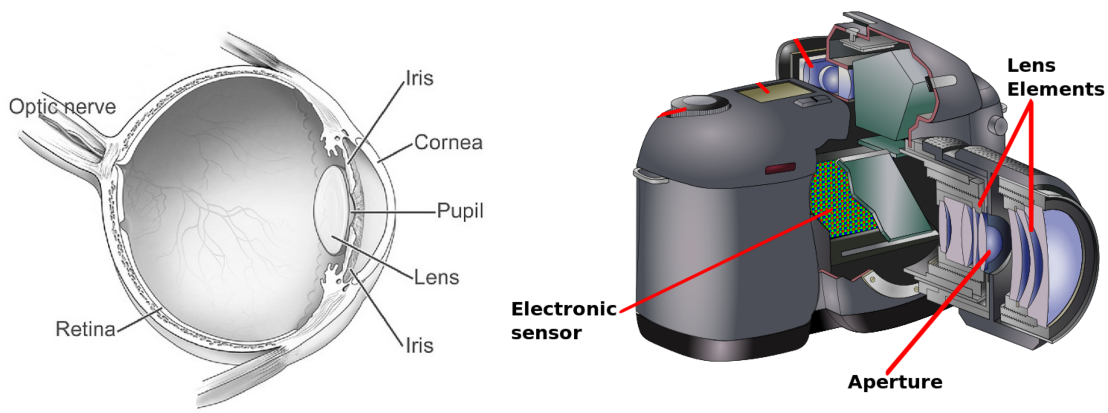
5 Camera Models

Having studied how to model, represent, and manipulate 3D objects so that we can build a scene, we will not take a look at how cameras work. The image below shows the fundamental components of a camera - note that we do not make a distinction between biological cameras and human-made devices. From the point of view of image formation, the same principles apply to both.



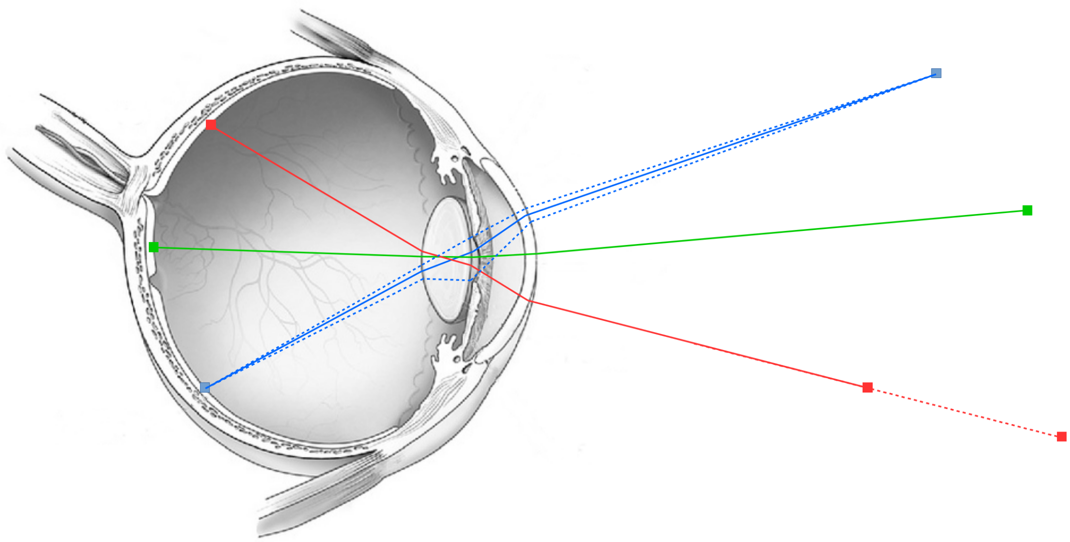
A human eye [Source: Public domain] and a digital camera [Source: Wikipedia, author: Astrocog]

The essential components of a camera include:

- A lens whose function is to bend light rays so as to ensure objects of interest are in focus in the resulting image
- An aperture that controls the amount of light entering the camera
- An imaging surface that senses light and converts it into a signal

These components are clearly labeled in the digital camera diagram. In the case of the human eye the retina is the imaging surface, the pupil is the aperture (its size is controlled by the iris), and the lens, cornea, and humours filling the cavities all act as light bending elements - they form the actual lens for the eye. Because for most artificial cameras the image surface is flat we often refer to it as the *image plane*.

An image is the result of light rays that make their way to the imaging surface at the back of the camera. These light rays have been bounced off, reflected, and/or refracted by objects in the scene, their colour changed in the process due to the properties of the material each object is made of. The key to understanding image formation is light from objects located at different spatial locations with regard to the camera will hit the imaging surface at specific locations determined by the configuration of the lens, aperture, and shape of the imaging surface itself.

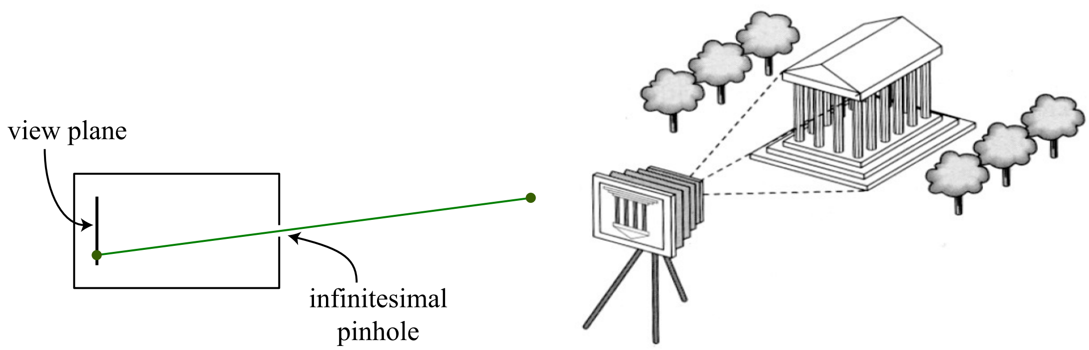


Light from objects at different locations hits the imaging surface at specific points. The resulting pattern of light forms the image recorded by the eye.

The diagram above illustrates different paths light can take from objects (in this case simple colour dots) in the scene to the imaging plane. Light from each object follows a specific path determined by their location and the configuration of the camera's components. For every point in the image, there are multiple paths light can take to reach the imaging plane as determined by the size of the aperture (this is shown for the blue dot using dashed lines). The purpose of the lens system is to ensure all of the light paths corresponding to an object that is in focus will terminate at the same location in the imaging surface. We will see that this is not the case for rays coming from objects not in focus. Note that light rays from different objects that are along the same line of sight from the camera will end up at the same image location.
In what follows, we will build a model of a camera that will allow us to determine mathematically the correspondence between points in a scene (corresponding to object surface locations) and their *projection* onto the image plane. We will begin with the simplest camera model, the pinhole camera.

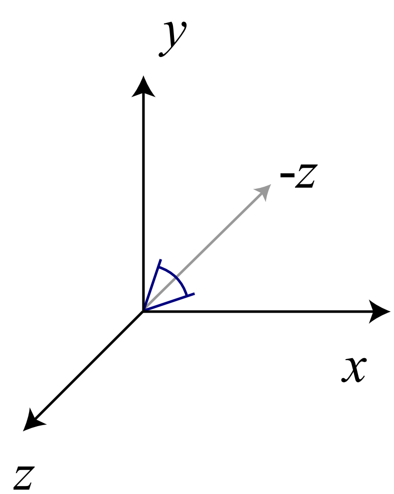
5.1 Pinhole Camera Model

A **pinhole** camera is an idealized camera model where the aperture size approaches zero (it is an infinitesimally small pinhole) and there is no lens.

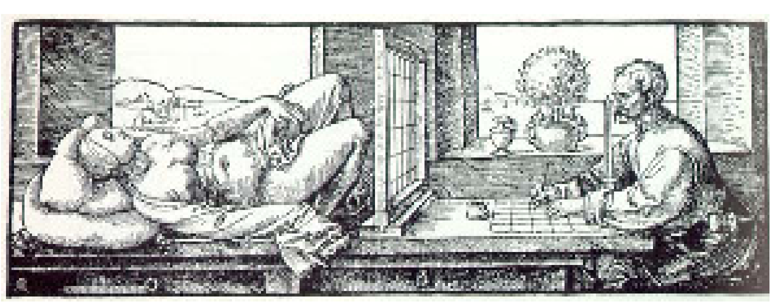


Light from a point travels along a unique straight path through the pinhole onto the view plane, therefore no lens is needed. The object is imaged upside-down on the image plane.

Note:
We use a right-handed coordinate system for the camera, with the x-axis as the horizontal direction and the y-axis as the vertical direction. This means that the optical axis (gaze direction) is the negative z-axis.

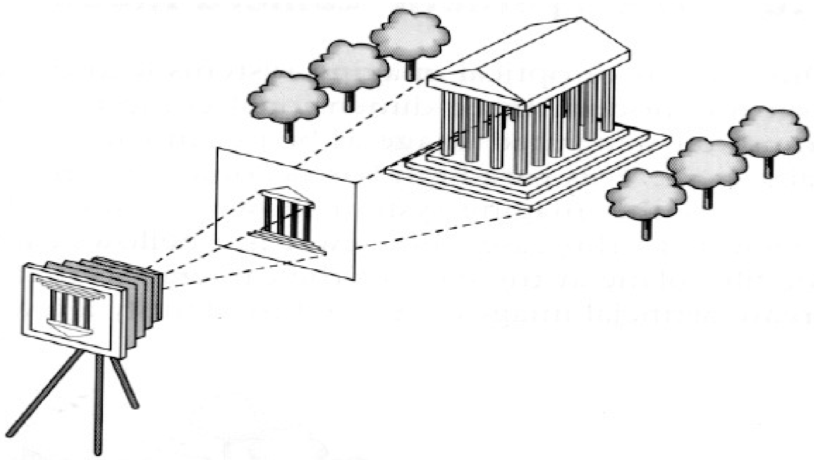


Here is another way of thinking about the pinhole model. Suppose you view a scene with one eye looking through a square window, and draw a picture of what you see through the window:



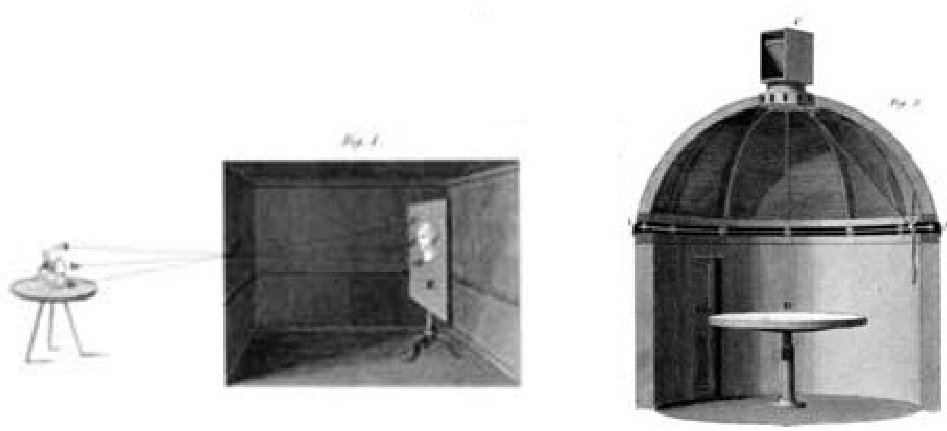
(Engraving by Albrecht Dürer, 1525).

The image you'd get corresponds to drawing a ray from the eye position and intersecting it with the window. This is equivalent to the pinhole camera model, except that the view plane is in front of the eye instead of behind it, and the image appears right-side-up, rather than upside down. (The eye point here replaces the pinhole). To see this, consider tracing rays from scene points through a view plane behind the eye point and one in front of it:



For the remainder of these notes, we will consider this camera model, as it is somewhat easier to think about.

Aside:
The earliest cameras were room-sized pinhole cameras, called *camera obscura*s. You would walk in the room and see an upside-down projection of the outside world on the far wall. The word *camera* is Latin for "room;" *camera obscura* means "dark room."

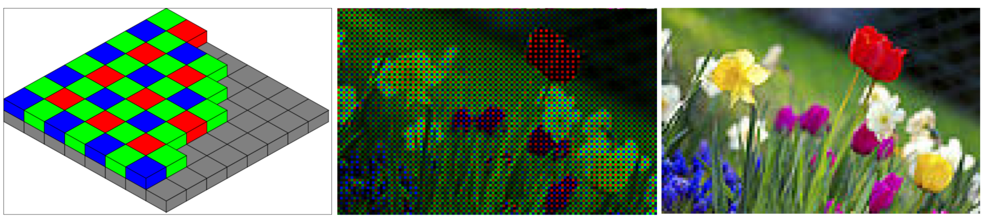


18th-century camera obscuras. The camera on the right uses a mirror in the roof to project images of the world onto the table, and viewers may rotate the mirror.

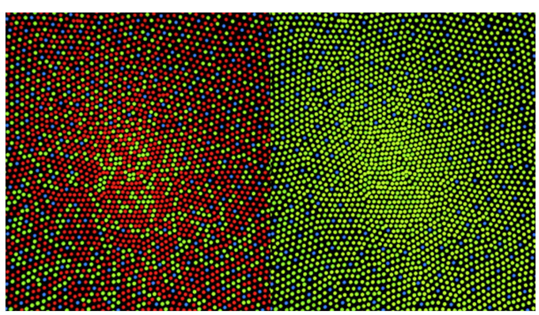
Aside:

In a conventional camera, the view plane contains a strip of film coated with photo-sensitive chemicals. In a digital camera, the view plane contains an array of semiconductor-based photodetectors. In the human eye, the imaging surface is the retina, it is curved, and and contains a dense array of cells with photoreactive molecules.

An interesting observation is that for both digital camera sensors and the human retina, each photodetector will respond to only one of red, blue, or green coloured light. The image is really a mosaic of three individual colour responses, which then gets interpolated (by the camera software, or the human brain) to fill-in colour information all across the scene.



A digital camera's sensor (gray) is attached to a colour filter array. Each photoreceptor will therefore respond to light of the corresponding colour only. The actual image resulting from this is a mosaic (center) of red, green, and blue dots. Camera software interpolates the individual colour measurements to determine the full colour of each pixel (left). [Source: Wikipedia, authors: C. Burnett, Anita Martinz]

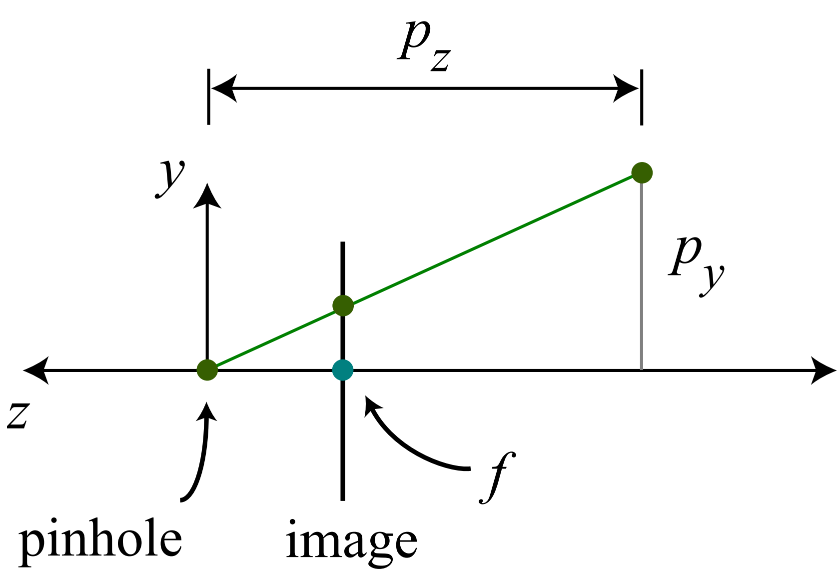


The human eye also contains a mosaic of phtoreceptors sensitive to either red, green, or blue. The brain interpolates colour from this mosaic of responses. The distribution of photoreceptors varies between people and also between each eye of a single person. Therefore variations in colour perception are common. In some cases, the variations are large enough that they give rise to conditions such as colour blindness (shown on the left mosaic, note the absence of red photoreceptors). [Source: Wikipedia, author: Mark Fairchild]

5.2 Camera Projections

A camera projection describes the mapping from points in space to coordinates on the image plane. While many models of camera projection with specific applications have been developed, there are two that are extensively used in computer graphics: *perspective projection* and *orthographic projection*.

5.3 Perspective Projection

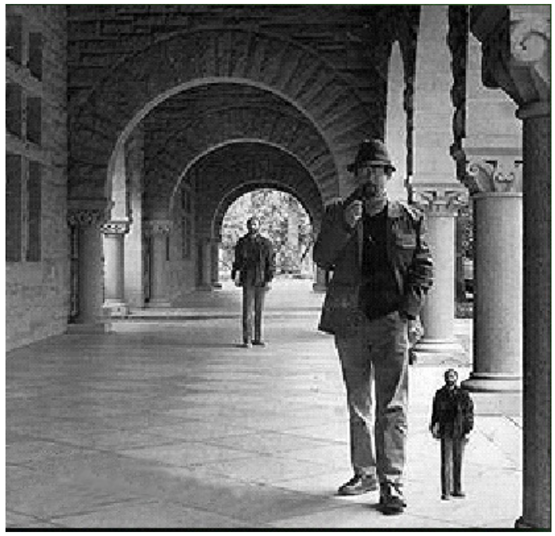


Perspective projection

Consider a point $\bar{p}$ in 3D space oriented with the camera at the origin, which we want to project onto the view plane. To project p_y to y, we can use similar triangles to get $y = \frac{f}{p_z}p_y$. This is **perspective projection**.

Note that $f < 0$, and the focal length is $|f|$.

In perspective projection, distant objects appear smaller than near objects:



The man without the hat appears to be two different sizes, even though the two images of him have identical sizes when measured in pixels. In 3D, the man without the hat on the left is about 18 feet behind the man with the hat. This shows how much you might expect size to change due to perspective projection.

Above we found the form of the perspective projection using the idea of similar triangles. Here we consider a complementary algebraic formulation. To begin, we are given

- a point $\bar{p}^c$ in camera coordinates (uvw space),
- center of projection (eye or pinhole) at the origin in camera coordinates,
- image plane perpendicular to the z-axis, through the point $(0, 0, f)$, with $f < 0$, and
- line of sight is in the direction of the negative z-axis (in camera coordinates),

we can find the intersection of the ray from the pinhole to $\bar{p}^c$ with the view plane.

The ray from the pinhole to $\bar{p}^c$ is $\bar{r}(\lambda) = \lambda(\bar{p}^c - \bar{0})$.

The image plane has normal $(0, 0, 1) = \vec{n}$ and contains the point $(0, 0, f) = \bar{f}$. So a point $\bar{x}^c$ is on the plane when $(\bar{x}^c - \bar{f}) \cdot \vec{n} = 0$. If $\bar{x}^c = (x^c, y^c, z^c)$, then the plane satisfies $z^c - f = 0$.

To find the intersection of the plane $z^c = f$ and ray $\vec{r}(\lambda) = \lambda\bar{p}^c$, substitute $\vec{r}$ into the plane equation. With $\bar{p}^c = (p^c_x, p^c_y, p^c_z)$, we have $\lambda p^c_z = f$, so $\lambda^* = f/p^c_z$, and the intersection is

$$\vec{r}(\lambda^*) = \left(f\frac{p^c_x}{p^c_z}, f\frac{p^c_y}{p^c_z}, f \right) = f\left(\frac{p^c_x}{p^c_z}, \frac{p^c_y}{p^c_z}, 1 \right) \equiv \bar{x}^*. \tag{1}$$

The first two coordinates of this intersection $\bar{x}^*$ determine the image coordinates.

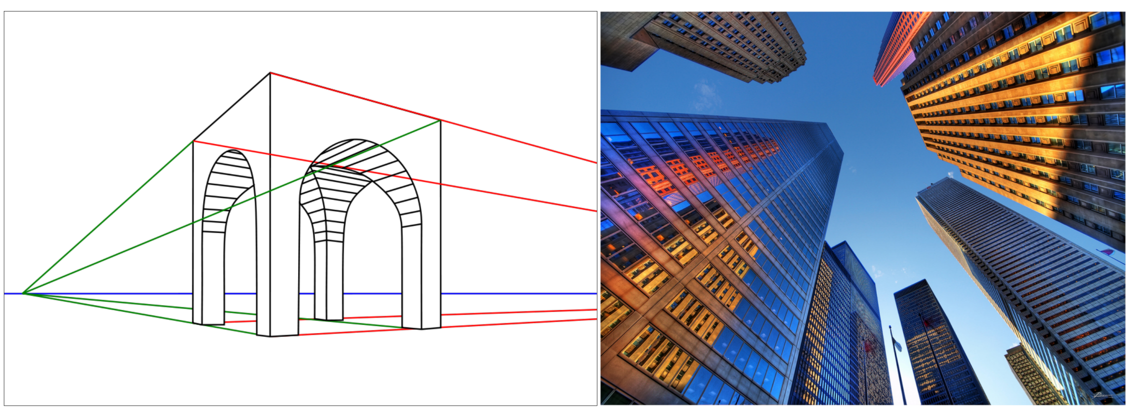
2D points in the image plane can therefore be written as

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \frac{f}{p_z^c} \begin{bmatrix} p_x^c \\ p_y^c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \frac{f}{p_z^c} \bar{p}^c.$$

The **perspective projection** of $\bar{p}^c$ on the image plane is $(x^*, y^*, 1)$.

Note:
Two important properties of perspective projection are:

- Perspective projection preserves linearity. In other words, the projection of a 3D line is a line in 2D. This means that we can render a 3D line segment by projecting the endpoints to 2D, and then draw a line between these points in 2D.
- Perspective projection does not preserve parallelism: two parallel lines in 3D do not necessarily project to parallel lines in 2D. When the projected lines intersect, the intersection is called a **vanishing point**, since it corresponds to a point infinitely far away. Exercise: when do parallel lines project to parallel lines and when do they not?



A sketch and a photograph showing the mapping of lines to lines, and the convergence of parallel lines under perspective projection. [Respective sources: Wikipedia, Flickr. Authors: Unknown, Paul Bica]

Aside:
The discovery of linear perspective, including vanishing points, formed a cornerstone of Western painting beginning at the Renaissance. On the other hand, defying realistic perspective was a key feature of Modernist painting.

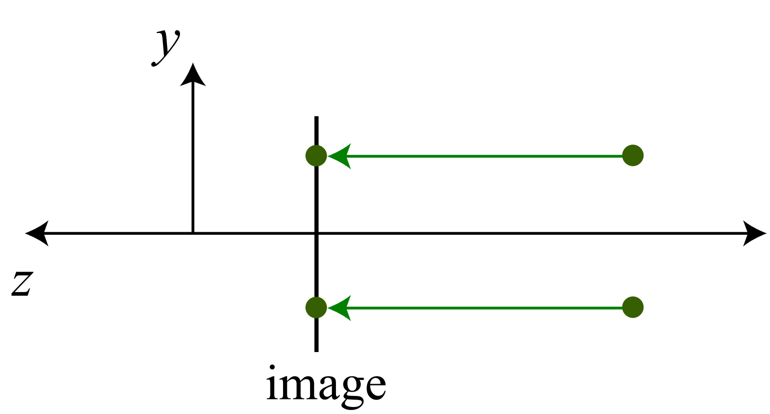
To see that linearity is preserved, consider that rays from points on a line in 3D through a pinhole all lie on a plane, and the intersection of a plane and the image plane is a line. That means to draw polygons, we need only to project the vertices to the image plane and draw lines between them.

5.4 Orthographic Projection

For objects sufficiently far away, rays are nearly parallel, and variation in p_z is insignificant.

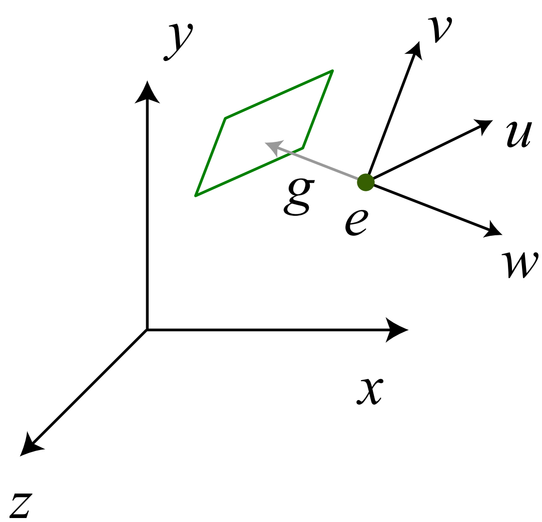
Here, the baseball players appear to be about the same height in pixels, even though the batter is about 60 feet away from the pitcher. Although this is an example of perspective projection, the camera is so far from the players (relative to the camera focal length) that they appear to be roughly the same size.

In the limit, $y = \alpha p_y$ for some real scalar α. This is **orthographic projection**:



5.5 Camera Position and Orientation

Assume camera coordinates have their origin at the "eye" (pinhole) of the camera, $\bar{e}$.



Let $\vec{g}$ be the gaze direction, so a vector perpendicular to the view plane (parallel to the camera z-axis) is

$$\vec{w} = \frac{-\vec{g}}{\|\vec{g}\|} \tag{2}$$

We need two more orthogonal vectors $\vec{u}$ and $\vec{v}$ to specify a camera coordinate frame, with $\vec{u}$ and $\vec{v}$ parallel to the view plane. It may be unclear how to choose them directly. However, we can instead specify an "up" direction. Of course this up direction will not be perpendicular to the gaze direction.

Let $\vec{t}$ be the "up" direction (e.g., toward the sky so $\vec{t} = (0, 1, 0)$). Then we want $\vec{v}$ to be the closest vector in the viewplane to $\vec{t}$. This is really just the projection of $\vec{t}$ onto the view plane. And of course, $\vec{u}$ must be perpendicular to $\vec{v}$ and $\vec{w}$. In fact, with these definitions it is easy to show that $\vec{u}$ must also be perpendicular to $\vec{t}$, so one way to compute $\vec{u}$ and $\vec{v}$ from $\vec{t}$ and $\vec{g}$ is as follows:

$$\vec{u} = \frac{\vec{t} \times \vec{w}}{\|\vec{t} \times \vec{w}\|} \qquad\qquad \vec{v} = \vec{w} \times \vec{u} \tag{3}$$

Of course, we could have use many different "up" directions, so long as $\vec{t} \times \vec{w} \neq 0$.

Using these three basis vectors, we can define a **camera coordinate system**, in which 3D points are represented with respect to the camera's position and orientation. The camera coordinate system has its origin at the eye point $\bar{e}$ and has basis vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, corresponding to the x, y, and z axes in the camera's local coordinate system. This explains why we chose $\vec{w}$ to point away from the image plane: the right-handed coordinate system requires that z (and, hence, $\vec{w}$) point away from the image plane.

Now that we know how to represent the camera coordinate frame within the world coordinate frame we need to explicitly formulate the rigid transformation from world to camera coordinates. With this transformation and its inverse we can easily express points either in world coordinates or camera coordinates (both of which are necessary).

To get an understanding of the transformation, it might be helpful to remember the mapping from points in camera coordinates to points in world coordinates. For example, we have the following correspondences between world coordinates and camera coordinates: Using such correspondences

Camera coordinates (x_c, y_c, z_c)	World coordinates (x, y, z)
$(0, 0, 0)$	$\bar{e}$
$(0, 0, f)$	$\bar{e} + f\vec{w}$
$(0, 1, 0)$	$\bar{e} + \vec{v}$
$(0, 1, f)$	$\bar{e} + \vec{v} + f\vec{w}$

it is not hard to show that for a general point expressed in camera coordinates as $\bar{p}^c = (x_c, y_c, z_c)$, the corresponding point in world coordinates is given by

$$\begin{aligned} \bar{p}^w &= \bar{e} + x_c\vec{u} + y_c\vec{v} + z_c\vec{w} && (4)\\ &= \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \bar{p}^c + \bar{e} && (5)\\ &= M_{cw}\bar{p}^c + \bar{e}. && (6) \end{aligned}$$

where

$$M_{cw} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \tag{7}$$

Note: We can define the same transformation for points in homogeneous coordinates:

$$\hat{M}_{cw} = \begin{bmatrix} M_{cw} & \bar{e} \\ \vec{0}^T & 1 \end{bmatrix}.$$

Now, we also need to find the inverse transformation, i.e., from world to camera coordinates. Toward this end, note that the matrix M_{cw} is orthonormal. To see this, note that vectors $\vec{u}$, $\vec{v}$ and, $\vec{w}$ are all of unit length, and they are perpendicular to one another. You can also verify this by computing $M_{cw}^T M_{cw}$. Because M_{cw} is orthonormal, we can express the transformation from world to camera coordinates as

$$\begin{aligned} \bar{p}^c &= M_{cw}^T(\bar{p}^w - \bar{e}) \\ &= M_{wc}\bar{p}^w - \bar{d}\,, \end{aligned}$$

where $M_{wc} = M_{cw}^T = \begin{bmatrix} \vec{u}^T \\ \vec{v}^T \\ \vec{w}^T \end{bmatrix}$. (why?), and $\bar{d} = M_{cw}^T \bar{e}$.

In homogeneous coordinates, $\hat{p}^c = \hat{M}_{wc}\hat{p}^w$, where

$$\begin{aligned} \hat{M}_{wc} &= \begin{bmatrix} M_{wc} & -M_{wc}\bar{e} \\ \vec{0}^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} M_{wc} & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} I & -\bar{e} \\ \vec{0}^T & 1 \end{bmatrix}. \end{aligned}$$

This transformation takes a point from world to camera-centered coordinates.

5.6 Homogeneous Perspective

The mapping of $\bar{p}^c = (p_x^c, p_y^c, p_z^c)$ to $\bar{x}^* = \frac{f}{p_z^c}(p_x^c, p_y^c, p_z^c)$ is just a form of scaling transformation. However, the magnitude of the scaling depends on the depth p_z^c. So it's not linear.

Fortunately, the transformation can be expressed linearly (ie as a matrix) in homogeneous coordinates. To see this, remember that $\hat{p} = (\bar{p}, 1) = \alpha(\bar{p}, 1)$ in homogeneous coordinates. Using this property of homogeneous coordinates we can write $\bar{x}^*$ as

$$\hat{x}^* = \left(p_x^c, p_y^c, p_z^c, \frac{p_z^c}{f} \right).$$

As usual with homogeneous coordinates, when you scale the homogeneous vector by the inverse of the last element, when you get in the first three elements is precisely the perspective projection. Accordingly, we can express $\hat{x}^*$ as a linear transformation of $\hat{p}^c$:

$$\hat{x}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \hat{p}^c \equiv \hat{M}_p \hat{p}^c.$$

Try multiplying this out to convince yourself that this all works.
Finally, $\hat{M}_p$ is called the homogeneous perspective matrix, and since $\hat{p}^c = \hat{M}_{wc}\hat{p}^w$, we have $\hat{x}^* = \hat{M}_p \hat{M}_{wc} \hat{p}^w$.

5.7 Pseudodepth

After dividing by its last element, $\hat{x}^*$ has its first two elements as image plane coordinates, and its third element is f. We would like to be able to alter the homogeneous perspective matrix $\hat{M}_p$ so that the third element of $\frac{p_z^c}{f}\hat{x}^*$ encodes depth while keeping the transformation linear.

Idea: Let $\hat{x}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 1/f & 0 \end{bmatrix} \hat{p}^c$, so $z^* = \frac{f}{p_z^c}(ap_z^c + b)$.

What should a and b be? We would like to have the following two constraints:

$$z^* = \begin{cases} -1 & \text{when } p_z^c = f \\ 1 & \text{when } p_z^c = F \end{cases},$$

where f gives us the position of the **near plane**, and F gives us the z coordinate of the **far plane**.

So $-1 = af + b$ and $1 = af + b\frac{f}{F}$. Then $2 = b\frac{f}{F} - b = b\left(\frac{f}{F} - 1\right)$, and we can find

$$b = \frac{2F}{f - F}.$$

Substituting this value for b back in, we get $-1 = af + \frac{2F}{f-F}$, and we can solve for a:

$$\begin{aligned} a &= -\frac{1}{f}\left(\frac{2F}{f-F} + 1\right) \\ &= -\frac{1}{f}\left(\frac{2F}{f-F} + \frac{f-F}{f-F}\right) \\ &= -\frac{1}{f}\left(\frac{f+F}{f-F}\right). \end{aligned}$$

These values of a and b give us a function $z^*(p_z^c)$ that increases monotonically as p_z^c decreases (since p_z^c is negative for objects in front of the camera). Hence, z^* can be used to sort points by depth.

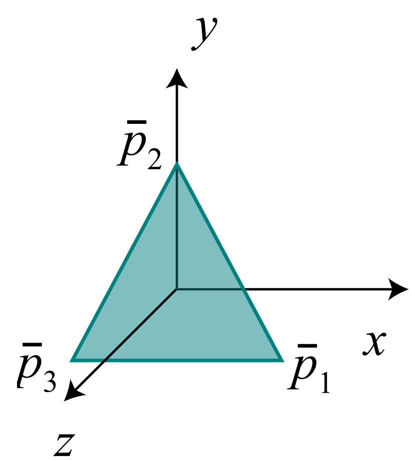
Why did we choose these values for a and b? Mathematically, the specific choices do not matter, but they are convenient for implementation. These are also the values that OpenGL uses.

What is the meaning of the near and far planes? Again, for convenience of implementation, we will say that only objects between the near and far planes are visible. Objects in front of the near plane are behind the camera, and objects behind the far plane are too far away to be visible. Of course, this is only a loose approximation to the real geometry of the world, but it is very convenient for implementation. The range of values between the near and far plane has a number of subtle implications for rendering in practice. For example, if you set the near and far plane to be very far apart in OpenGL, then Z-buffering (discussed later in the course) will be very inaccurate due to numerical precision problems. On the other hand, moving them too close will make distant objects disappear. However, these issues will generally not affect rendering simple scenes. (For homework assignments, we will usually provide some code that avoids these problems).

5.8 Projecting a Triangle

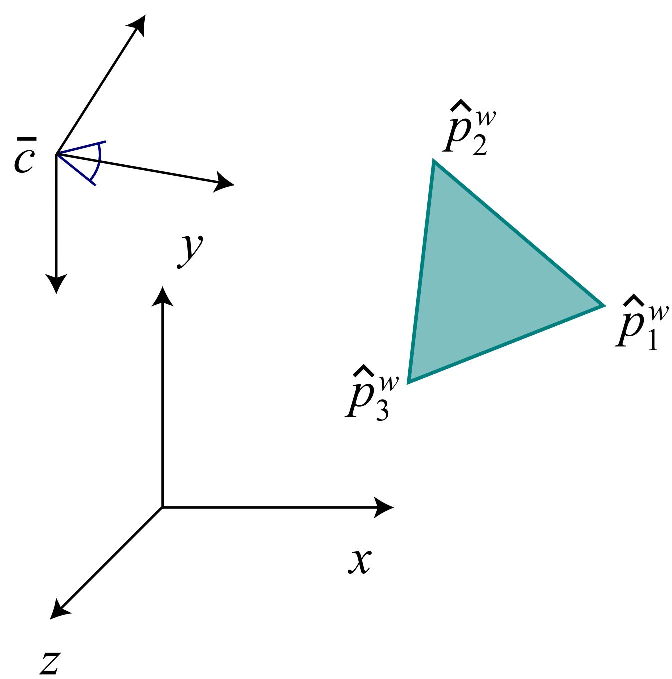
Let's review the steps necessary to project a triangle from object space to the image plane.

1. A triangle is given as three vertices in an object-based coordinate frame: $\bar{p}_1^o$, $\bar{p}_2^o$, $\bar{p}_3^o$.



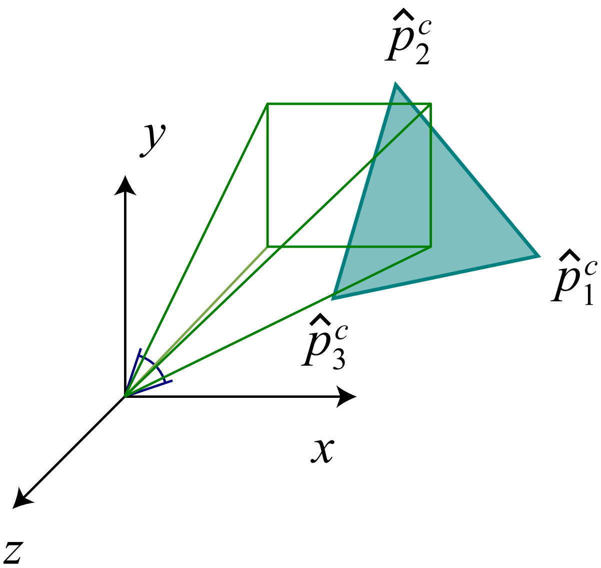
A triangle in object coordinates.

2. Transform to world coordinates based on the object's transformation: $\hat{p}_1^w$, $\hat{p}_2^w$, $\hat{p}_3^w$, where $\hat{p}_i^w = \hat{M}_{ow}\hat{p}_i^o$.



The triangle projected to world coordinates, with a camera at $\bar{c}$.

3. Transform from world to camera coordinates: $\hat{p}_i^c = \hat{M}_{wc}\hat{p}_i^w$.



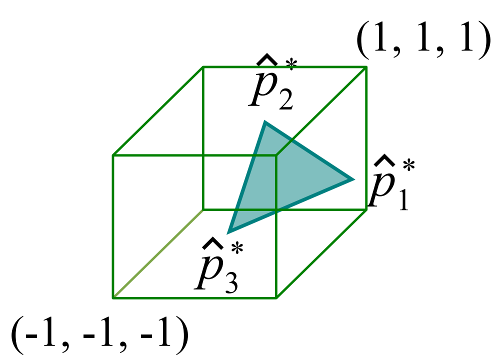
The triangle projected from world to camera coordinates.

4. Homogeneous perspective transformation: $\hat{x}_i^* = \hat{M}_p\hat{p}_i^c$, where

$$\hat{M}_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 1/f & 0 \end{bmatrix}, \text{ so } \hat{x}_i^* = \begin{bmatrix} p_x^c \\ p_y^c \\ ap_z^c + b \\ \frac{p_z^c}{f} \end{bmatrix}.$$

5. Divide by the last component:

$$\begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = f \begin{bmatrix} \frac{p_x^c}{p_z^c} \\ \frac{p_y^c}{p_z^c} \\ \frac{ap_z^c+b}{p_z^c} \end{bmatrix}.$$



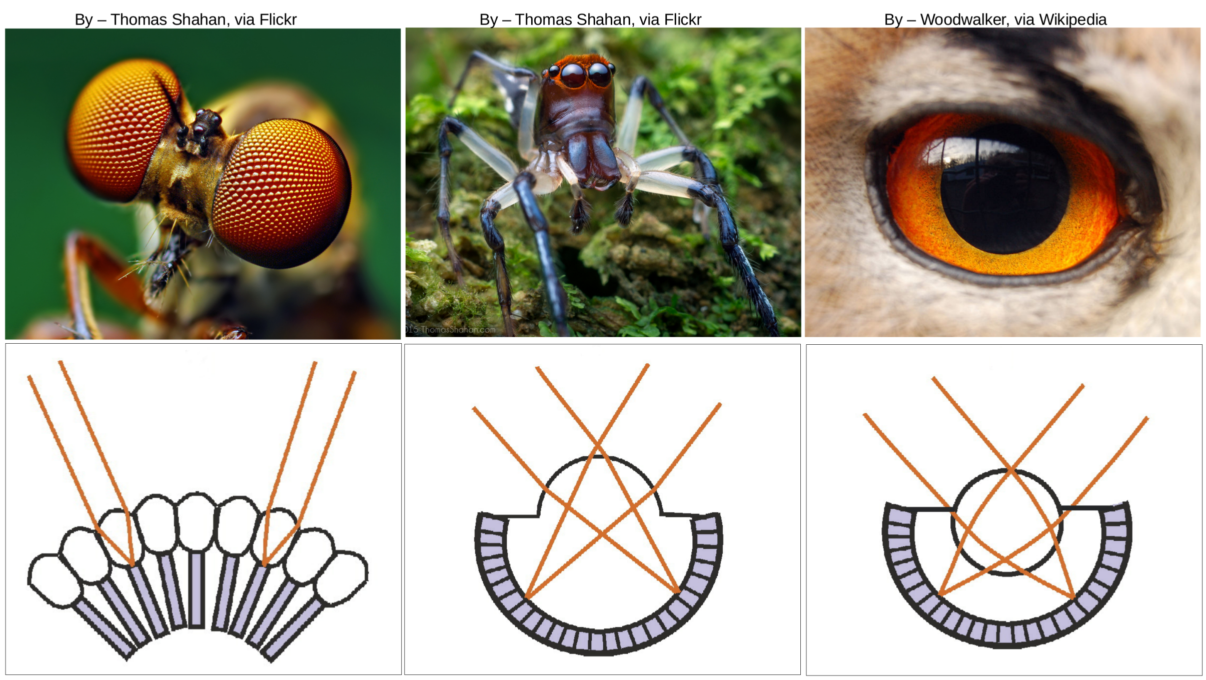
The triangle in normalized device coordinates after perspective division.

Now (x^*, y^*) is an image plane coordinate, and z^* is pseudodepth for each vertex of the triangle.

The pinhole camera model is used in many computer graphics applications where a realistic simulation of the camera itself is not of interest. However, real cameras have a finite-size aperture that introduces the problem of having to focus rays of light arriving anywhere within its area. Let's have a quick look at a simple but more realistic model for a camera with an aperture and lens. Such a model will become important when we move on to study photo-realistic image rendering.

5.9 Thin Lens Model

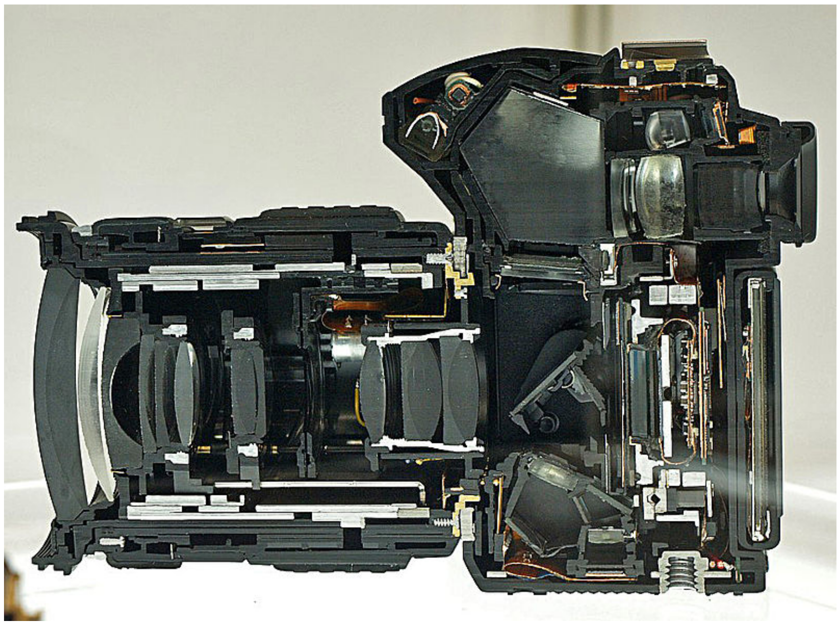
Most modern cameras use a lens to focus light onto the view plane (i.e., the sensory surface). This is done so that one can capture enough light in a sufficiently short period of time that the objects do not move appreciably, and the image is bright enough to show significant detail over a wide range of intensities and contrasts.
Lens models can be quite complex, biological eyes have evolved specialized structures to provide their host animals with the visual information needed for their survival.



Diagrams adapted from: Michael F. Land, *"The Optical Structures of Animal Eyes"*, Current Biology, 15(9), 2005

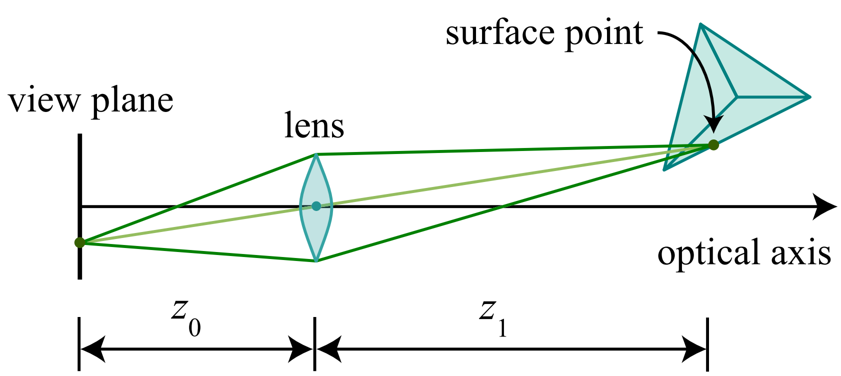
Examples of different lens structures found in the animal world. From left to right: An insect's compound eye, a spider's simple corneal eye (no lens), and an owl's complex eye with distinct lens.

Artificial cameras typically contain complicated lens arrangements intended to permit the user to change the field of view (zooming in or out), and to manage chromatic aberration. Typical designs contain multiple simple lenses that work together to achieve focusing as shown below.



Cross section of a digital SLR camera. Note the complex arrangement of individual glass components making up the camera's lens. [Source: Wikipedia, author: Hanabi123]

Here we consider perhaps the simplest case, known widely as the thin lens model. In the thin lens model, rays of light emitted from a point travel along paths through the lens, converging at a point behind the lens. The key quantity governing this behaviour is called the *focal length* of the lens. The focal length,, $|f|$, can be defined as distance behind the lens to which rays from an infinitely distant source converge in focus.



More generally, for the thin lens model, if z_1 is the distance from the center of the lens (i.e., the nodal point) to a surface point on an object, then for a focal length $|f|$, the rays from that surface point will be in focus at a distance z_0 behind the lens center, where z_1 and z_0 satisfy the thin lens equation:

$$\frac{1}{|f|} = \frac{1}{z_0} + \frac{1}{z_1} \tag{8}$$

We will come back to the thin-lens model when we look at photo-realistic image rendering, where it will be essential for simulating the depth-of-field effect of real cameras - depth-of-field refers to the existence of regions in the image that are in focus and appear sharp, whereas parts of the scene not in focus appear blurry.

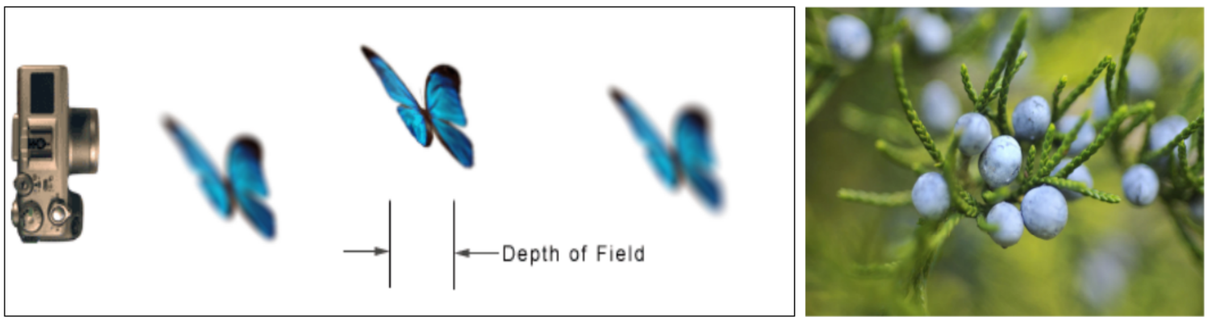


Illustration of the Depth-of-Field effect. Only the scene components within a small distance from the focus point will be sharp. Everything else is blurred. [Source: Wikipedia, authors: Jared Benedict, Denis Gazso]