The Complexity and Proof Complexity of the Comparator Circuit Value Problem

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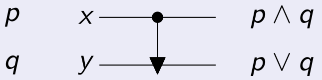
Outline of the talk

1. Define comparator circuits
2. Define CC as the class of problems reducible to $\textsc{Ccv}$ (the comparator circuit value problem)
3. Give interesting complete problems for CC
4. Introduce universal comparator circuits, with resulting robustness properties of CC.
5. Introduce a theory VCC and a propositional proof system CCFrege for CC.
6. Support the conjecture that CC and NC are incomparable using oracle separations.

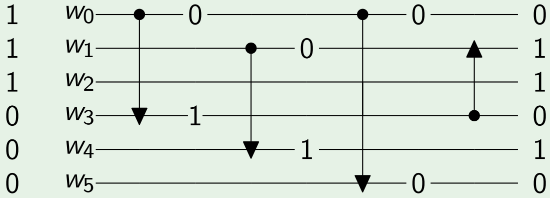
Comparator Circuits

- Originally invented for sorting, e.g.,
  - Batcher's $\mathcal{O}(\log^2 n)$-depth sorting networks ('68)
  - Ajtai-Komlós-Szemerédi (AKS) $\mathcal{O}(\log n)$-depth sorting networks ('83)
- Can also be considered as Boolean circuits.

Comparator gate

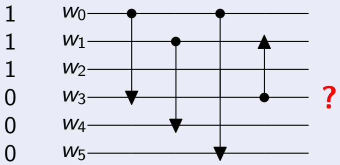


Example



Comparator Circuit Value ($\mathrm{C{\scriptsize CV}}$) Problem (decision)

Given a comparator circuit with specified Boolean inputs, determine the output value of a designated wire.



Comparator Circuit complexity class

1. CC $=$ {decision problems AC^0 many-one-reducible to $\mathrm{C{\scriptsize CV}}$}
2. Subramanian ['90] Defined CC using log space many-one reducibility
3. We introduce universal comparator circuits and use them to show that the two definitions coincide.
4. Subramanian showed

$$\boxed{\mathsf{NL} \subseteq \mathsf{CC} \subseteq \mathsf{P}}$$

NL is nondeterministic log space

- Recall NL $\subseteq$ CC $\subseteq$ P
- But also NL $\subseteq$ NC $\subseteq$ P
  where NC (the parallel class) contains the problems solvable by uniform polysize polylog depth Boolean circuit families.
- NC contains all context-free languages, and matrix powering and determinants over $\mathbb{Z}, \mathbb{Q}$ etc.

Conjecture

NC and CC are incomparable. (So in particular CC $\subsetneq$ P.)

Intuitively, we think CC $\subsetneq$ P because each of the two comparator gate outputs in a comparator circuit is limited to fan-out one. (More later. . .)

Example Complete Problems for CC

- CCV
- Stable Marriage Problem
- Lexicographical first maximal matching
- Telephone connection problem
- Others . . .

Stable Marriage Problem (search version) (Gale-Shapley '62)

- Given n men and n women together with their preference lists
- Find a stable marriage between men and women, i.e.,
  1. a perfect matching
  2. satisfies the stability condition: no two people of the opposite sex like each other more than their current partners
  3. A stable marriage always exists, but may not be unique.

Stable Marriage Problem (decision version)

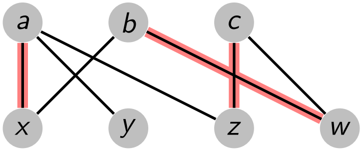
Is a given pair of (m, w) in the man-optimal (woman-optimal) stable marriage?

The Stable Marriage problem has been used to pair medical interns with hospital residencies in the USA.

Lex-first maximal matching problem (CC-Complete)

Lex-first maximal matching

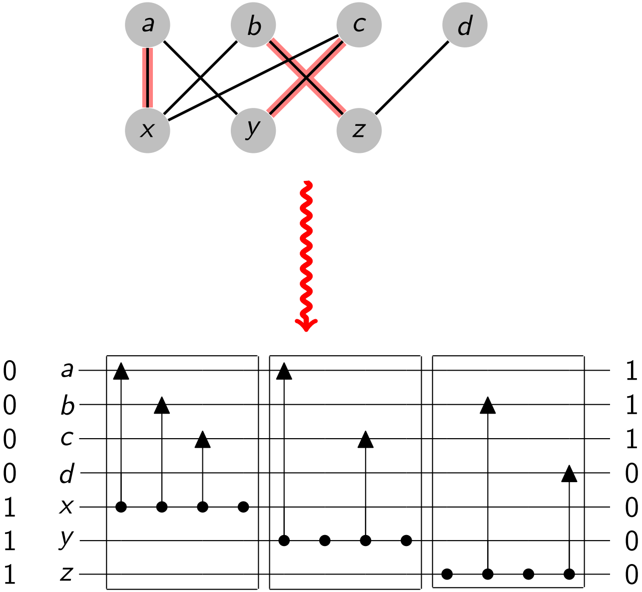
- Let G be a bipartite graph.
- Successively match the bottom nodes $x, y, z, \ldots$ to the least available top node



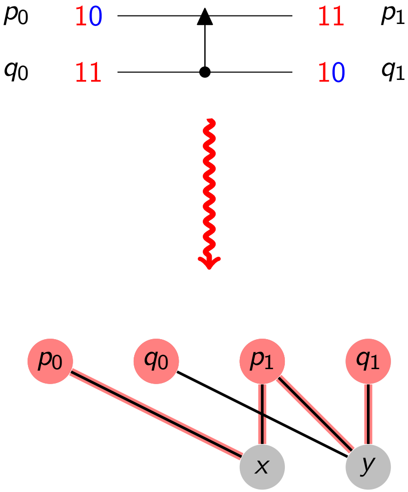
Lex-first maximal matching decision problems

- Edge Is a given edge $\{u, v\}$ in the lex-first maximal matching of G?
- Vertex Is a given (top) vertex v in the lex-first maximal matching of G?
- The problems are equivalent.

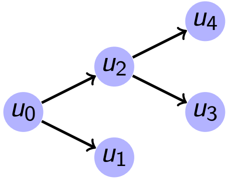
Reducing vertex lex-first maximal matching to $\textsc{Ccv}$



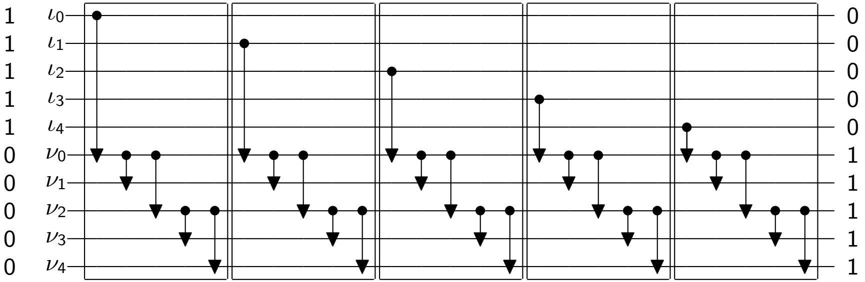
Reducing CCV to lex-first maximal matching



NL $\subseteq$ CC



- This result is due to Feder [1992].
- Dai Lê has a neat proof (See the appendix to our recent arXiv paper.)
- Show $stCONN \leq_m^{AC^0}$ CCV.
- May assume that the given directed graph $G = (V, E)$ has edges of the form (u_i, u_j), where $i < j$.



Two-Sorted Notation

- $x, y, z, \ldots$ denote elements of $\mathbb{N}$ (presented in unary)
- $X, Y, Z, \ldots$ denote binary strings
- $|X|$ denotes the length of X.
- A complexity class is a set of relations of the form $R(\vec{x}, \vec{X})$
- AC^0 many-one reducibility
  $R_1(X) \leq_m^{AC^0} R_2(X)$ iff there exists an AC^0 function $F(X)$ such that

$$R_1(X) \leftrightarrow R_2(F(X))$$

- Thus CC is the class of relations $R(\vec{x}, \vec{X})$ that are AC^0 many-one reducible to $\textsc{Ccv}$.

Function Classes

- Given a class C of relations, we associate a class FC of functions as follows.
- A function F taking strings to strings is in FC iff
  1. $|F(X)| = |X|^{O(1)}$ (p-bounded)
  2. The bit graph $B_F(i, X)$ is in C
- Here $B_F(i, X)$ holds iff the ith bit of $F(X)$ is 1.

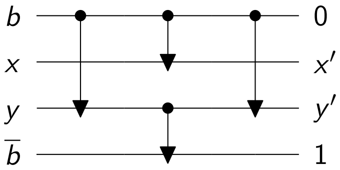
Is FCC closed under composition?

- This question was left open in our earlier paper in CSL 2011 paper (before Yuval Filmus joined our project)
- Suppose $F(X) = G(H(X))$. Let $Y = H(X)$.
- The bit graph of $G(Y)$ is AC^0-reducible to Ccv.
- Thus the circuit computing $G(Y)$ is described by $Y' = \mathsf{AC}^0(Y)$.
- But $Y = H(X)$ is the output of another comparator circuit.

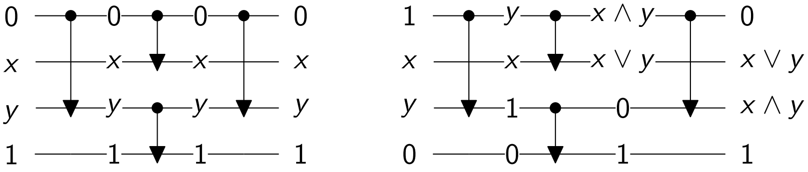
So we need a universal comparator circuit, taking Y' as input, to compute $G(Y)$.

Universal comparator circuits [Filmus]

Here is a gadget which allows a conditional application of a comparator to two of its inputs x, y, depending on whether b is 0 or 1.



Operation of the gadget:



Universal comparator circuits

- In order to simulate a single arbitrary comparator in a circuit with m wires we put in $m(m-1)$ gadgets in a row, for the $m(m-1)$ possible comparators.
- Simulating n comparators requires $m(m-1)n$ gadgets.
- Thus there is an AC^0 function UNIV such that if m, n are arbitrary parameters, then

$$U = \mathrm{UNIV}(m, n) = \langle m', n', U' \rangle$$

is a universal circuit with m' wires and n' gates which simulates all comparator networks with at most m wires and at most n comparators.

$$\begin{aligned} m' &= 2m(m-1)n + m \\ n' &= 4m(m-1)n \end{aligned}$$

Applications of universal comparator circuits

- FCC is closed under composition.
- CC is closed under (many-one) log-space reducibility.
- This is becasue NL $\subseteq$ CC, so FCC includes all log space functions. And FCC is closed under composition.
- If $R(X) \leftrightarrow \mathrm{C{\scriptsize CV}}(F(X))$, where F is log-space computable, then

$$\chi_R(X) = \chi_{\mathrm{C{\scriptsize CV}}}(F(X))$$

where χ_R is the characteristic function of R.

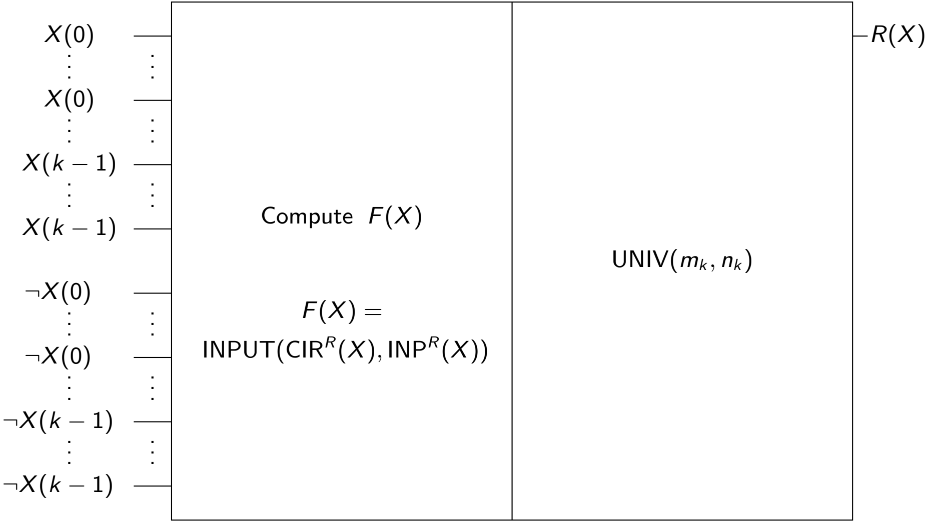
Applications of universal comparator circuits Cont'd

- $R(X)$ is in CC iff there is an AC^0-uniform family $\{C_k^R\}_{k\in\mathbb{N}}$ of comparator circuits, where C_k computes $R(X)$ for $|X| = k$.
- The direction $\Leftarrow$ is immediate.
- Proof of direction $\Rightarrow$: This is clear if $R(X)$ is in AC^0. (An AC^0 circuit converts into a polysize tree circuit, which converts to a comparator circuit.)
- If $R(X) \in \mathrm{CC}$, then

$$R(X) \leftrightarrow \textsc{Ccv}(F(X))$$

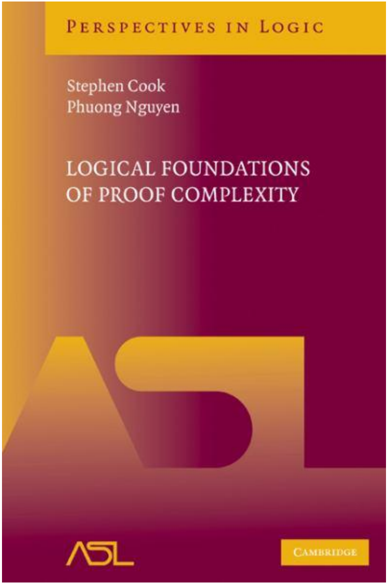
for some AC^0 function $F(X)$. Apply a universal circuit to the output of $F(X)$.

The circuit C_k computing $R(X)$ for $|X| = k$



Theory VCC **for the class** CC

- Reference:



Two-sorted language $\mathcal{L}_A^2$ (Zambella '96)

Vocabulary $\mathcal{L}_A^2 = [0, 1, +, \cdot, |\ |;\ \in, \leq, =_1, =_2]$

- Standard model $\mathbb{N}_2 = \langle \mathbb{N}, \text{finite subsets of } \mathbb{N} \rangle$
- $0, 1, +, \cdot, \leq, =$ have usual meaning over $\mathbb{N}$
- $|X| =$ length of X
- Set membership $y \in X$

Note

The natural inputs for Turing machines and circuits are finite strings.

- "number" variables $x, y, z, \ldots$ (range over $\mathbb{N}$)
- "string" variables $X, Y, Z, \ldots$ (range over finite subsets of $\mathbb{N}$)
- Number terms are built from $x, y, z, \ldots, 0, 1, +, \cdot$ and $|X|, |Y|, |Z|, \ldots$
- The only string terms are variable $X, Y, Z, \ldots$

Definition (Σ_0^B formula)

1. All the number quantifiers are bounded.
2. No string quantifiers (free string variables are allowed)

Two-sorted complexity classes

A two-sorted complexity class consists of relations $R(\vec{x}, \vec{X})$, where

- $\vec{x}$ are number arguments (in unary) and $\vec{X}$ are string arguments

Definition (Two-sorted AC^0)

A relation $R(\vec{x}, \vec{X})$ is in AC^0 iff some alternating Turing machine accepts R in time $\mathcal{O}(\log n)$ with a constant number of alternations.

Σ_0^B-Representation Theorem [from Immerman FO]

$R(\vec{x}, \vec{X})$ is in AC^0 iff it is represented by a Σ_0^B-formula $\varphi(\vec{x}, \vec{X})$.

Useful consequences

1. Don't need to work with uniform circuit families or alternating Turing machines when defining AC^0 functions or relations.
2. Useful when working with AC^0-reductions

The theory V^0 for AC^0 reasoning

Theories developed using Cook-Nguyen method extend V^0.

The axioms of V^0

1. 2-**BASIC axioms**: essentially the axioms of Robinson arithmetic plus
   - the defining axioms for $\leq$ and the string length function $|\ |$
   - the axiom of extensionality for finite sets (bit strings).
2. Σ_0^B-**COMP** (Comprehension): for every Σ_0^B-formula $\varphi(z)$ without X,

$$\exists X \leq y \, \forall z < y \big(X(z) \leftrightarrow \varphi(z) \big)$$

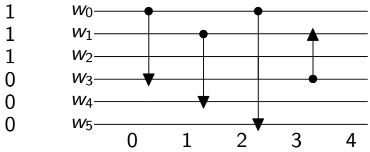
The Σ_0^B-IND scheme is provable in V^0

1. $\big[\varphi(0) \wedge \forall x \big(\varphi(x) \rightarrow \varphi(x+1)\big)\big] \rightarrow \forall x \varphi(x)$, where $\varphi \in \Sigma_0^B$.
2. The provably total functions in V^0 are precisely FAC^0.

The two-sorted theory VCC [using the Cook-Nguyen method]

- VCC has vocabulary $\mathcal{L}_A^2$
- Axiom of VCC $=$ Axiom of V^0 $+$ one additional axiom asserting the existence of a solution to the $\textsc{Ccv}$ problem.

Asserting the existence of a solution to Ccv



- X encodes a comparator circuit with m wires and n gates
- Y encodes the input sequence
- Z is an $(n+1) \times m$ matrix, where column i of Z encodes values layer i

The following Σ_0^B formula $\delta_{\mathsf{CCV}}(m, n, X, Y, Z)$ states that Z encodes the correct values of all the layers of the Ccv instance encoded in X and Y:

$$\forall k < m\big(Y(k) \leftrightarrow Z(0,k)\big) \wedge \forall i < n \forall x < m \forall y < m,$$

$$(X)^i = \langle x, y\rangle \rightarrow \left[\begin{array}{ll} & Z(i+1,x) \leftrightarrow \big(Z(i,x) \wedge Z(i,y)\big) \\ \wedge & Z(i+1,y) \leftrightarrow \big(Z(i,x) \vee Z(i,y)\big) \\ \wedge & \forall j < m\Big[(j \neq x \wedge j \neq y) \rightarrow \big(Z(i+1,j) \leftrightarrow Z(i,j)\big)\Big] \end{array}\right]$$

$$\mathsf{VCC} = \mathsf{V}^0 + \exists Z \leq \langle m, n+1\rangle + 1,\ \delta_{\mathsf{CCV}}(m, n, X, Y, Z)$$

Properties of VCC

From long version of our CSL 2011 paper

- The provably total functions of VCC comprise FCC.
- VCC admits induction on CC concepts.
- VCC extends VNC^1. (Recall $NC^1 \subseteq CC$.)
- VCC proves that Lex-first Max Matching, and Stable Marriage, are complete for CC.

Associate proof system CFrege with VC [Ch. 10, CN 2010]

- Each Σ_0^B formula $\varphi(X)$ translates into a polysize family $\{\varphi(X)[n]\}_{n\in\mathbb{N}}$ of bounded depth propositional formulas.
  - Here $\varphi(X)[n]$ expresses $\varphi(X)$ for $|X| = n$, using atoms p_i^X for the bits of X. (This method due to [Paris/Wilkie]).
- If $\varphi(X)$ is true, then each translated formula $\varphi(X)[n]$ is a tautology.
- If C is a circuit class such as $\mathrm{AC}^0, \mathrm{NC}^1, \mathrm{P}$ then CFrege is AC^0-Frege, Frege, EFrege, respectively.
- The lines in the CFrege-proof represent Boolean circuits of the appropriate kind (bounded-depth, formulas, circuits) respectively.
- A proof of $\varphi(X)$ in the theory VC translates into a polysize family of CFrege proofs of the tautologies $\{\varphi(X)[n]\}_{n\in\mathbb{N}}$
- The theory VC proves the soundness of CFrege.
- CFrege is the strongest proof system whose soundness is provable in VC.

Suggestion for proof system CCFrege

- CCFrege is EFrege with restrictions on introduction of extension variables.
- Each extension variable is the value of some wire segment in a comparator circuit whose inputs do not involve extension variables.
- The extension variables are $w_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, where the comparator circuit has m wires and n gates.
- w_{ij} is the value of the jth segment of wire i, where each wire gets a new segment after *every* gate.
- Let a_j be the wire number corresponding to the AND of gate j, and let o_j be the wire number corresponding to the OR of gate j.
  Thus $a_j \neq o_j$, and $0 \leq a_j, o_j \leq m$

Defining formulas for the extension variables w_{ij}

- $w_{i0} \leftrightarrow A_i$, $1 \leq i \leq m$, where A_i has no extension variables.
- $w_{i,j+1} \leftrightarrow w_{ij}$ if $i \neq a_j, i \neq o_j$
- $w_{i,j+1} \leftrightarrow (w_{ij} \wedge w_{o_j,j})$ if $i = a_j$
- $w_{i,j+1} \leftrightarrow (w_{ij} \vee w_{a_j,j})$ if $i = o_j$

Properties of CCFrege

Claim:

CCFrege corresponds to VCC:

1. A proof of a Σ_0^B-formula $\varphi(X)$ in the theory VCC translates into a polysize family of CCFrege proofs of the tautologies $\{\varphi(X)[n]\}_{n\in\mathbb{N}}$
2. The theory VCC proves the soundness of CCFrege.
3. CCFrege is the strongest proof system whose soundness is provable in VCC.

Proof of (2)

- Given a CCFrege proof, VCC can evaluate the extension variables in terms of the values for the input variables, using its axiom asserting the existence of values for the wires of a comparator circuit.
- VCC proves by induction that all formulas in the proof are true.

Conjecture: NC and CC are incomparable

- Lex-First Max Matching (LFMM) is in CC.

Conjecture

LFMM is not in NC.
(The obvious algorithm for LFMM is sequential.)

- The function $A \rightsquigarrow A^n$ (where A is an $n \times n$ integer matrix) is in NC^2 , but we do not know how to put it in CC.

Why do we think $\mathsf{NC}^2 \subsetneq \mathsf{CC}$?

- NC^2-gates have multiple fan-out, but each end of a comparator gate has fan-out one.
- If either input of a comparator gate is 'flipped', then exactly one output is flipped.
  Thus comparator gates are 1-Lipschitz.
- Flipping an input to a gate generates a unique flip-path in the circuit from that gate to some output of the circuit.
- But flipping an input to an NC^2-gate can generate many parallel flip-paths.

Relativized CC and NC are incomparable

Oracle gates for comparator circuits

- The oracle $\alpha : \{0,1\}^* \to \{0,1\}^*$ is length preserving.
- $\alpha_n : \{0,1\}^n \to \{0,1\}^n$ is the restriction of α to n.
- An oracle gate α_n can be inserted anywhere in a relativized comparator circuit: select any n wires as inputs to the gate and any n wires as outputs.
- To make α_n gates look more like comparator gates, we require that α_n have the 1-Lipschitz property.
- We allow $\neg$ gates in relativized CC(α) circuits.
(We can allow them in comparator circuits without changing CC.)
- Changing one input to one α_n gate produces a unique flip path in the circuit from that gate to the outputs of the circuit.

Theorem

There is a relation $R_1(\alpha)$ computable by a polysize family of comparator oracle circuits by which cannot be computed by any $\mathsf{NC}(\alpha)$ *circuit family (even when α is restricted to be 1-Lipschitz).*

Proof Idea.

- $\alpha_n^k(\vec{0})$ is easily computed by relativized comparator circuits, but requires depth k circuits [ACN 07].
- The hard part is proving the depth lower bound when α is 1-Lipschitz.

$\square$

Theorem

There is a relation $R_2(\alpha)$ computable by an $\mathsf{NC}^2(\alpha)$ circuit family but not computable by any polysize family of comparator oracle circuits (even when α is restricted to be 1-Lipschitz).

Proof Idea.

- Let $\alpha_i^k : \{0,1\}^{dn} \to \{0,1\}$ be a Boolean oracle.
- Define a function $y = f[(\alpha_1^1, \ldots, \alpha_n^1), \ldots, (\alpha_1^m, \ldots, \alpha_n^m)]$ as follows:

$$\begin{aligned} x_i^k &= \alpha_i^k(\overbrace{x_1^{k+1}, \ldots, x_1^{k+1}}^{d \text{ times}}, \ldots, \overbrace{x_n^{k+1}, \ldots, x_n^{k+1}}^{d \text{ times}}), && k \in [m], i \in [n], \\ x_i^{m+1} &= 0, && i \in [n], \\ y &= x_1^1 \oplus \cdots \oplus x_n^1. \end{aligned}$$

$\square$

Conclusion

The complexity class CC is interesting because

- It is robust (closed under a variety of reductions).
- It has interesting complete problems.
- It appears to be a proper subset of P and incomparable with NC (and SC).
- It has a theory VCC which captures reasoning in CC and proves basic properties of CC.
- It has an associated propositional proof system CCFrege.