely β) is an internal vertex of a long path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$. To see this consider the subgraph $\tilde{H}$ of H induced on $\tilde{R}_{x,y} \cup R_{x,z} \cup P_{y,z}$, for any path $R_{x,z} \in \mathcal{R}_{x,z}$ and any $P_{y,z} \in \mathcal{P}_{y,z}$ (such paths $R_{x,z}$ and $P_{y,z}$ exist by definition of H). Now, $\tilde{H}$ is a connected two-long-sided $\{2,2,2\}$ graph with an AT $\{x,y,z\}$ containing α, β, and thus, by Observation 1, Lemma 1 (1) is contradicted. It is easy to see that $\tilde{H}$ is a $\{2,2,2\}$ graph, since no internal vertex of $\tilde{R}_{x,y}$ can be adjacent to an internal vertex of $R_{x,z}$ (otherwise we would have a contradiction with definitions of $\tilde{\mathcal{R}}_{x,y}$ and $\mathcal{R}_{x,z}$, or $\tilde{H}$ would not be HDP).

If $\tilde{\mathcal{R}}_{x,y} \neq \emptyset$ and it has a long path, then it is not the case that one of α, β (say α) is an internal vertex of some long path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$ and the other one (namely β) is an internal vertex of some path $R_{x,z} \in \mathcal{R}_{x,z}$ for the following reason. Similar to the above, take the subgraph $\tilde{H}$ of H induced on $\tilde{R}_{x,y} \cup R_{x,z} \cup P_{y,z}$, for any $P_{y,z} \in \mathcal{P}_{y,z}$. Now, $\tilde{H}$ is a connected two-long-sided $\{2,2,2\}$ graph containing α, β, and thus, by Observation 1, Lemma 1 (2) is contradicted.

If both $\tilde{\mathcal{R}}_{x,y} \neq \emptyset$ and $\tilde{\mathcal{R}}_{x,z} \neq \emptyset$, and if both have long paths, then it is not the case that one of α, β (say α) is an internal vertex of some long path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$ and the other one (namely β) is an internal vertex of some long path $\tilde{R}_{x,z} \in \tilde{\mathcal{R}}_{x,z}$ for the following reason. Similar to the above, take the subgraph $\tilde{H}$ of H induced on $\tilde{R}_{x,y} \cup \tilde{R}_{x,z} \cup P_{y,z}$, for any $P_{y,z} \in \mathcal{P}_{y,z}$. Now, $\tilde{H}$ is a connected two-long-sided (2,2,2) graph containing α, β, and thus, by Observation 1, Lemma 1 (2) is contradicted.

Similar to the proof of Theorem 4, it is not the case that one of α, β is in M_2 and the other one is in non-neighborhood of x on a long path in $\mathcal{P}_{x,y}$, or in $\mathcal{P}_{x,z}$, it is not the case that one of α, β is in M_1 and the other one is in non-neighborhood of y on a long path in $\mathcal{P}_{x,y}$, and it is not the case that one of α, β is in M_3 and the other one is in non-neighborhood of z on a long path in $\mathcal{P}_{x,z}$.

The only options for DP vertices α, β are that either one of them is in $N[x]$ and the other one is in M_2, or that one of them is in $N[z]$ and the other one is in M_3, or that one of them is in $N[y]$ and the other one is in M_1.

Examples showing that each of these three types of DPs can occur are given in Figure 5. $\Box$

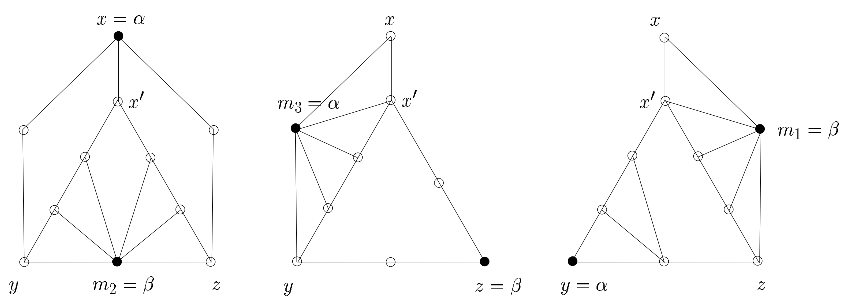

Figure 5: Examples of positions of dominating pairs of $\{1, 2, 2\}$ graphs. α and β are DP vertices.

Also, if we fix the position of DP vertices in a frame HDP graph, their position determines the structure of the graph [10]. For example, it is easy to see that in a no-long-sided $\{2, 2, 2\}$ graph with an AT $\{x, y, z\}$, a DP (α, β), and sets $M_i, i \in \{1, 2, 3\}$ of mid-vertices of $\mathcal{P}_{x,y}$, $\mathcal{P}_{x,z}$ and $\mathcal{P}_{y,z}$ respectively, if $\alpha \in M_i$ and $\beta \in M_j$, where $i \neq j$ and $i, j \in \{1, 2, 3\}$, then every vertex in M_k, for $k \in \{1, 2, 3\} \setminus \{i, j\}$, must either be adjacent to α or to β.

We now turn to proving that the diameter of a frame HDP graph is less than or equal to 5. To prove this, we first establish the following two claims and two lemmas.

Claim 11 *In a two-long-sided $\{2, 2, 2\}$ graph H, all vertices of distance $i \geq 3$ from x on $P_{x,y}$, if they exist, must be adjacent to all vertices in M, where M is the set of mid-vertices of all paths in $\mathcal{P}_{y,z}$ of H. By symmetry, the same holds for the vertices of distance i from x on $P_{x,z}$.*

Proof: Assume to the contrary. Thus, there exists a vertex u of distance i from x on $P_{x,y}$, where $i \geq 3$, that is not adjacent to a vertex $v \in M$. Note that u is not adjacent to x', where x' is the neighbor of x on $P_{x,z}$, and $P_{x,z}$ is a long path in $\mathcal{P}_{x,z}$, which exists since H is a two-long-sided

$\{2, 2, 2\}$ graph. Let $P_{y,z} \in \mathcal{P}_{y,z}$ be the path that contains v. Consider the subgraph $\tilde{H}$ of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Denote by (α, β) a DP of $\tilde{H}$. From Theorem 4, one of $\{\alpha, \beta\}$ is in $N[x]$ and the other one is in M in $\tilde{H}$.

It is not the case that one of α and β is equal to v and the other one belongs to $\{x, x'\}$, since otherwise the path between them induced on $\{vz\} \cup P_{x,z}$ would miss u. Similarly, it is not the case that one of α and β is equal to v and the other one belongs to $P_{x,y} \cap N(x)$ for the following reason. Assume to the contrary. Thus, without loss of generality assume that $\alpha = v$ and $\beta \in P_{x,y} \cap N(x)$. Since $\beta x \in E$, the path from α to β induced on $\{vz\} \cup P_{x,z} \cup \{x\beta\}$ misses u. Thus, $\tilde{H}$ does not have a DP contradicting H being HDP. □

Claim 12 *In a $\{1, 2, 2\}$ graph H, every non-neighbor of x' in a long path in $\mathcal{R}_{x',y}$ must be universal to M. By symmetry, the same holds for $\mathcal{R}_{x',z}$.*

Proof: Assume to the contrary. Thus, there exists a non-neighbor v of x' on a long path $P_{x',y} \in \mathcal{R}_{x',y}$ that is not adjacent to a vertex $m \in M$. Let $\tilde{H}$ be the subgraph of H induced on $P_{x',y} \cup \{x\} \cup P_{y,z} \cup P_{x',z}$, where $P_{x',z}$ is any path in $\mathcal{R}_{x',z}$, and $m \in P_{y,z}$. Let (α, β) be a DP of $\tilde{H}$. Since $\tilde{H}$ is 1-disjoint with respect to x, by Claim 2, one DP vertex of $\tilde{H}$ is in $\tilde{H} \setminus C$ and the other one is in C, where C is the connected component of $\tilde{H} \setminus \{x'\}$ containing y, z. Without loss of generality assume $\alpha \in \tilde{H} \setminus C = \{x, x'\}$ (note that since $\tilde{H}$ is a frame HDP graph, by Claim 2, $\tilde{H} \setminus C = \{x, x'\}$). Then, by Claims 7 and 8, β is the midpoint of $P_{y,z}$ in $\tilde{H}$, i.e., $\beta = m$. Since $v \in P_{x',y} \setminus \{x', y\}$, by Claim 6 and the assumption that $vx' \notin E$, v is not adjacent to any vertex in $P_{x',z}$. Now, the path from α to β induced on $\{\beta z\} \cup P_{x',z} \cup \{x\}$ does not hit v contradicting (α, β) being a DP of $\tilde{H}$. □

Lemma 2 *Let G be a $\{2, 2, 2\}$ graph. Then $diam(G) \leq 5$.*

Proof: By Claim 5, $\mathcal{P}_{y,z}$ consist of P_3s only. Let $P = x_0, x_1, \ldots, x_\ell$ be a path in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z}$, where $x_0 = x$.

By Claim 11, all vertices x_i, for $i \geq 3$ are adjacent to all mid-vertices m of paths in $\mathcal{P}_{y,z}$. Hence, $d(x_i, m) \leq 4$ for all x_i of P. This implies $d(x_i, x_j) \leq 5$ for x_i, x_j in P.

Now consider any other path $Q = x'_0, x'_1, \ldots, x'_k$ in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z}$, where $x'_0 = x$. All that remains to be shown is that $d(x_i, x'_j) \leq 5$ for all x_i, x'_j contained in P, Q correspondingly. By Claim 11,

$d(x_i, x'_j) \leq 5$ for $i \geq 1$, $j \geq 2$, as well as for $i \geq 2$, $j \geq 1$. Since $x_0 = x'_0 = x$, $d(x_0, x'_j) \leq 5$, $d(x'_0, x_j) \leq 5$ for all j, and $d(x_1, x'_1) \leq 2$. Consequently, $diam(G) \leq 5$. □

An example of a $\{2, 2, 2\}$ graph with diameter 5 is presented in Figure 6.

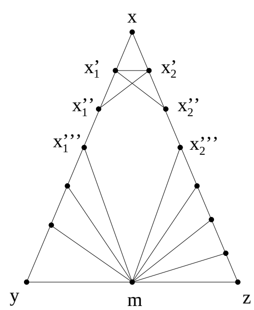

Figure 6: An example of a $\{2, 2, 2\}$ graph with diameter 5 and diametral points x and y.

Lemma 3 *Let G be a $\{1, 2, 2\}$ graph. Then $diam(G) \leq 5$.*

Proof: Similar to the proof of Lemma 2, using Claim 8 and Claim 12. □

An example of a $\{1, 2, 2\}$ graph with diameter 5 is presented in Figure 7.

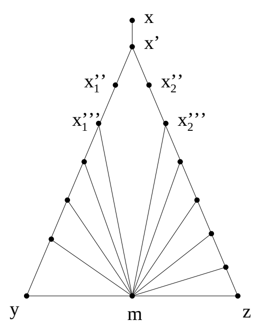

Figure 7: An example of a $\{1, 2, 2\}$ graph with diameter 5 and the diametral points x and y.

Theorem 6 *The diameter of a frame HDP graph is less than or equal to 5.*

Proof: Follows directly from Lemmas 2 and 3. □

Deogun and Kratsch's Polar Theorem holds only for weak dominating pair graphs with diameter at least 5 [3]. Since we have shown that the diameter of a frame HDP graph is always less than or equal to 5, their Polar Theorem works only for those frame HDP graphs with diameter 5. The following holds for all frame HDP graphs.

Lemma 4 (Polar Lemma) *Let G be a frame HDP graph of diameter at least 3, and let Z be the set of all dominating pairs of G with V_z being the set of all vertices of Z. There exists a partition of V_z into sets P and Q such that $(p, q) \in Z$ implies $p \in P$ and $q \in Q$.*

Proof: Let $\{x, y, z\}$ be an AT of G. Denote by M_2 the set of mid-vertices of all P_3s in $\mathcal{P}_{y,z}$, by M_1 the set of mid-vertices of all P_3s in $\mathcal{P}_{x,z}$, and by M_3 the set of mid-vertices of all P_3s in $\mathcal{P}_{x,y}$. The only way the above partition would not be possible is if $\{(a, b), (b, c), (c, a)\} \subseteq Z$. This can never happen in frame HDP graphs of diameter at least 3 for the following reasons. From Theorems 4 and 5 we know that in frame HDP graphs for each DP either one DP vertex is in $N[x]$ and the other one is in M_2, or one is in $N[y]$ and the other one is in M_1, or one is in $N[z]$ and the other one is in M_3. Assume that $\{(a, b), (b, c), (c, a)\} \subseteq Z$. Without loss of generality let $a \in N[x]$ and $b \in M_2$. By Theorems 4 and 5 since $(b, c) \in Z$ and since $b \in M_2$, we know that c must be in $N[x]$. But now we see that $(c, a) \in Z$ and c, a both belong to $N[x]$ contradicting Theorems 4 and 5; note that $N[x]$ does not dominate since $diam(G) \geq 3$. $\square$

The other direction of Lemma 4 does not hold for frame HDP graphs of diameter smaller than 5. An example is presented in Figure 8, where $Z = \{(x, m), (y, w_2), (w_2, m)\}$, $P = \{m, y\}$, $Q = \{x, w_2\}$, and $(y, x) \notin Z$.

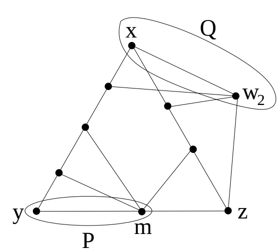

Figure 8: A counter example for the reverse of Lemma 4.

4 Further Results and Future Directions

We now describe some open problems in this area, and also study whether various properties of AT-free graphs generalize to HDP graphs.

Notice that HDP graph recognition is in coNP. It is easy to find a short proof that a graph $G = (V, E)$ is not HDP by showing that a particular induced subgraph has the property that for every pair of vertices u, v, there is some other vertex w and a u, v-path missing w. The complexity of HDP graph recognition is one of the topics for future research; possibly it is coNP-complete.

Deogun and Kratsch [3] characterized chordal HDP graphs in the following way.

Theorem 7 [3] *A chordal graph G is HDP if and only if it does not contain the graphs A_1 and $B_n (n \geq 1)$ as induced subgraphs (see Fig. 9).*

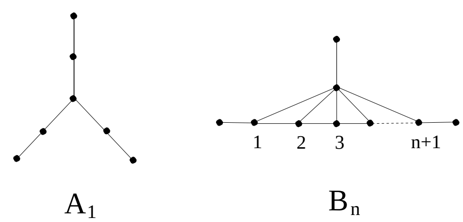

Figure 9: Forbidden induced subgraphs for chordal HDP graphs.

We use this characterization to find a polynomial time algorithm for recognizing chordal HDP graphs.

Theorem 8 *Chordal HDP graphs can be recognized in polynomial time.*

Proof: A polynomial time algorithm for recognizing chordal HDP graphs first determines if a graph $G = (V, E)$ is chordal by using Rose, Tarjan, and Lueker's linear time algorithm [4]. If G is chordal, then the algorithm checks in time $O(|V|^7)$ if G contains a subgraph A_1 from Theorem 7. If G does not contain a subgraph A_1, then the algorithm determines in polynomial time, whether G contains a subgraph B_i, for $i \geq 1$. This is done as follows. First all ATs of G are determined. Then for each AT and each ordering $\{x, y, z\}$ of the AT vertices a set $S = N(x) \setminus (N(y) \cup N(z))$ is determined. For each $s \in S$ a set $C = N(s) \setminus (N[x] \cup (N(y) \cap N(z)))$ is defined. Now, it is checked whether y and z are in the same connected component of the graph induced by $C \cup \{y, z\}$ in G and, if this is the case, a corresponding y, z-path P is determined. It follows by the construction, that the graph induced by P, s, x forms an induced subgraph B_i of G. Reversely, if G contains a B_i, it follows immediately that the algorithm will find it. If the algorithm exhausts all ATs $\{x, y, z\}$ and all AT vertices failing to find a B_n, then G is a chordal HDP graph. An example illustrating this Algorithm is presented in Figure 10. □

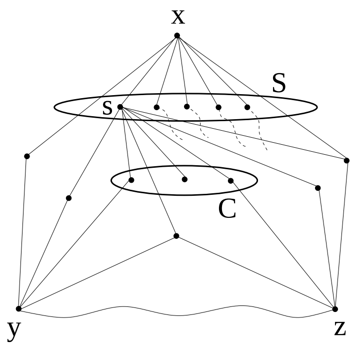

Figure 10: An example illustrating the $B_n (n \geq 1)$ recognition algorithm.

4.1 AT-free results that do not extend to HDPs

In addition to proving the existence of a dominating pair and characterizing AT-free graphs, Corneil, Olariu, and Stewart [1] showed how to augment an arbitrary AT-free graph to obtain a new AT-free graph. They called a vertex *pendant* if it is of degree one, and they said that a vertex v of an AT-free graph G *pokable* if the graph G' obtained from G by adding a pendant vertex adjacent to v is AT-free. They referred to a dominating pair (α, β) as *pokable* if both α and β are pokable vertices, and they proved that every connected AT-free graph contains a pokable dominating pair, and that every connected AT-free graph which is not a clique contains a nonadjacent pokable dominating pair. They used this result to prove the Composition Theorem for AT-free graphs, which says that for any two AT-free graphs G_1 and G_2 with pokable dominating pairs (x_1, y_1) and (x_2, y_2) respectively, the graph G constructed from G_1 and G_2 by identifying vertices x_1 and x_2 is AT-free. Unfortunately, this is not true for HDP graphs. Figure 11 gives an example of an HDP graph with no pokable DPs. Part (a) of the figure shows a graph G with its DP vertices α and β, and parts (b) and (c) show that the addition of a pendant vertex u to α makes G non-HDP, since the deletion of vertices β, v_5, v_6, v_7 from $G \cup \{u\}$ yields a non-HDP graph, and that the addition of a pendant vertex w to β makes G non-HDP, since the deletion of vertices α, v_6, v_7 from $G \cup \{w\}$ yields a non-HDP graph.

Notice that even when pokable DP vertices of HDP graphs exist, such as degree 2 vertices of 3-suns, their identification does not yield an HDP graph (see Figure 12). Thus, our intuition is that the structure of frame HDP graphs is very close to that of HDP graphs. However, this remains a major open question.

Recently, Corneil, Olariu, and Stewart discovered a linear time algorithm for finding dominating

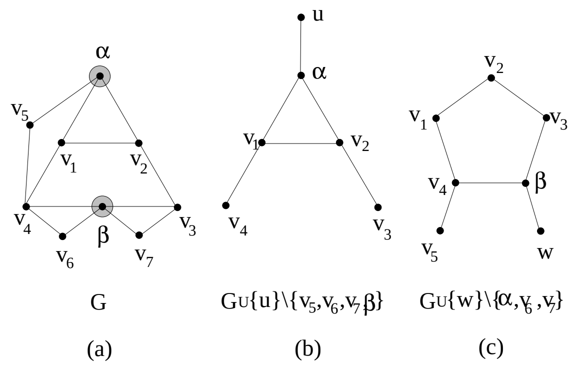

Figure 11: A counter example to pokability of DP vertices of an HDP graph.

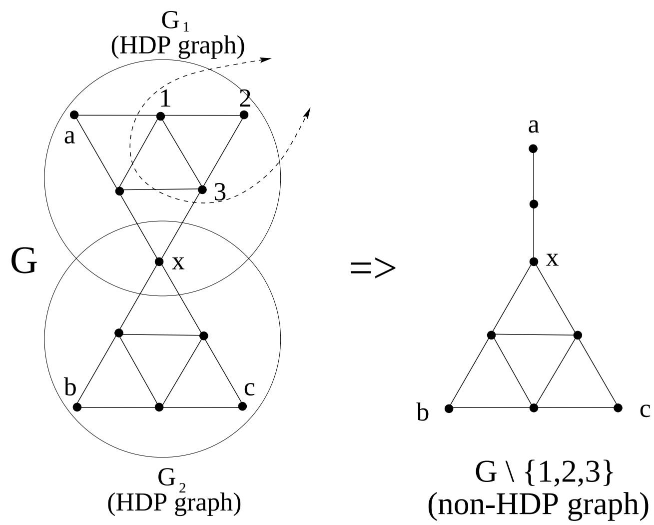

Figure 12: A counter example to the Composition Conjecture of HDP graphs; by removing vertices $1, 2$ and 3 from G we get a $\{1, 2, 2\}$ graph with an AT $\{a, b, c\}$, a non-path disjoint vertex a, and ax is not an edge, contradicting Claim 1.

pairs in AT-free graphs [2]. Their algorithm is based on two sweeps of the lexicographic breadth-first search (LBFS) algorithm: let y be the last vertex of an arbitrary LBFS, and let z be the last vertex of any LBFS starting at y; then (y, z) is a pokable DP of G. In addition, they described how to use this algorithm to find in linear time all dominating pairs in a connected AT-free graph with diameter greater than or equal to three, even though there may be $O(|V|^2)$ dominating pairs. To see why this algorithm does not work for HDP graphs, first recall Gallai's definition of a *knotting graph* of G [5].

Definition 3 *For a given graph $G = (V, E)$, the corresponding* knotting graph *is $K[G] = (V_K, E_K)$, where V_K and E_K are defined as follows. For each vertex v of G, there are copies $v_1, v_2, ..., v_{i_v}$ in*

V_K, where i_v is the number of connected components of $\overline{N(v)}$, the complement of the graph induced by $N(v)$. For each edge vw of E there is an edge v_iw_j in E_K, where w is contained in the i^{th} connected component of $\overline{N(v)}$ and v is contained in the j^{th} connected component of $\overline{N(w)}$.

Using this definition, Köhler characterized a dominating pair of a graph G as a pair of vertices (a, b) of G such that each common neighbor x of a and b in $\overline{G}$ has two different copies in the knotting Graph $K[\overline{G}]$ that are adjacent to a copy of a and b, correspondingly [7]. We use this characterization to first prove that there exists an infinite family of HDP graphs with all of their DPs adjacent, and then to show that there exists an infinite family of HDP graphs with a non-adjacent DP for which the 2-sweep LBFS algorithm does not find DP vertices. To prove this we define a graph K_n^+ in the following way. Consider a circular order of the vertices of K_n. A graph that consists of a K_n and an independent set U of n vertices each of which is adjacent to $n-2$ consecutive vertices of the K_n and no two vertices of U are adjacent to the same $n-2$ consecutive vertices of K_n, is called a K_n^+ *graph*, where $n \geq 3$. Examples of a K_n^+ graphs for $n = 4, 5$ are given in Figure 13. Notice that K_4^+ is HDP and that its only DPs are (α, β) and $(\tilde{\alpha}, \tilde{\beta})$, which are both adjacent DPs. Also, notice that K_3^+ is not HDP.

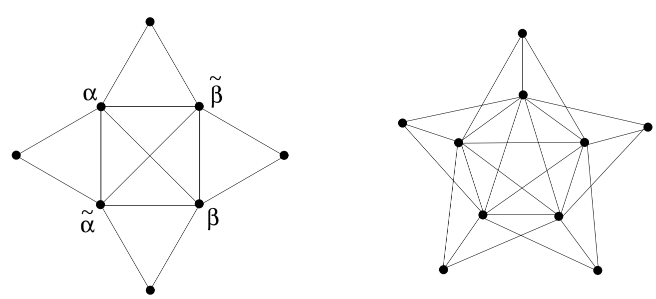

Figure 13: K_4^+ and K_5^+ graphs.

Claim 13 *K_n^+ graphs, for $n \geq 4$, are HDP and all of their DPs are adjacent.*

Proof: Consider the circular order of the K_n vertices in K_n^+. Every two non-consecutive vertices on the cycle of K_n in K_n^+ are DP vertices, and these are all the DP vertices of K_n^+. This is because the knotting Graph $K[\overline{K_n^+}]$ consists of a K_n and an independent set of vertices (an example for $n = 6$ is presented in Figure 14), and therefore, by Köhler's [7] characterization of DPs, these are all the DP vertices of K_n^+.

Now we prove that K_n^+ graphs, for $n \geq 4$ are HDP, i.e., that all connected induced subgraphs of K_n^+ are HDP. Since the only DP vertices of K_n^+ are the non-consecutive vertices of the clique cycle, if these vertices are present in an induced subgraph H of K_n^+, they are DP vertices of H. Therefore, we only need to prove that induced subgraphs of K_n^+ that contain either 0 vertices of the clique, or 1 vertex of the clique, or 2 consecutive vertices of the clique cycle, are HDP. Clearly, $K_n^+ \setminus K_n$ is an independent set and therefore an HDP. If an induced subgraph H of K_n^+ contains only one vertex of the clique, then H is a star $K_{1,n-2}$, which is HDP. If an induced subgraph H of K_n^+ contains two consecutive vertices of the clique cycle, then H is equal to one of the following graphs:

(a) paths P_2, P_3, P_4,

(b) a set of 1 to $n-3$ triangles with a common edge,

(c) a set of 1 to $n-3$ triangles with a common edge and one extra vertex adjacent to a vertex of the common edge,

(d) a set of 1 to $n-3$ triangles with a common edge and two extra vertices, one adjacent to one vertex of the common edge and the other vertex adjacent to the other vertex of the common edge.

All of the graphs (a)-(d) are AT-free and therefore HDP. Therefore, every K_n^+, for $n \geq 4$, is HDP. $\square$

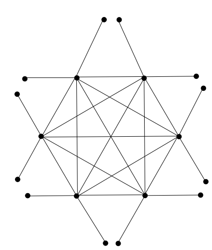

Figure 14: The knotting graph $K[\overline{K_6^+}]$.

Claim 14 *There exists an infinite family of HDP graphs with a non-adjacent DP for which the 2-sweep LBFS algorithm does not find DP vertices.*

Proof: Consider graph K_n^+, $n \geq 4$, with an extra vertex u universal to all vertices of U, and call such a graph K_n^*. Denote by $1, 2, 3, ..., n$ the vertices on the cycle of K_n in K_n^*, and by $a, b, c, \ldots$ vertices of the independent set U of K_n^*. Using the knotting graph technique, we determine that all DPs of K_n^* consist of the vertex u and a vertex of the cycle of K_n, i.e., all DPs of K_n^* are $(u, 1)$, $(u, 2)$, $(u, 3)$, $\ldots$, (u, n).

To prove that K_n^*, $n \geq 4$, are HDP, we need to prove that all connected induced subgraphs of K_n^* have DPs. Since $(u,1),(u,2),(u,3),\ldots,(u,n)$ are all DPs of K_n^*, it is enough to consider only connected induced subgraphs of K_n^* that do not contain these DPs. That is, we need to determine that the subgraphs $K_n^* \setminus \{u\}$, $K_n^* \setminus \{1,2,...,n\}$, and $K_n^* \setminus \{1,2,...,n,u\}$ of K_n^* are HDP. Clearly, $K_n^* \setminus \{u\} = K_n^+$, and therefore they are HDP, by Claim 13. $K_n^* \setminus \{1,2,...,n\}$ is a star $K_{1,n}$ and therefore is HDP. Also, $K_n^* \setminus \{1,2,...,n,u\} = U$ which is an independent set and therefore is HDP. Therefore, all K_n^*, $n \geq 4$, are HDP.

Clearly, every LBFS that starts at a vertex $v \in U$ and first visits the neighbors of v that are the clique vertices, i.e., in $\{1,2,...,n\}$, ends at a vertex in U which cannot be a DP vertex of K_n^*. Therefore, a 2-sweep LBFS cannot be used to find DPs of K_n^*, for $n \geq 4$. That is, there exists an infinite family of HDP graphs that have a non-adjacent DP for which the 2-sweep LBFS algorithm does not find DP vertices. □

Note that for a graph G that is either a K_n^+, or a K_n^*, $n \geq 4$, and for a DP $\{\alpha,\beta\}$ of G, $N[\alpha] \cup N[\beta] = V(G)$. Since for these graphs the 2-sweep LBFS algorithm did not find DPs, the question is whether the 2-sweep LBFS algorithm finds DPs for HDP graphs for which $N[\alpha] \cup N[\beta] \neq V(G)$, for all DPs $\{\alpha,\beta\}$. However, it can be seen that adding a clone w of the vertex u in K_n^*, $n \geq 4$, creates an infinite family of HDP graphs for which all DPs are non-adjacent, $N[\alpha] \cup N[\beta] \neq V(G)$ for all DPs $\{\alpha,\beta\}$, and still the 2-sweep LBFS does not find DPs.

4.2 A hierarchy above permutation graphs

It was mentioned before that HDP graphs are not closed under complements. An example is C_7 which is not HDP, but whose complement is AT-free, and therefore HDP. This motivates the definition of *coHDP* graphs, the complements of HDP graphs. Notice that HDP $\cap$ coHDP $\neq \emptyset$, since both C_6 and $\overline{C}_6$ are HDP, and therefore,$\{C_6, \overline{C}_6\} \subseteq$ coHDP. Remember that permutation graphs are those graphs which are at the same time comparability and cocomparability [9] and that C_5 is not a permutation graph. Also, if we call *coAT-free* the complements of AT-free graphs, we know that AT-free graphs strictly contain cocomparability graphs [6], coAT-free graphs strictly contain comparability graphs, and C_5 belongs to AT-free $\cap$ coAT-free, while C_6 does not. Clearly, AT-free $\subset$ HDP and coAT-free $\subset$ coHDP, and we have seen that $C_6 \in$ HDP $\cap$ coHDP. Thus, it might be interesting to look into the hierarchy of graph classes in the intersections AT-free ∩ coAT-free, and HDP ∩ coHDP (see Figure 15).

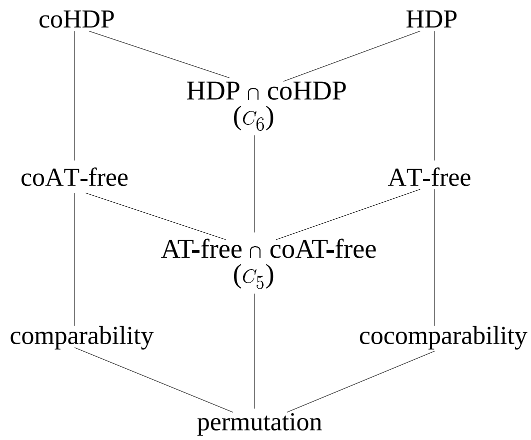

Figure 15: A hierarchy of graph classes around HDP and coHDP graphs.

5 Acknowledgments

The authors wish to thank the National Sciences and Research Council of Canada for financial support, and the Fields Institute for their hospitality.

References

[1] D. G. Corneil, S. Olariu, and L. Stewart. Asteroidal triple-free graphs. *SIAM J. Discrete Math.*, 10(3):399–430, August 1997.

[2] D. G. Corneil, S. Olariu, and L. Stewart. Linear time algorithms for dominating pairs in asteroidal triple-free graphs. *SIAM J. Comput.*, 28(4):1284–1297, 1999.

[3] J.S. Deogun and D. Kratsch. Dominating pair graphs. Upublished.

[4] G.S. Lueker D.J. Rose, R.E. Tarjan. Algorithmic aspects of vertex elimination on graphs. *SIAM J. Comput.*, 5:266–283, 1976.

[5] T. Gallai. Transitiv orientierbare Graphen. *Acta Math. Acd. Sci. Hungar.*, 18:25–66, 1967. See: A Translation of Gallai's Paper: 'Transitiv Orientierbare Graphen' by F. Maffray and M. Preissmann, *Perfect Graphs*, ed J. L. Ramirez Alfonsin and B. A. Reed. John Wiley and Sons, 2001, 25-66.

[6] M. C. Golumbic, C. L. Monma, and W. T. Trotter. Tolerance graphes. *Discrete Appl. Math.*, 9:157–170, 1984.

[7] E. G. Köhler. *Graphs Without Asteroidal Triples*. PhD thesis, TU Berlin, 1999.

[8] C. G. Lekkerkerker and J. Ch. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51:45–64, 1962.

[9] A. Pnueli, A. Lempel, and S. Even. Transitive orientation of graphs and indentification of permutation graphs. *Canadian Journal of Mathematics*, 23:160–175, 1971.

[10] N. Pržulj. Minimal Hereditary Dominating Pair Graphs. Master's thesis, Dept. of Computer Science, University of Toronto, January 2000. http://www.cs.toronto.edu/˜natasha/.

[11] D. B. West. *Introduction to Graph Theory*. Prentice Hall, 1996.