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# MINIMUM AVERAGE TIME BROADCAST GRAPHS

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ABSTRACT

We initiate the study of minimum average time broadcast graphs - those graphs on $n$ vertices with the fewest edges in which every vertex can broadcast in minimum average time. We find minimum average time broadcast graphs for all even $n$ and for more than half of all odd $n$. In addition, we give some upper and lower bounds on the number of edges in such graphs for all $n$.

*Keywords*: broadcasting, average time

## 1. Introduction and Definitions

Given a graph $G = (V, E)$ and a vertex $u \in V$, *broadcasting* is the process of disseminating a piece of information from vertex $u$ (called the *originator*) to every other vertex in the graph where, in each time unit, any vertex which knows the information can pass the information to at most one of it's neighbors. The set of calls used to disseminate the information is called a *broadcast scheme.*

A *broadcast graph* on $n$ vertices is a graph which allows any vertex to broadcast in time $\lceil \log n \rceil$. A *minimum broadcast graph* on $n$ vertices is a broadcast graph with the minimum number of edges over all broadcast graphs on $n$ vertices. This minimum number of edges is denoted $B(n)$. The study of minimum broadcast graphs and $B(n)$ has a long history. See [6] for a survey on this and related problems and [5] for a recent reference on the construction of minimum broadcast graphs.

In this paper, we are interested in broadcasting under a slightly different time constraint. In particular, we wish to minimize the average time at which a vertex is informed during a broadcast. Broadcasting in a tree under this model was studied by Koh and Tcha [7].

A broadcast scheme can be represented by a spanning tree $T_u$ of $G$ rooted at the originator $u$ with each vertex $v$ labeled with $t(v)$, the time that $v$ receives the message under that scheme. We assume that $t(u) = 0$ for originator $u$. The *average time of a broadcast scheme* is $b_a(T_u) = (\sum_{v \in V} t(v)) \; / \; |V|$. For each originator $u$, we want to determine $\min b_a(T_u)$ over all possible broadcast trees rooted at $u$, that is we want to determine the *minimum average broadcast time for originator* $u$, $b_a(u) = \min\{b_a(T_u)\}$. We define the *minimum average broadcast time for graph* $G$ to be $b_a(G) = \max_{u \in V}\{b_a(u)\}$, that is, the largest minimum average broadcast time for any originator in $G$. Since $b_a(K_n)$ is the minimum of $b_a(G)$ for all graphs $G$ on $n$ vertices, we let $\beta_a(n) = b_a(K_n)$ and say that $\beta_a(n)$ is the *minimum average broadcast time* for $n$ vertices.

A *minimum average time broadcast graph* (*matbg*) is a graph on $n$ vertices for which $b_a(G) = \beta_a(n)$ and such that for any $G'$ on $n$ vertices with $b_a(G) = b_a(G')$, $G$ has no more edges than $G'$. In other words, matbg's are those graphs on $n$ vertices with the fewest edges which allow broadcasting in minimum average time from any originator. We use $B_a(n)$ to denote the number of edges in a minimum average time broadcast graph on $n$ vertices.

In Section 2, we determine a formula for $\beta_a(n)$ and give some general bounds on $B_a(n)$. In Section 3, we present exact values of $B_a(n)$ for some $n$.

## 2. Bounds

We begin by determining the minimum average broadcast time for $n$ vertices.

**Lemma 2.1** *For* $n = 2^k + i$, *where* $k = \lfloor \log n \rfloor$ *and* $0 \leq i < 2^k$, $\beta_a(n) = k - 1 + \frac{2i+1}{n}$.

**Proof.** To minimize the average time a vertex is informed, it suffices to minimize the sum of the times that all of the vertices are informed. Since the number of informed vertices is 1 at time 0 and can at most double in each time unit after that, the sum of the times that vertices are informed in any broadcasting scheme must be at least $\sum_{j=1}^{k} j2^{j-1} + i(k+1) = ((k-1)2^k + 1) + i(k+1)$. This can be achieved in a complete graph on $n$ vertices, so the minimum average broadcast time for $n$ vertices is $\beta_a(n) = \frac{((k-1)2^k+1)+i(k+1)}{n} = k - 1 + \frac{2i+1}{n}$. □

The value of $B(n)$ provides a simple lower bound on $B_a(n)$.

**Lemma 2.2** *For any* $n \geq 1$, $B_a(n) \geq B(n)$.

**Proof.** A broadcast scheme that completes in minimum average time must inform its last vertices at time $\lceil \log n \rceil$. Thus, such a scheme is also a minimum time broadcast scheme and any graph that allows minimum average time broadcast must also allow minimum time broadcast. The result follows. □

Another simple lower bound on $B_a(n)$ is obtained by considering the shape of broadcast trees.

**Theorem 2.1** *For* $n = 2^k + i$, *where* $k = \lfloor \log n \rfloor$ *and* $0 \leq i < 2^k$, $B_a(n) \geq \lceil \frac{kn}{2} \rceil$.

**Proof.** To broadcast in minimum average time, it is necessary that the number of informed vertices double in each of the first $k$ time units and that the remaining vertices are informed in the following time unit. A minimum average time broadcast scheme from vertex $u$ must correspond to a spanning subtree of the graph $G$ rooted at $u$ which consists of a "core" binomial tree of $2^k$ vertices plus $n - 2^k$ additional edges, each connecting a new vertex to a distinct vertex of the core binomial tree. The root of this tree must have degree at least $k$. Since there must exist such a tree rooted at every vertex in the graph, $B_a(n) \geq \frac{kn}{2}$. In fact, since the number of edges must be an integer, $B_a(n) \geq \lceil \frac{kn}{2} \rceil$. □

We can improve this lower bound when $n$ is one less than a power of 2.

**Theorem 2.2** $B_a(2^{k+1} - 1) \geq \frac{1}{2}((2^{k+1} - 1)k + \lceil \frac{2^{k+1}-1}{k+2} \rceil)$, *for* $k \geq 1$.

**Proof.** In a minimum average time broadcasting scheme on $2^{k+1} - 1$ vertices, $2^k$ vertices must be informed by time $k$ and the remaining $2^k - 1$ vertices must be informed at time $k + 1$. This implies that either the originator or the vertex that it calls at time 1 must be of degree $\geq k + 1$ and the other of these two vertices must have degree $\geq k$. Thus, in an matbg on $2^{k+1} - 1$ vertices, every vertex must be of degree $\geq k$ and any vertex of degree $k$ must be adjacent to a vertex of degree $\geq k + 1$. Let $n = 2^{k+1} - 1$ and let $w_1, w_2, ..., w_{n'}$ denote the vertices of degree greater than $k$, where $n' \leq n$ in some matbg on $n$ vertices. Let $e_i = deg(w_i) - k$ for $1 \leq i \leq n'$. Each vertex $w_i$ of degree greater than $k$ can be adjacent to at most $k + e_i$ vertices of degree $k$. Thus, the number of vertices of degree $k$ can be at most $\sum_{1 \leq i \leq n'}(k + e_i) \geq n - n'$. It follows that $\sum_{1 \leq i \leq n'} e_i \geq n - n'(k + 1)$. Since each $e_i \geq 1$, $\sum_{1 \leq i \leq n'} e_i \geq n'$. Combining these two inequalities and the fact that the $e_i$ are integers, we obtain $\sum_{1 \leq i \leq n'} e_i \geq \lceil \frac{n}{k+2} \rceil$. The number of edges in such a graph must be at least $\frac{1}{2}(nk + \sum_{1 \leq i \leq n'} e_i)$. Thus, $B_a(2^{k+1} - 1) \geq \frac{1}{2}((2^{k+1} - 1)k + \lceil \frac{2^{k+1}-1}{k+2} \rceil)$.
□

A simple construction provides an upper bound on $B_a(n)$ for all $n$.

**Theorem 2.3** *For* $n = 2^k + i$, *where* $k = \lfloor \log n \rfloor$ *and* $0 \leq i < 2^k$, $B_a(2^k + i) \leq k(2^{k-1} + i)$.

**Proof.** To construct a graph $G$, on $n$ vertices with $k(2^{k-1} + i)$ edges in which we can broadcast in minimum average time from any originator, begin by labeling the vertices $a_1, a_2, ...a_{2^k}$ and $b_1, b_2, ...b_i$. Construct an matbg $A$ on the vertices $a_1, a_2, ...a_{2^k}$. (Such an matbg has $k2^{k-1}$ edges, as will be shown in Theorem 3.5. It is also important to note that the edges of $A$ comprise $k$ complete matchings.) For every edge $(a_j, a_m)$ in $A$, where $1 \leq j \leq i$ and $1 \leq m \leq 2^k$, add the edge $(b_j, a_m)$ to $G$.

To broadcast from vertex $a_j$ of $G$ in minimum average time, broadcast first in $A$ according to a minimum average time scheme for $A$ and then, at time $k + 1$, the $b_j$ vertices can be informed by edges corresponding to a single matching in $A$. If the originator of the broadcast is $b_j$, broadcast according to the scheme for $A$ substituting $b_j$ for $a_j$ and, at time $k + 1$, complete the broadcast using the edges corresponding to a single matching in $A$. □

## 3. Exact Values

For $n$ less than 9 and for $n = 2^k$, it is easy to show that the minimum broadcast graphs on $n$ vertices are also minimum average time broadcast graphs.

**Theorem 3.4** *For* $1 \leq n \leq 8$, $B_a(n) = B(n)$.

**Proof.** For these small values of $n$, minimum average time broadcast schemes can be easily constructed for minimum broadcast graphs. (These graphs can be found in [4].) The result follows. □

**Theorem 3.5** $B_a(2^k) = B(2^k) = k2^{k-1}$, *for* $k \geq 1$.

**Proof.** For $n = 2^k$, a minimum time broadcast scheme is also a minimum average time broadcast scheme since $2^j$ vertices must be informed at each time $j$ for $j = 1, 2, ..., k$. □

The known minimum broadcast graphs for $n$ which is 1 less than a power of 2 are also matbg's.

**Theorem 3.6** $B_a(2^{k+1} - 1) = \frac{1}{2}((2^{k+1} - 1)k + \lceil \frac{2^{k+1}-1}{k+2} \rceil)$, *for* $1 \leq k \leq 5$.

**Proof.** Theorem 2.2 gives the lower bound. The matching upper bound comes from the known minimum broadcast graphs on 3, 7, 15, 31, and 63 vertices. For 3, 7, and 15, minimum average time broadcast schemes can be easily constructed for the minimum broadcast graphs given in [4]. It is also easy to verify that the broadcast schemes given for the minimum broadcast graphs on 31 and 63 vertices (which are can be found in [2] and [8], respectively) are minimum average time schemes. □

In fact, minimum average time broadcast graphs can be found for all even $n$.

**Theorem 3.7** $B_a(n) = \lceil \frac{kn}{2} \rceil$, *for even* $n = 2^k + i$, *where* $k = \lfloor \log n \rfloor$ *and* $2 \leq i < 2^k$.

**Proof.** The lower bound comes from Theorem 2.1. $KG_n$, the modified Knödel graph on $n$ vertices, is a graph on $n$ vertices with $\lceil \frac{kn}{2} \rceil$ edges which allows minimum average time broadcast from any originator. The vertices of $KG_n$ are labeled $(x, j)$ where $x \in Z_{\frac{n}{2}}$ and $j \in Z_2$. In this paper, the values of $x$ will be those representatives of $Z_{\frac{n}{2}}$ between 0 and $\frac{n}{2} - 1$, the values of $j$ will be 0 and 1, and the operations are modulo $\frac{n}{2}$ or modulo 2. The edges of $KG_n$ are $[(x, 0), (x + 2^i, 1)]$ for all $x$ and all $i$ where $0 \leq i \leq k - 1$. The edges of the form $[(x, 0), (x + 2^i, 1)]$ compose a perfect matching and are called *edges of dimension* $i$. To broadcast in $KG_n$ from any vertex in minimum average time, each informed vertex can call it's dimension $j - 1$ neighbor at each time $1 \leq j \leq k$. This assures that $2^k$ vertices are informed at time $k$. Finally, at time $k + 1$, the remaining vertices can be called by their dimension 0 neighbors. (See [5] or [1] for additional details.) The result follows. □

Some additional values for $B_a(n)$ can be determined using different constructions for matbgs.

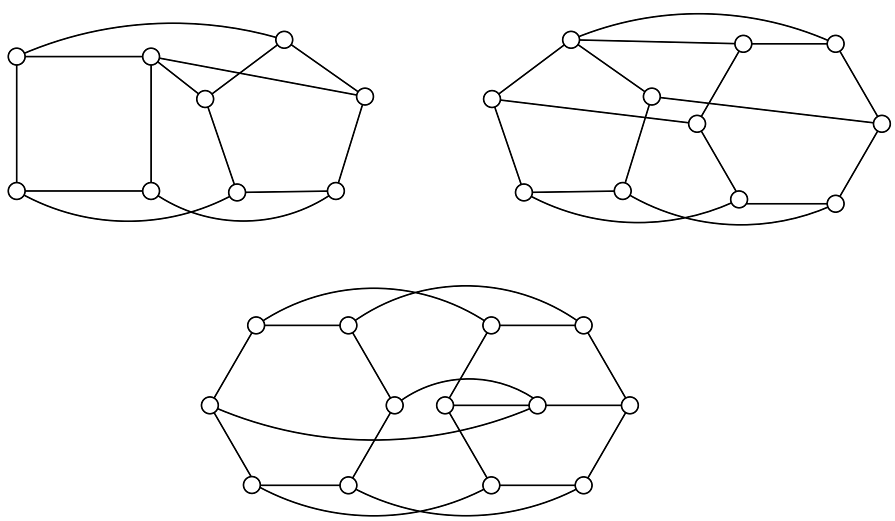

Fig. 1. Minimum Average Time Broadcast Graphs on 9, 11, and 13 Vertices.

**Theorem 3.8** *For* $9 \leq n \leq 13$, $B_a(n) = \lceil \frac{3n}{2} \rceil$.

**Proof.** Theorem 2.1 gives $B_a(n) \geq \lceil \frac{3n}{2} \rceil$. Minimum average time broadcast graphs for odd $n$, $9 \leq n \leq 13$, with $\lceil \frac{3n}{2} \rceil$ edges are shown in Figure 1.

For $9 \leq n \leq 12$, the graphs are constructed by connecting matbg's $A$ and $B$ on $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ vertices, respectively, with $\lceil \frac{n}{2} \rceil$ "cross" edges such that each vertex in $A$ is connected to a vertex in $B$ and vice versa. To achieve minimum average time broadcast in the resulting graph, at time 1 the originator sends the message to a cross edge neighbor. Each matbg of "half" size can then broadcast in minimum average time independently.

For $n = 13$, a slight modification is required to this method. For $n = 13$, matbg's on 6 and 7 vertices are joined by 6 "cross" edges as shown in Figure 1. If the originator is any of the 11 vertices on a cross edge, at time 1 the message is sent on a cross edge and then each smaller matbg can broadcast in minimum average time independently. The other two vertices (which are isomorphic) can broadcast in minimum average time using the scheme in Figure 2. In the figure, the two black vertices represent the originator and the vertex it calls at time 1. The other vertices are labeled with the time they receive the message. □

For some values of $n$, we can construct matbg's which are multiple fixed step

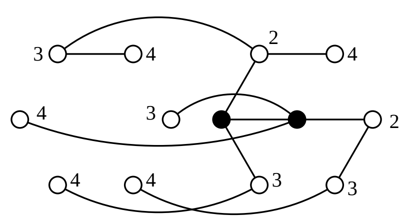


Fig. 2. Broadcast Scheme on 13 Vertex matbg.

graphs. The undirected multiple fixed step graph $G(n; s_1, s_2, ..., s_p)$ is a graph on $n$ vertices whose labels are integers modulo $n$. The edges of such a graph connect each vertex $i$ to $i \pm s_j \pmod n$ for each $1 \leq j \leq p$. When describing broadcast schemes for these graphs, we will refer to vertex $i + s_j \pmod n$ as vertex $i$'s $+s_j$ *neighbor* and to vertex $i - s_j \pmod n$ as vertex $i$'s $-s_j$ *neighbor*. The edges arising from a given $s_j$ are called the $\pm s_j$ *edges*. (See [3] for further information on these graphs.)

**Theorem 3.9** *For* $2^k + 1 \leq n \leq 2^k + 2^{k-1}$ *with even* $k \geq 4$, $B_a(n) = \lceil \frac{kn}{2} \rceil$.

**Proof.** The lower bound comes from Theorem 2.1. For each $n$ in this range, the undirected multiple fixed step graph $G(n; 1, 4, 16, ..., 2^{k-2})$ is an matbg. To broadcast in minimum average time from any vertex $i$, $i$ calls $i+1$ at time 1. At time 2, $i$ calls $i-1$ and $i+1$ calls $i+2$. At time 3, all 4 informed vertices call their $+4$ neighbors. Thus, the 8 consective vertices $i-1, i, i+1, ..., i+6$ are informed. At time 4, the 4 most clockwise informed vertices $(i+3, i+4, ..., i+6)$ each call their $+4$ neighbors and the other informed vertices call their $-4$ neighbors. This results in 16 consecutive informed vertices. By using the $\pm 16$ edges similarly, in two more time units we can obtain 64 consecutive informed vertices. (That is, all of the informed vertices call their $+16$ neighbors at time 5. At time 6, the newly informed vertices call their $+16$ neighbors while those informed before time 5 call their $-16$ neighbors.) Continue this process until we have $2^k$ consecutive informed vertices at time $k$. The remaining $n - 2^k$ uninformed vertices can be informed at time $k+1$ using the $\pm 2^{k-2}$ edges. An example is shown for $n = 21$ in Figure 3. In the figure, the two black vertices represent the originator and the vertex it calls at time 1. The other vertices are labeled with the time they receive the message. □

A slight modification of this technique allows us to obtain some additional values for $B_a(n)$.

**Theorem 3.10** *For* $2^k + 2^{k-1} + 1 \leq n \leq 2^k + 2^{k-1} + 3$ *with even* $k \geq 4$, $B_a(n) = \lceil \frac{kn}{2} \rceil$.

**Proof.** The lower bound comes from Theorem 2.1. For each $n$ in this range, the

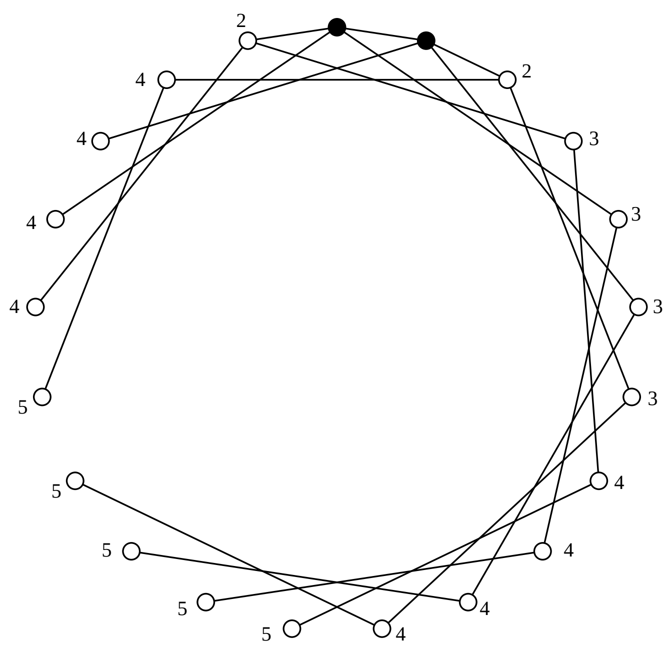


Fig. 3. Broadcasting Scheme for 21 Vertices.

undirected multiple fixed step graph $G(n; 1, 4, 16, ..., 2^{k-4}, 2^{k-2}+1)$ is an matbg. To broadcast in minimum average time from any vertex $i$, use the $\pm 1, \pm 4, \pm 16, ..., \pm 2^{k-4}$ edges to inform $2^{k-2}$ consecutive vertices by time $k-2$ as in the proof of Theorem 3.9. At time $k-1$, all informed vertices call their $+(2^{k-2}+1)$ neighbors. This leaves one isolated uninformed vertex between two groups of $2^{k-2}$ consecutive informed vertices. At time $k$, inform 2 more sets of $2^{k-2}$ consecutive vertices using the $\pm(2^{k-2}+1)$ edges. At this point, there are 3 isolated uninformed vertices dividing the four groups of $2^{k-2}$ consecutive informed vertices. There are $n - 2^k - 3$ additional consecutive uninformed vertices between the two groups of $2^{k-2}$ that were informed at time $k$. Without loss of generality, let these vertices be labeled clockwise by $0, 1, 2, ..., n - 2^k - 4$. At time $k+1$, the vertices $1, 2, ..., 2^k - 5$ can be informed using the $\pm(2^{k-2}+1)$ edges. The remaining vertices $(0, n - 2^k - 4$, and the three isolated uninformed vertices) can be informed using the $\pm 1$ edges. An example is shown for $n = 25$ in Figure 4. In the figure, the two black vertices represent the originator and the vertex it calls at time 1. The other vertices are labeled with the time they receive the message. □

We can use the technique used in Theorem 3.8 to construct matbgs for $9 \leq n \leq$

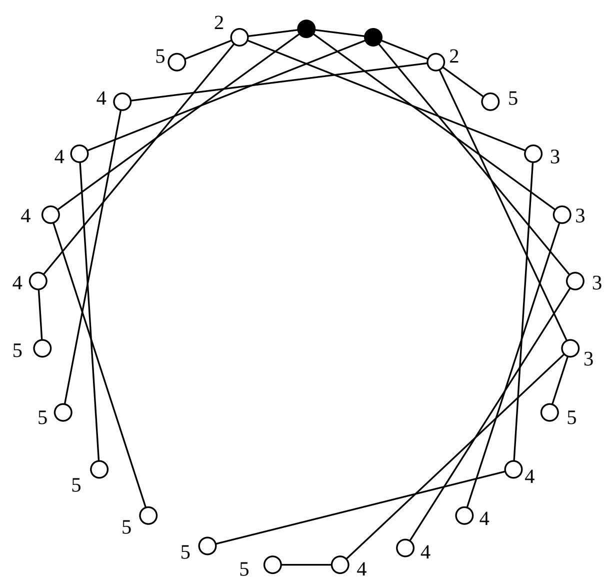


Fig. 4. Broadcasting Scheme for 25 Vertices.

12 combined with the results from Theorems 3.5, 3.9, and 3.10 to determine $B_a(n)$ for some additional values of $n$.

**Theorem 3.11** *For* $2^k + 1 \leq n \leq 2^k + 2^{k-1} + 6$ *with odd* $k \geq 5$, $B_a(n) = \lceil \frac{kn}{2} \rceil$.

**Proof.** The lower bound comes from Theorem 2.1. To show that $B_a(n) \leq \lceil \frac{kn}{2} \rceil$, we construct an matbg $G$ with $n$ vertices and $\lceil \frac{kn}{2} \rceil$ edges by connecting an matbg $A$ on $\lceil \frac{n}{2} \rceil$ vertices and an matbg $B$ on $\lfloor \frac{n}{2} \rfloor$ vertices with $\lceil \frac{n}{2} \rceil$ "cross" edges so that each vertex of $A$ is connected to a vertex of $B$ and vice versa. From Theorems 3.5, 3.9, and 3.10, we know that $B_a(\lceil \frac{n}{2} \rceil) = \lceil \frac{(k-1)\lceil n/2 \rceil}{2} \rceil$ and that $B_a(\lfloor \frac{n}{2} \rfloor) = \lceil \frac{(k-1)\lfloor n/2 \rfloor}{2} \rceil$. The resulting graph $G$ has $\lceil \frac{(k-1)\lceil n/2 \rceil}{2} \rceil + \lceil \frac{(k-1)\lfloor n/2 \rfloor}{2} \rceil + \lceil \frac{n}{2} \rceil = \lceil \frac{kn}{2} \rceil$ edges. To broadcast in minimum average time in $G$, at time 1 the originator sends the message to its neighbor in the other matbg of "half" size. Each matbg of half size can then broadcast in minimum average time independently. □

## 4. Summary

We have given some general bounds on $B_a(n)$ and have determined the exact values of $B_a(n)$ for all even $n$. We have determined the exact values of $B_a(n)$ for odd $n$ in the range $2^k \leq n \leq 2^k + 2^{k-1} + 3$ when $k \geq 4$ is even and in the range $2^k \leq n \leq 2^k + 2^{k-1} + 6$ when $k \geq 5$ is odd, and for some additional small values of

$n$. It remains an open question to determine the value of $B_a(n)$ for the remaining odd $n$.

Table 1 shows the known values of $B_a(n)$ and $B(n)$ for $1 \leq n \leq 64$.

Table 1. Known values of $B_a(n)$ and $B(n)$ for $1 \leq n \leq 64$

| $n$ | $B_a(n)$ | $B(n)$ | $n$ | $B_a(n)$ | $B(n)$ | $n$ | $B_a(n)$ | $B(n)$ | $n$ | $B_a(n)$ | $B(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 17 | 34 | 22 | 33 | 83 | ? | 49 | 123 | ? |
| 2 | 1 | 1 | 18 | 36 | 23 | 34 | 85 | ? | 50 | 125 | ? |
| 3 | 2 | 2 | 19 | 38 | 25 | 35 | 88 | ? | 51 | 128 | ? |
| 4 | 4 | 4 | 20 | 40 | 26 | 36 | 90 | ? | 52 | 130 | ? |
| 5 | 5 | 5 | 21 | 42 | 28 | 37 | 93 | ? | 53 | 133 | ? |
| 6 | 6 | 6 | 22 | 44 | 31 | 38 | 95 | ? | 54 | 135 | ? |
| 7 | 8 | 8 | 23 | 46 | ? | 39 | 98 | ? | 55 | ? | ? |
| 8 | 12 | 12 | 24 | 48 | ? | 40 | 100 | ? | 56 | 140 | ? |
| 9 | 14 | 10 | 25 | 50 | ? | 41 | 103 | ? | 57 | ? | ? |
| 10 | 15 | 12 | 26 | 52 | 42 | 42 | 105 | ? | 58 | 145 | 121 |
| 11 | 17 | 13 | 27 | 54 | 44 | 43 | 108 | ? | 59 | ? | 124 |
| 12 | 18 | 15 | 28 | 56 | 48 | 44 | 110 | ? | 60 | 150 | 130 |
| 13 | 20 | 18 | 29 | ? | 52 | 45 | 113 | ? | 61 | ? | 136 |
| 14 | 21 | 21 | 30 | 60 | 60 | 46 | 115 | ? | 62 | 155 | 155 |
| 15 | 24 | 24 | 31 | 65 | 65 | 47 | 118 | ? | 63 | 162 | 162 |
| 16 | 32 | 32 | 32 | 80 | 80 | 48 | 120 | ? | 64 | 192 | 192 |

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