New maximal arcs in Desarguesian planes

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Abstract

Constructions are described of maximal arcs in Desarguesian projective planes utilizing sets of conics on a common nucleus in $\mathrm{PG}(2,q)$. Several new infinite families of maximal arcs in $\mathrm{PG}(2,q)$ are presented and a complete enumeration is carried out for Desarguesian planes of order 16, 32 and 64. For each arc we list the order of its stabilizer and the numbers of subarcs it contains. Maximal arcs may be used to construct interesting new partial geometries, 2-weight codes and resolvable Steiner 2-designs.

1 Introduction

A *maximal* $\{k;n\}$*-arc* $\mathcal{K}$ in a projective plane of order q is a subset of $k = nq + n - q$ points such that every line meets $\mathcal{K}$ in 0 or n points. A line that meets $\mathcal{K}$ is called a *secant*, otherwise it is *external*. The number n of arc points on a secant is called the *degree* of $\mathcal{K}$, $2 \leq n \leq q$. The set of lines external to $\mathcal{K}$ is a maximal $\{k';n'\}$-arc $\mathcal{K}'$ in the dual plane, where $k' = n'q + n' - q$,
will study maximal arcs in Desarguesian projective planes $\mathrm{PG}(2,q)$. It has been shown [2] that no nontrivial maximal arcs exist in $\mathrm{PG}(2,q)$ if q is odd. For q even a necessary and sufficient condition for the existence of a degree n maximal arc in $\mathrm{PG}(2,q)$ is that $n|q$ where $2 \leq n < q$ [11].
Maximal arcs give rise to many interesting incidence structures such as partial geometries, Steiner 2-designs and linear codes. So for example, a maximal $\{k;n\}$-arc $\mathcal{K}$ in a projective plane π of order q yields a resolvable Steiner 2-design $S(2,n,k)$ on the arcs points with blocks induced by the secants. It is well known [16] that on the complementary set of points $\pi \setminus \mathcal{K}$ the secants induce a partial geometry $\mathrm{pg}(q-n, q-m, q-m-n+1)$, where $m = q/n$. If on the other hand $\mathrm{PG}(2,q)$ is embedded as a hyperplane in

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$\mathrm{PG}(3,q)$ a maximal arc $\mathcal{K}$ yields a partial geometry $\mathrm{pg}(q^2-1,k-1,n-1)$ with points $\mathrm{PG}(3,q)\setminus\mathrm{PG}(2,q)$ and lines induced by those lines of $\mathrm{PG}(3,q)$ which intersect $\mathrm{PG}(2,q)$ in a point of $\mathcal{K}$ [4]. Since $\mathcal{K}$ is a 2-intersection set in $\mathrm{PG}(2,q)$ the coordinates of the arc points can be regarded as columns of the generator matrix of an optimal 2-weight code [3, 5].
There are three basic general constructions for maximal arcs in $\mathrm{PG}(2,q)$.

1. In 1969 Denniston [6] used a special pencil of conics to construct maximal arcs in Desarguesian planes of even order. Let $\xi^2+\alpha\xi+1$ be an irreducible polynomial over $\mathrm{GF}(q)$, and let $\mathcal{F}$ be the set of conics given by the pencil

$$F_\lambda : x^2+\alpha xy+y^2+\lambda z^2=0, \quad \lambda\in\mathrm{GF}(q)\cup\{\infty\}.$$

   Using homogeneous coordinates over $\mathrm{GF}(q)$ to represent $\mathrm{PG}(2,q)$ we see that F_0 is the point $(0,0,1)$, F_∞ is the line $z^2=0$ and every other conic in the pencil is non-degenerate and has nucleus F_0. Following [7] we call F_λ the *standard pencil*. Denniston showed that if A is an additive subgroup of $\mathrm{GF}(q)$ of order n then the set of points $\mathcal{K}$ of all conics F_λ for $\lambda\in A$, form a maximal $\{q(n-1)+n;n\}-$arc. These arcs in $\mathrm{PG}(2,q)$ are called Denniston maximal arcs and are characterized by invariance under a linear cyclic collineation group of order $q+1$ [1]. Their full stabilizers have been completely described [8].

2. In 1974 Thas [15] used ovoids and spreads in generalized quadrangles $W(q)$ to construct maximal arcs of degree q in planes of order q^2. Let $\mathrm{PG}(3,q)$ be embedded as a hyperplane H in $\mathrm{PG}(4,q)$. Let $\mathcal{O}$ be an ovoid and $\mathcal{S}$ a spread of H such that each line of $\mathcal{S}$ has exactly one point in common with $\mathcal{O}$. If $x\in\mathrm{PG}(4,q)\setminus H$ then the union of points $\mathcal{K}$ on the lines joining x and $\mathcal{O}$ form a maximal $\{q^3-q^2+q;q\}-$arc in the translation plane π defined by $\mathcal{S}$. The known ovoids of $\mathrm{PG}(3,q)$, q even are the elliptic quadrics and the Tits ovoids defined for $q=2^{2m+1}$, $m\geq 1$. For a plane π to be Desarguesian $\mathcal{S}$ must be a regular spread. It is known [8] that any elliptic quadric-regular spread pair gives rise to a Denniston maximal arc. Penttila [8] found that there are exactly two orbits on Tits ovoids in the stabilizer of a regular spread yielding two families of Thas maximal arcs of degree q in $\mathrm{PG}(2,q^2)$, $q=2^{2m+1}$ associated with Tits ovoids.

3. In 1980 Thas [17] employed quadrics and spreads in projective spaces to construct degree q^{t-1} maximal arcs in symplectic translation planes of order q^t. Let $\mathrm{Q}^- = \mathrm{Q}^-(2t-1,q)$ be a non-singular elliptic quadric

and let $\mathcal{S}$ be a $(t-1)$-spread in $\mathrm{PG}(2t-1,q)$ (i.e., a partition into $(t-1)$-dimensional subspaces) of which the restriction to Q^- is a $(t-2)$-spread. Embed $\mathrm{PG}(2t-1,q)$ as a hyperplane H in $\mathrm{PG}(2t,q)$. If $x \in \mathrm{PG}(2t,q) \setminus H$ then the union of points $\mathcal{K}$ on the lines joining x and Q^- form a maximal $\{q^{2t-1}-q^t+q^{t-1}; q^{t-1}\}-$arc in the translation plane π defined by $\mathcal{S}$. We note that for $t=2$ we obtain Construction 2 in which the ovoid is an elliptic quadric. Again as before, if $\mathcal{S}$ is regular then $\mathcal{K}$ is a Denniston maximal arc [8]. A $(t-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(2t-1,q)$ inducing a $(t-2)$-spread in Q^- is equivalent to a $(t-2)$-system *contained by* a $(t-1)$-system of the symplectic polar space $W_{2t-1}(q)$ (see [9], [10] and [14] for more details).

The rest of the paper is organized as follows. In Section 2 we give conditions for a set of conics on a common nucleus to generate a maximal arc. Admissible sets are defined recursively from a basis utilizing a special composition of conics. A concise algebraic description of the pencils is presented in terms of sparse polynomials. Several infinite families of new maximal arcs of degree q and $2q$ in $\mathrm{PG}(2,q^2)$ are described in Section 3. The last section contains an exhaustive enumeration of maximal arcs based on pencils of conics in Desarguesian planes of order $q = 16$, 32 and 64. For each maximal arc we list the order of its stabilizer and the number of subarcs contained in it. We conclude with some conjectures and open problems.

2 Sets of conics

In a $\mathrm{PG}(2,2^m)$ represented via homogeneous coordinates over $\mathrm{GF}(2^m)$ let (a,b,c) denote points, $[u,v,w]$ lines and let incidence be given by the usual inner product $au+bv+cw=0$. Let $\mathcal{F}$ bet the set of conics

$$F_{\alpha,\beta,\lambda} : \alpha x^2 + xy + \beta y^2 + \lambda z^2 = 0, \quad \lambda \in \mathrm{GF}(q) \cup \{\infty\}, \tag{1}$$

where α, β are non-zero elements of $\mathrm{GF}(2^m)$ such that the quadratic polynomial $\alpha\xi^2 + \xi + \beta$ is irreducible over $\mathrm{GF}(2^m)$, or equivalently, satisfying $Tr_{2^m/2}(\alpha\beta) = 1$. Here $Tr_{2^m/2}(a)$ stands for the trace of $a \in \mathrm{GF}(2^m)$ over $\mathrm{GF}(2)$. We see that $F_{\alpha,\beta,0} := F_0$ is the point $(0,0,1)$, $F_{\alpha,\beta,\infty} := F_\infty$ is the line at infinity $z^2=0$ and every other conic is non-degenerate with nucleus F_0. We note that for fixed α, β satisfying $Tr_{2^m/2}(\alpha\beta)=1$ the set $\mathcal{F}$ is isomorphic to Denniston's standard pencil.
Given two non-degenerate conics $F_{\alpha,\beta,\lambda}$, $F_{\alpha',\beta',\lambda'} \in \mathcal{F}$ with $\lambda \neq \lambda'$ define a composition

$$F_{\alpha\oplus\alpha',\beta\oplus\beta',\lambda\oplus\lambda'} \;=\; F_{\alpha,\beta,\lambda} \;\oplus\; F_{\alpha',\beta',\lambda'}$$

$$\alpha \oplus \alpha' = \frac{\alpha\lambda + \alpha'\lambda'}{\lambda + \lambda'}, \quad \beta \oplus \beta' = \frac{\beta\lambda + \beta'\lambda'}{\lambda + \lambda'}, \quad \lambda \oplus \lambda' = \lambda + \lambda'. \tag{2}$$

Clearly, $\oplus$ is commutative and since $(\alpha \oplus \alpha') \oplus \alpha'' = \alpha \oplus (\alpha' \oplus \alpha'') = (\alpha\lambda + \alpha'\lambda' + \alpha''\lambda'')/(\lambda + \lambda' + \lambda'')$ it is also associative. Moreover, $F_{\alpha,\beta,\lambda} \oplus F_{\alpha,\beta,\lambda'} = F_{\alpha,\beta,\lambda+\lambda'}$, implies that $\oplus$ is idempotent in the first two parameters.

Lemma 2.1 *The conics $F_{\alpha,\beta,\lambda}$, $F_{\alpha',\beta',\lambda'} \in \mathcal{F}$ and $F_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'}$, $\lambda \neq \lambda'$ are mutually disjoint if $Tr_{2^m/2}((\alpha \oplus \alpha')(\beta \oplus \beta')) = 1$.*

Proof. Our aim is to map the three conics into a standard pencil. Consider a collineation H of $\mathrm{PG}(2, 2^m)$ represented by the matrix

$$H = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & a & 0 \\ b & c & 1 \end{pmatrix}$$

which maps (x, y, z) to $(x/a, ay, bx + cy + z)$. Let

$$A = \frac{\alpha'\lambda + \alpha\lambda'}{\lambda + \lambda'}, \quad B = \frac{\beta'\lambda + \beta\lambda'}{\lambda + \lambda'}.$$

It is easily verified that if $a = \sqrt{A}$, $b = \sqrt{(\alpha/A + 1)/\lambda}$ and $c = \sqrt{A(\beta + B)/\lambda}$ then H maps the conics $F_{\alpha,\beta,\lambda}$, $F_{\alpha',\beta',\lambda'}$ and $F_{\alpha,\beta,\lambda} \oplus F_{\alpha',\beta',\lambda'}$ to $F_{1,\epsilon,\lambda}$, $F_{1,\epsilon,\lambda'}$ and $F_{1,\epsilon,\lambda+\lambda'}$, respectively, which are contained in a standard pencil provided that $\epsilon = AB$ has unit trace. But

$$Tr_{2^m/2}(\epsilon) = Tr_{2^m/2}(AB) = Tr_{2^m/2}(\alpha\beta) + Tr_{2^m/2}(C)$$

$$= Tr_{2^m/2}(\alpha'\beta') + Tr_{2^m/2}(C) = Tr_{2^m/2}(\alpha \oplus \alpha')(\beta \oplus \beta')) = 1$$

where $C = ((\alpha\beta' + \beta\alpha')\lambda\lambda' + (\alpha\beta + \alpha\beta')\lambda'^2)/(\lambda^2 + \lambda'^2)$. □

In order to state our main result we require the notion of a closure. A set of non-degenerate conics $\mathcal{F}$ of the form (1) is said to be closed under composition $\oplus$ if for any two different conics F, F' in $\mathcal{F}$ their composition $F \oplus F'$ is also in $\mathcal{F}$. We recall that any $F_{\alpha,\beta,\lambda}$ in $\mathcal{F}$ has $Tr_{2^m/2}(\alpha\beta) = 1$.

Lemma 2.2 *Suppose that $\mathcal{F}$ containing N conics is closed under composition. Let $F' = F_{\alpha',\beta',\lambda'} \notin \mathcal{F}$, be a conic with $Tr_{2^m/2}(\alpha'\beta') = 1$ such that $Tr_{2^m/2}((\alpha \oplus \alpha')(\beta \oplus \beta')) = 1$ for every $F_{\alpha,\beta,\lambda} \in \mathcal{F}$. Then the closure $\mathcal{F}' = \langle \mathcal{F} \cup \{F'\} \rangle$ contains $2N + 1$ conics and $\mathcal{F}' = \{F,\ F',\ F \oplus F' \mid F \in \mathcal{F}\}$.*

Proof. The claim follows from the fact that $\mathcal{F}$ is closed under composition and that

$$(F \oplus F') \oplus (G \oplus F') = F \oplus G.$$

$\square$

Starting from a single conic repeated applications of Lemma 2.2 yield the following result.

Corollary 2.3 *A closed set $\mathcal{F}$ of conics on a common nucleus contains $2^d - 1$ conics, $1 \le d \le m$.*

Our next result shows that sets of conics closed under composition can be used to construct maximal arcs.

Theorem 2.4 *Let $\mathcal{F}$ be a closed set of $2^d - 1$ conics with a common nucleus F_0 in $\mathrm{PG}(2, 2^m)$, $1 \le d \le m$. Then the set of points of all conics in $\mathcal{F}$ together with F_0 form a maximal $\{2^{m+d} - 2^m + 2^d; 2^d\}$-arc $\mathcal{K}$ in $\mathrm{PG}(2, 2^m)$.*

Proof. We will show that every line of $\mathrm{PG}(2, 2^m)$ meets $\mathcal{K}$ in 0 or 2^d points. Since $Tr_{2^m/2}(\alpha\beta) = 1$, the line at infinity F_∞ is external to $\mathcal{K}$. Every other line of the plane meets F_∞ in one of its points $(1, 0, 0)$, $(a, 1, 0)$, $a \in \mathrm{GF}(2^m)$ and hence belongs to the set of lines $\{[0, 1, 0],\ [0, b, 1],\ [1, a, b] \mid b \in \mathrm{GF}(2^m)\}$. The lines $[0, 1, 0]$ and $[1, a, 0]$ meet the common nucleus F_0 and are therefore tangent to every conic in $\mathcal{F}$. Consequently, they meet $\mathcal{K}$ in $|\mathcal{F}|+1 = 2^d$ points. Any of the remaining lines meets a conic $F \in \mathcal{F}$ in either 0 or 2 points. A line $[0, b, 1]$ or $[1, a, b]$ with points $(x, 1, b)$ or $(ax, x + b, 1)$, $x \in \mathrm{GF}(2^m)$ is disjoint from $F = F_{\alpha,\beta,\lambda}$ if and only if the quadratic equation $\alpha x^2 + x + \beta + \lambda b^2 = 0$ or $\alpha a^2 x^2 + ax(x + b) + \beta(x + b)^2 + \lambda = 0$ has no solution in $\mathrm{GF}(2^m)$, respectively. This is equivalent to the condition $Tr_{2^m/2}(\alpha, \beta, \lambda) = 0$ of the form

$$Tr_{2^m/2}(\alpha\lambda b^2) = 0, \quad or \quad Tr_{2^m/2}(\frac{\alpha a^2 + a + \beta}{a^2 b^2}\lambda) = 0. \tag{3}$$

Using (2) it is easily verified that in both cases

$$Tr_{2^m/2}(\alpha \oplus \alpha', \beta \oplus \beta', \lambda \oplus \lambda') \;=\; Tr_{2^m/2}(\alpha, \beta, \lambda) + Tr_{2^m/2}(\alpha', \beta', \lambda'). \tag{4}$$

Hence, if a line L does not meet both F and F' then it also does not meet $F \oplus F'$. There are two ways for L to interact with the conics in $\mathcal{F}$. If $Tr_{2^m/2}(F) = 0$ for all $F \in \mathcal{F}$ then L is disjoint from $\mathcal{K}$. On the other hand, if $Tr_{2^m/2}(F) = 1$ for some $F \in \mathcal{F}$ then using the recursive doubling argument of Lemma 2.2 and (4) we see that there are exactly 2^{d-1} conics in $\mathcal{F}$ with trace 1. Since by Lemma 2.1 the conics are mutually disjoint, L intersects $\mathcal{K}$ in 2^d points. $\square$

A more explicit form can be given to a closed set of conics $\mathcal{F} = \{F_{\alpha,\beta,\lambda}\}$ by using polynomials to interpolate the coefficients α, β in terms λ.

Theorem 2.5 *Let $p(\lambda) = \sum_{i=0}^{d-1} a_i \lambda^{2^i-1}$ and $q(\lambda) = \sum_{i=0}^{d-1} b_i \lambda^{2^i-1}$ be polynomials with coefficients in $\mathrm{GF}(2^m)$. For an additive subgroup A of order 2^d in $\mathrm{GF}(2^m)$ let $\mathcal{F} = \{F_{p(\lambda),q(\lambda),\lambda} \mid \lambda \in A\}$ be a set of conics (1) with a common nucleus F_0. If $Tr_{2^m/2}(p(\lambda)q(\lambda)) = 1$ for every $\lambda \in A$ then the set of points on all conics in $\mathcal{F}$ together with F_0 form a maximal $\{2^{m+d} - 2^m + 2^d; 2^d\}$-arc $\mathcal{K}$ in $\mathrm{PG}(2, 2^m)$. If both $p(\lambda)$, $q(\lambda)$ are of degree ≤ 1 in λ then $\mathcal{K}$ is a Denniston maximal arc .*

Proof. The result follows from Theorem 2.4 provided that polynomials $p(\lambda)$ and $q(\lambda)$ satisfy composition. Since $\lambda^{2^i} + \lambda'^{2^i} = (\lambda + \lambda')^{2^i}$ for any $\lambda, \lambda' \in \mathrm{GF}(2^m)$ we have

$$p(\lambda) \oplus p(\lambda') = \frac{p(\lambda)\lambda + p(\lambda')\lambda'}{\lambda + \lambda'} = \sum_{i=0}^{d-1} a_i \frac{\lambda^{2^i} + \lambda'^{2^i}}{\lambda + \lambda'} = \sum_{i=0}^{d-1} a_i (\lambda + \lambda')^{2^i - 1}$$

and the same is true for $q(\lambda)$. Consequently,

$$F_{p(\lambda),q(\lambda),\lambda} \oplus F_{p(\lambda'),q(\lambda'),\lambda'} = F_{p(\lambda)\oplus p(\lambda'),q(\lambda)\oplus q(\lambda'),\lambda\oplus\lambda'} = F_{p(\lambda+\lambda'),q(\lambda+\lambda'),\lambda+\lambda'}.$$

Hence addition in A corresponds to composition in $\mathcal{F}$.
If $p(\lambda) = a_0 + a_1\lambda$, $q(\lambda) = b_0 + b_1\lambda$ then the collineation H of Lemma 2.1 with $a = \sqrt{a_0}$, $b = \sqrt{a_1/a_0}$ and $c = \sqrt{a_0, b_1}$ maps every conic in $\mathcal{F}$ to $F_{1,\epsilon,\lambda}$, $\epsilon = \sqrt{a_0 b_0}$ of the standard pencil. $\square$

The 2^d-degree maximal arcs in $\mathrm{PG}(2, 2^m)$ described in Theorem 2.5 are uniquely determined by the $2d$ coefficients $a_i, b_i \in \mathrm{GF}(2^m)$, $i = 0, \cdots, d-1$, which in some way reflect the arc symmetry. In general, a maximal arc $\mathcal{K}$ of degree 2^d constructed from a closed set of conics inherits the rich subgroup structure of the additive subgroup A of $\mathrm{GF}(2^m)$. Since A is elementary abelian the subarcs of $\mathcal{K}$ are maximal of degree 2^s for all s, $1 \leq s \leq d$ and form a projective space $\mathrm{PG}(d-1, 2)$. Note that all degree 4 arcs obtained from the above Theorems are by Lemma 2.1 of Denniston type. On the other hand, there exist closed sets of conics which are maximal and can not be extended any further.

3 New families of maximal arcs

Let w be a primitive element of $\mathrm{GF}(2^{2m})$, $m > 1$. The set $G_{2^m} = \{w^{i(2^m+1)} \mid i = 0, \cdots, 2^m - 2\} \cup \{0\}$ is a subfield of order 2^m of $\mathrm{GF}(2^{2m})$. Since $w^{i(2^m+1)2^m} =$

$w^{i(2^m+1)}$ we see that $x = x^{2^m}$ for any element $x \in G_{2^m}$. Using the transitivity of trace

$$Tr_{2^{2m}/2}(x) = Tr_{2^m/2}(Tr_{2^{2m}/2^m}(x)) = Tr_{2^m/2}(x + x^{2^m}) = 0.$$

Consequently, $Tr_{2^{2m}/2}(x) = 0$ for any element x of the subfield G_{2^m}.

Theorem 3.1 *Let w be a primitive element of* $\mathrm{GF}(2^{2m})$*, $\epsilon \in \mathrm{GF}(2^{2m})$ an element with $Tr_{2^{2m}/2}(\epsilon) = 1$, and let $q(\lambda) = \sum_{i=0}^{m-1} b_i\lambda^{2^i-1}$ be a polynomial with coefficients $b_0 = \epsilon$, $b_i \in \{0,1\}$, for $i = 1, \cdots, m-1$. Then the conics $\mathcal{F}_{\mathbf{b}} = \{F_{1,q(\lambda),\lambda} \mid \lambda \in G_{2^m}\}$ generate maximal arcs of degree 2^m in* $\mathrm{PG}(2, 2^{2m})$.

Proof. To satisfy the condition of Theorem 2.5 we must check the trace of $q(\lambda)$. Since λ is in the subfield G_{2^m} and the coefficients of nonzero powers of λ are either 0 or 1 we have

$$Tr_{2^{2m}/2}(\sum_{i=0}^{d-1} b_i\lambda^{2^i-1}) = \sum_{i=0}^{d-1} Tr_{2^{2m}/2}(b_i\lambda^{2^i-1}) = Tr_{2^{2m}/2}(\epsilon) = 1.$$

□

For example, in GF(64) we can take a primitive element satisfying $w^6+w = 1$ and $\epsilon = w^5$ to generate a set of 7 conics

$$\{x^2 + xy + (w^{27i} + w^5)y^2 + w^{9i}z^2 = 0 \mid i = 0, \cdots, 6\}$$

corresponding to a maximal arc of degree 8 in $\mathrm{PG}(2, 64)$ with a stabilizer of order 12.

In GF(256) we can take a primitive element satisfying $w^{25} + w = 1$ and $\epsilon = w^5$ to generate 3 sets of 15 conics

$$\{x^2 + xy + (w^{51i} + w^5)y^2 + w^{17i}z^2 = 0 \mid i = 0, \cdots, 15\}$$

$$\{x^2 + xy + (w^{119i} + w^5)y^2 + w^{17i}z^2 = 0 \mid i = 0, \cdots, 15\}$$

$$\{x^2 + xy + (w^{119i} + w^{51i} + w^5)y^2 + w^{17i}z^2 = 0 \mid i = 0, \cdots, 15\}$$

yielding 3 non-isomorphic arcs of degree 16 in $\mathrm{PG}(2, 256)$. Each of these arcs contains 15 maximal subarcs of degree 8, most with small stabilizers.

Let G_{2^m} be a subfield of index 2 of $\mathrm{GF}(2^{2m})$, for some $m > 1$, and denote by

$$C = \{x \mid x^2 + x + c = 0,\ c \in G_{2^m},\ x \in \mathrm{GF}(2^{2m})\} \tag{5}$$

the set of all solutions in $\mathrm{GF}(2^{2m})$ of quadratic equations $x^2 + x + c = 0$ with c in the subfield G_{2^m}. Since $Tr_{2^{2m}/2}(c) = 0$ this equation has always 2

solutions and so $|C| = 2^{m+1}$. Clearly, $G_{2^m} \subset C$, C is closed under addition and contains its squares $c^2 \in C$.
Next we construct maximal arcs of degree 2^{m+1} in $\mathrm{PG}(2, 2^{2m})$ utilizing C as an additive subgroup of $\mathrm{GF}(2^{2m})$.

Theorem 3.2 *Let $\epsilon \in \mathrm{GF}(2^{2m})$ be an element with $Tr_{2^{2m}/2}(\epsilon) = 1$, and let $C = \{x \mid x^2 + x + c = 0,\ c \in G_{2^m},\ x \in \mathrm{GF}(2^{2m})\}$. If $q(\lambda) = \lambda^{2^{m-1}-1} + \epsilon$ then the conics $\mathcal{F} = \{F_{1,q(\lambda),\lambda} \mid \lambda \in C\}$ generate a maximal arc in $\mathrm{PG}(2, 2^{2m})$ of degree 2^{m+1}.*

Proof. To satisfy the condition of Theorem 2.5

$$Tr_{2^{2m}/2}(x^{2^{m-1}-1} + \epsilon) = Tr_{2^{4m+2}/2}(x^{2^{m-1}-1}) + 1 = 1,$$

we must show that in $\mathrm{GF}(2^{2m})$ the trace of $x^{2^{m-1}-1}$ over $\mathrm{GF}(2)$ is 0 for all $x \in C$.
If $x^2 + x = 0$ then $x = 0, 1$ and so $x^{2^{m-1}} \in \{0, 1\}$ which have both 0 trace in $\mathrm{GF}(2^{2m})$. Let $c = w^{i(2^m+1)}$, $0 \le i \le 2^m - 2$ be any other element of G_{2^m}. Reducing modulo $x^2 + x + c$ yields

$$x^{2^{m-1}-1} = x + (x+1)\sum_{j=1}^{m-2} c^{2^j-1} = \sum_{j=1}^{m-2} c^{2^j-1} + c\sum_{j=1}^{m-2} c^{2^j-1}.$$

Since the first sum is in G_{2^m}, $Tr_{2^{2m}/2}(x^{2^{m-1}-1}) = Tr_{2^{2m}/2}(c\sum_{j=1}^{m-2} c^{2^j-1})$. Using the expansion $x + x^{2^m} = (x + x^2) + (x^2 + x^4) + \cdots + (x^{2^{m-1}} + x^{2^m}) = c + c^2 + \cdots + c^{2^{m-1}}$ the trace over $\mathrm{GF}(2^m)$ is

$$Tr_{2^{2m}/2^m}(c\sum_{j=1}^{m-2} c^{2^j-1}) = (x + x^{2^m})\sum_{j=1}^{m-2} c^{2^j-1} = \sum_{j=1}^{m-2}(c^{2^j-1} + c^{2^{m-1}+2^{j-1}-1}).$$

But in $\mathrm{GF}(2^{2m})$ we have $(c^{2^{m-1}+2^{j-1}-1})^2 = c^{2^m+2^j-2} = c^{2^j-1}w^{i(2^{2m}-1)} = c^{2^j-1}$. Consequently,

$$Tr_{2^m/2}(\sum_{j=1}^{m-2} c^{2^j-1}) = Tr_{2^m/2}(\sum_{j=1}^{m-2} c^{2^{m-1}+2^{j-1}-1}),$$

and the desired result follows from the transitivity of trace

$$Tr_{2^{2m}/2}(x^{2^{m-1}-1}) = Tr_{2^m/2}(Tr_{2^{2m}/2^m}(c\sum_{j=1}^{m-2} c^{2^j-1})) = 0.$$

$\square$

For example, in GF(64) with a primitive element satisfying $w^6 + w = 1$, $\epsilon = w^5$ and $C = \{w^{9i}, w^{9i} + w^{21} \mid i = 0, \cdots, 6\} \cup \{w^{21}\}$ we obtain a set of 15 conics

$$\{x^2 + xy + (w^5 + \lambda^3)y^2 + \lambda z^2 = 0 \mid \lambda \in C\}$$

corresponding to a maximal arc of degree 16 in PG$(2, 64)$ with a stabilizer of order 12. This arc contains 5 non-isomorphic maximal 8-arcs with stabilizers of order 12,6,4,4 and 2 including our previous example.
In GF(256) we can take a primitive element satisfying $w^{25} + w = 1$, $\epsilon = w^5$ and $C = \{w^{17i}, w^{17i} + w^7 \mid i = 0, \cdots, 14\} \cup \{w^7\}$ to obtain a set of 31 conics

$$\{x^2 + xy + (w^5 + \lambda^7)y^2 + \lambda z^2 = 0 \mid \lambda \in C\}$$

corresponding to a maximal arc of degree 32 in PG$(2, 256)$ which contains 31 non-self dual maximal subarcs of degree 16.
Our final family of maximal arcs have degree 2^{m+2} and live in $PG(2, 2^{4m+2})$.

Theorem 3.3 *Let $\epsilon \in \mathrm{GF}(2^{4m+2})$ be an element of trace 1 over* GF(2), *and let $C = \{x \mid x^2+x+c = 0,\ c \in G_{2^{2m+1}},\ x \in \mathrm{GF}(2^{4m+2})\}$. If $q(\lambda) = \lambda^3+\epsilon$ then the conics $\mathcal{F} = \{F_{1,q(\lambda),\lambda} \mid \lambda \in C\}$ generate a maximal arc in* PG$(2, 2^{4m+2})$ *of degree 2^{2m+2}.*

Proof. To satisfy the condition of Theorem 2.5

$$Tr_{2^{4m+2}/2}(x^3 + \epsilon) = Tr_{2^{4m+2}/2}(x^3) + 1 = 1,$$

we must show that in GF(2^{4m+2}) the trace of x^3 over GF(2) is 0 for all $x \in C$. If $x^2 + x = 0$ then $x = 0, 1$ and so $x^3 \in \{0, 1\}$. If $x^2 + x = 1$ then $x^3 = x(x + 1) = 1$. But both 0 and 1 have zero trace in GF(q^2), where we use $q = 2^{2m+1}$ to simplify notation. Let $c = w^{i(q+1)}$ be any other element of G_q. Since $x^2 + x = c$ we have $x^3 = xc + x + c$. Using trace transitivity we obtain

$$Tr_{q^2/2}(x^3) = Tr_{q^2/2}(xc + x) = Tr_{q/2}(Tr_{q^2/q}(xc + x)) = Tr_{q/2}((1 + c)(x + x^q)).$$

Write $x+x^q = c+(c^2+c^4)+\cdots+(c^{2^{2m-1}}+c^{2^{2m}})$ and $(x+x^q)c = c^2+\sum_{j=1}^{2m} c^{2^j+1}$. Then since $Tr_{q/2}(x + x^2) = 0$ we obtain

$$Tr_{q/2}((1 + c)(x + x^q)) = Tr_{q/2}(\sum_{j=1}^{2m} c^{2^j+1}) = Tr_{q/2}(\sum_{j=1}^{m}(c^{2^j+1} + c^{2^{2m-j+1}+1})).$$

Since $(c^{2^{2m-j+1}+1})^{2^j} = c^{2^j+1+q-1} = c^{2^j+1}w^{i(q^2-1)} = c^{2^j+1}$ it follows from the definition of $Tr_{q/2}$ after some rearrangement of terms that

$$Tr_{q/2}(c^{2^{2m-j+1}+1}) = Tr_{q/2}(c^{2^j+1})$$

which implies that $Tr_{q^2/2}(x^3) = 0$. □

We note that the maximal arcs of degree 16 in PG$(2,64)$ constructed via Theorems 3.2 and 3.3 are identical.

4 Maximal arcs in small planes

In this section we present results of an exhaustive computer search for maximal arcs based on pencils of conics in Desarguesian planes of order 16, 32 and 64. To distinguish the arcs we have utilized two invariants which also can be used to find collineation stabilizers.

For small maximal arcs $\mathcal{K}$ we search for "good" quadrangles in $\mathcal{K}$, such that the three extra points of a Fano plane containing the quadrangle lie on a secant to $\mathcal{K}$. The numbers of such good quadrangles incident with arc points form an excellent vector invariant which has been used before for hyperovals [13]. Various refinements, such as counting Fano subplanes contained in $\mathcal{K}$, etc., may lead to faster algorithms in special cases.

For large degree arcs we employ invariants based on the structure of subarcs in a pencil. If $\mathcal{K}$ is of degree 2^d then it contains $(2^d-1)(2^{d-1}-1)/3$ maximal subarcs of degree 4 which by Lemma 2.1 are all of Denniston type and correspond to lines in the associated PG$(d-1,2)$ on conics. The number of different types of subarcs incident with a conic is a strong vector invariant. For example, a 16-degree maximal arc in PG$(2,64)$ contains 35 degree 4 subarcs which are of 4 types, and 15 degree 8 subarcs which belong to 38 types (see Tables 2-4).

4.1 Maximal arcs in PG$(2,16)$

The Desarguesian plane of order 16 contains two maximal arcs of degree 4 and 2 maximal arcs of degree 8. The 4-degree arcs and one of the 8-degree arcs are of Denniston type. With a primitive element in GF(16) satisfying $w^4+w=1$ these arcs have stabilizers of order $|G| = 68,\ 408,\ 16320$ and are generated by the sets $\{x^2+xy+w^3y^2+\lambda z^2 \mid \lambda \in A\}$, where A are the additive subgroups $<1,w>$, $<1,w^5>$ and $<1,w,w^2>$, respectively. The other degree 8 maximal arc is generated by

$$\{x^2+xy+(w^{11}+w^{10}\lambda+\lambda^3)y^2+\lambda z^2 \mid \lambda \in <1,w,w^2>\}.$$

Its dual is the Lunelli-Sce hyperoval with stabilizer of order $|G| = 144$.

4.2 Maximal arcs in $\mathrm{PG}(2, 32)$

The Desarguesian plane of order 32 contains one Denniston maximal arc of degree 4, 8 and 16, respectively. In GF(32) let w be a primitive element satisfying $w^{18} + w = 1$. Since 1 has unit trace in GF(32) the Denniston arcs are be generated by $\{x^2 + xy + y^2 + \lambda z^2 \mid \lambda \in A\}$, where A is $< 1, w >$, $< 1, w, w^2 >$ and $< 1, w, w^2, w^3 >$, respectively. The degree 4 and 8 arcs are duals of each other with $|G| = 68$, and the unique maximal arc of degree 16 is dual of the classical hyperoval with $|G| = 163680$.
In addition to these known maximal arcs there are 3 new maximal arcs of degree 8, their duals of degree 4, and one arc of degree 16. All 3 new 8-arcs have stabilizers of order 2 and are generated by

$$\{x^2 + xy + (w^k + w^l\lambda + w^m\lambda^3)y^2 + \lambda z^2 \mid \lambda \in < 1, w, w^9 >\},$$

with exponents $(k, l, m) = (12, 15, 4)$, $(5, 25, 14)$ and (6,19,8), respectively. The set of 15 conics

$$\{p(\lambda)x^2 + xy + q(\lambda)y^2 + \lambda z^2 \mid \lambda \in < 1, w, w^7, w^9 >\},$$

$$p(\lambda) = w^{25} + w^{16}\lambda + w^{10}\lambda^3 + w^{30}\lambda^7,\ q(\lambda) = w^{27} + w^5\lambda + w^{11}\lambda^3 + w^3\lambda^7,$$

generates a maximal arc of degree 16 which is the dual of Cherowitzo's hyperoval (see [12]) with stabilizer of order $|G| = 5$.
A randomized search has been carried out in $\mathrm{PG}(2, 32)$ for maximal arcs of degree 4. The only solutions found were dual to the 3 new degree 8 arcs and the Denniston arc with frequencies proportional to reciprocals of their stabilizer orders. It is therefore almost certain that except for hyperovals and their duals no other maximal arcs exist in $\mathrm{PG}(2, 32)$.

4.3 Maximal arcs in $\mathrm{PG}(2, 64)$

In the Desarguesian plane of order 64 there are 4 non-isomorphic Denniston maximal arcs of degree 4 and 7 of degree 8. Apart from their duals (and the known hyperovals) the only other previously known maximal arcs are two degree 8 arcs related to Tits ovoids (see Construction 2 of Thas in Section 1). Note that the 7 Denniston and 2 Thas degree 8 maximal arcs are all self-dual.
We have constructed 31 new non self-dual maximal arcs of degree 8 from pencils of conics and 87 new arcs of degree 16 most of which have trivial

No	(k,l,m)	(a,b,c)	$\|G\|$	D_1	D_2	D_3	D_4	E
1	(5, ,)	(6,15,24)	5460	0	7	0	0	1
2	(5, ,)	(0, 1, 4)	390	0	1	0	6	2
3	(5, ,)	(0, 1,22)	390	1	0	3	3	4
4	(5, ,)	(0, 1,19)	260	0	1	4	2	5
5	(5, ,)	(0, 1, 8)	260	1	1	1	4	6
6	(5, ,)	(0, 1, 9)	130	0	1	2	4	4
7	(5, ,)	(0, 1, 2)	130	0	0	2	5	3
8	(10,23,60)	(0, 1,14)	2	0	1	2	4	44
9	(51,54,28)	(0, 1, 4)	2	0	1	0	6	37
10	(34,21,58)	(0, 1, 9)	2	0	1	2	4	38
11	(30,35, 9)	(0, 1,21)	4	1	1	1	4	24
12	(42, 5,42)	(0, 1,19)	4	0	1	4	2	24
13	(5, , 0)	(0, 9,18)	12	0	7	0	0	11
14	(40, 0,45)	(1, 8, 9)	4	0	1	4	2	23
15	(59,60,34)	(0, 1, 5)	2	0	0	2	5	39
16	(47,45,19)	(0, 1, 3)	2	0	0	2	5	40
17	(62,45,27)	(1, 8,38)	4	0	1	4	2	25
18	(39,55,29)	(0, 1,19)	2	0	1	4	2	39
19	(10,23,60)	(0, 1, 3)	2	0	0	2	5	34
20	(50,33, 7)	(0, 1,21)	2	1	1	1	4	37
21	(15,22,59)	(0, 1,37)	2	0	1	2	4	43
22	(60, 1,38)	(0, 1,14)	2	0	1	2	4	40
23	(20,47,21)	(0, 1, 3)	2	0	0	2	5	42
24	(17,26, 0)	(0, 1,22)	2	1	0	3	3	43
25	(39,55,29)	(0, 1,22)	2	1	0	3	3	38
26	(47,45,19)	(0, 1,21)	2	1	1	1	4	39
27	(17, 2, 1)	(0,28,30)	2	0	1	2	4	47
28	(22, 8,45)	(0, 1, 8)	4	1	1	1	4	26
29	(10,23,60)	(0, 1,22)	6	1	0	3	3	16
30	(61,43,17)	(0, 1,17)	2	0	0	2	5	36
31	(34,21,58)	(0, 1,10)	2	0	1	0	6	36
32	(43,10,47)	(0, 1,17)	2	0	0	2	5	35
33	(25,20,57)	(0, 1,37)	2	0	1	2	4	40
34	(17,26, 0)	(0, 1,15)	2	0	1	2	4	42
35	(37,27, 1)	(0, 1, 3)	2	0	0	2	5	45
36	(15,46,42)	(0, 2,19)	2	0	1	4	2	40
37	(51, 3, 6)	(0, 1,10)	6	0	1	0	6	19
38	(60,36,18)	(0, 2,16)	4	1	1	1	4	23

Table 1: *Conics and invariants for maximal arcs of degree 8 in* PG(2, 64)

stabilizers. This brings the totals of known degree 4 and 8 maximal arcs in PG(2, 64) to 91 and 71, respectively.
As in Section 3, let w be a primitive element of GF(64) satisfying $w^6+w=1$. Denote by D_1, D_2, D_3, D_4 the 4 Denniston maximal arcs of degree 4 generated by $\{x^2+xy+w^5y^2+\lambda z^2 \mid \lambda \in A\}$, $A =< 1, w^{21} >, < 1, w^9 >, < 1, w^7 >, < 1, w >$ with stabilizers of order $|G|$=2340, 780, 260, 130, respectively.
In Table 1 we list exponents (k, l, m) of the coefficients of $q(\lambda)$ and exponents (a, b, c) of the generators of an additive subgroup A of GF(64) (where an empty space corresponds to a zero coefficient) for the Denniston (No.1-7) and new (No.8-38) maximal arcs of degree 8 generated by

$$\{x^2+xy+(w^k+w^l\lambda+w^m\lambda^3)y^2+\lambda z^2 \mid \lambda \in< w^a, w^b, w^c >\}.$$

No	k'	(k,l,m,n)	(a,b,c,d)	$\|G\|$	8-subarcs
1	0	(5, , ,)	(0, 1, 8,27)	2340	00609000000000000000000000000000000000
2	0	(5, , ,)	(0, 1, 4,23)	780	12033600000000000000000000000000000000
3	0	(5, , ,)	(0, 1, 2,19)	260	00241440000000000000000000000000000000
4	0	(5, , ,)	(0, 1, 2,12)	130	02112450000000000000000000000000000000
5	14	(15,41, 7,)	(14,22, 1,15)	12	00000000003000000000000060003000000003
6	14	(15,41, 7,)	(14,22,44,16)	12	00000000000013000000000000030000060020
7	0	(10,23,60,)	(0, 1,13,50)	4	00000002000100001020002000012000200200
8	0	(55,18, 9,45)	(0, 1, 8,52)	4	00001002222111001000000000000000200000
9	0	(25,11,38,42)	(0, 1, 9,10)	2	00000102000000000010000011010321010100
10	0	(62,35,24,38)	(0, 1,19,40)	4	00010000000110001000220000210020000002
11	0	(23,25,34,18)	(0, 1,13,53)	2	00100002100000120111111001000000000010
12	0	(31,12,54,54)	(0, 1,29,52)	2	00000011100000011001111010110000002010
13	0	(25,52,18, 9)	(0, 1, 5,44)	4	00010002000001200200000200010000022000
14	0	(11,44,57, 4)	(0, 1,59, 9)	4	00001000020000004000002020200002000000

Table 2: *Conics and invariants for maximal 16-arcs in* PG$(2,64)$*, part 1.*

For each arc we list the order $|G|$ of its stabilizer, the number N_i of Denniston 4-subarcs D_i, $i=1,\cdots,4$ and the number E of non-isomorphic degree 16 arcs it extends to. The maximal arc from Theorem 3.1 is arc No.13 in Table 1. We note that duals of the new 8-arcs do not contain any subarcs.
In Tables 2-4 we list exponents (k',l',m',n'), (k,l,m,n) of the coefficients of $p(\lambda)$, $q(\lambda)$ and exponents (a,b,c,d) of the generators of A in GF(64), respectively, for the 94 maximal arcs of degree 16 in PG$(2,64)$ generated by the conics

$$\{p(\lambda)x^2+xy+q(\lambda)y^2+\lambda z^2 \mid \lambda \in < w^a, w^b, w^c, w^d >\},$$

$$p(\lambda)=w^{k'}+w^{l'}\lambda+w^{m'}\lambda^3+w^{n'}\lambda^7, \quad q(\lambda)=w^k+w^l\lambda+w^m\lambda^3+w^n\lambda^7.$$

We also list stabilizer orders $|G|$ and distributions of 8-subarcs, where the k-th component contains the number of arcs No.k. The maximal arc from Theorem 3.2 is No.6 in Table 2. As it is seen from the tables, each arc has a unique "signature" of 8-subarcs making it a perfect invariant. We note that none of the 80 16-arcs in Tables 3 and 4 contains a Denniston 8-subarc and that all 80 arcs and their duals of degree 4 have trivial stabilizers in PG$(2,64)$. The distribution of Denniston 4-subarcs can be calculated from the distribution of 8-subarcs. It is found that a maximal 16-arc in PG$(2,64)$ contains either 1 or 5 D_1's, 3 or 7 D_2's, 9 or 13 D_3's and 18 or 22 D_4's with many arcs having the same distribution. Also note that each of the 38 maximal arcs of degree 8 is a subarc of some maximal arc of degree 16. Arcs No.24, 36, 38, 39 and 69 are completely "heterogeneous" since they contain 15 non-isomorphic subarcs of degree 8.

No	(k',l',m',n')	(k,l,m,n)	(a,b,c,d)	$\|G\|$	8-subarcs(first 7 0's omitted)
15	(41,18,24, 1)	(18,34,55,31)	(0, 1,13,51)	1	01210001001100120110000101010000
16	(20,49,55,32)	(53,56, 5,22)	(0, 1,13,10)	1	00201000300100200010010200010001
17	(46,20,26, 3)	(12,37,39,58)	(0, 1,13,44)	1	01000001000212010101000011001111
18	(61, 0, 6,46)	(23,25,34,18)	(0, 1,13,19)	1	20101011120100001011011000000000
19	(41,18,24, 1)	(12,37,39,58)	(0, 1,13,22)	1	10000000122111011100000012000000
20	(51,43,49,26)	(11,53,20,52)	(0, 1,13,47)	1	11001000000211020001110110100000
21	(42,11,17,57)	(13, 5,25,15)	(0, 1,13,58)	1	00100020202100001011010001110000
22	(24,57, 0,40)	(5,54,29,40)	(0, 1,13,45)	1	20010000001201010111000200020000
23	(54,26,32, 9)	(8,45,53,56)	(0, 1,13,43)	1	00100000003111101000012001011000
24	(16,23,29, 6)	(15, 3,45,45)	(0, 1,13, 9)	1	10010001100100101000001101111111
25	(29,50,56,33)	(26,21,18,16)	(0, 1,13, 7)	1	01200001110201001020000111000000
26	(55,30,36,13)	(47, 2,48,27)	(0, 1,13,36)	1	10210020002101012000001010000000
27	(19,46,52,29)	(34,60,32,50)	(0, 1,13,33)	1	00100001101100111001101201000001
28	(21,32,38,15)	(9,32,49,51)	(0, 1,13,28)	1	10001001011201002000100010201000
29	(44,27,33,10)	(18,34,55,31)	(0, 1,13,10)	1	11000001100210101011000100111000
30	(37,11,44,11)	(46,57, ,37)	(0, 1,29,46)	1	11100001010020011001000111020000
31	(11,55,25,55)	(6,25,61,10)	(0, 1,29,16)	1	00210100010022200000000200002000
32	(41,58,28,58)	(56,41,44,18)	(0, 1,29,38)	1	01000000000210100100002002111110
33	(41,58,28,58)	(16, 7,62,14)	(0, 1,29,47)	1	00102001001011101100000031001000
34	(44, 4,37, 4)	(34,39,36,34)	(0, 1,29,52)	1	01000000101013000100001111020010
35	(31, 1,34, 1)	(61,15, 6,42)	(0, 1,29,43)	1	00020000000040002420000000000010
36	(34,61,31,61)	(0,26,30,52)	(0, 1,29,54)	1	11000011000011010000111011110100
37	(35,43,13,43)	(11,23,24,21)	(0, 1,29,22)	1	00000000000040002220220000000010
38	(21, 9,42, 9)	(22,62,26,30)	(0, 1,29,31)	1	10101010101011001100001011011000
39	(18,57,27,57)	(35,10,60,25)	(0, 1,29,20)	1	01100000100011110101001010011110
40	(48,38, 8,38)	(46,57, ,37)	(0, 1,29,45)	1	02200011200221001001000000000000
41	(10,28,61,28)	(40,46,41, 5)	(0, 1,29,33)	1	20000100010020000002100200202000
42	(61,40,10,40)	(19,55,37,20)	(0, 1,29,46)	1	01300001000010001011001130001000
43	(21, 9,42, 9)	(26,51,22,60)	(0, 1,29,61)	1	00000002000010111011000221101000
44	(46,60,30,60)	(22,62,26,30)	(0, 1,29, 5)	1	10110001110110010002010000010200
45	(39,33,39,16)	(35,54, 8,10)	(0, 1,35,26)	1	01210001101101110010011000000100
46	(46,20,26, 3)	(7, 8,62,35)	(0, 1,35,44)	1	00000010100100011001001201210020
47	(12,55,61,38)	(13,59,61,17)	(0, 1,35,59)	1	00000000102002221010011000002000
48	(14,42,48,25)	(60, 0,39,55)	(0, 1,35,41)	1	10001011201011001001010000111000
49	(36,44,50,27)	(17,30,11,36)	(0, 1,35,24)	1	01110000210110110100000110200000
50	(57,48,54,31)	(27,16, ,59)	(0, 1,35,46)	1	01000002200001100111101001001100
51	(24,57, 0,40)	(49, 6,10,32)	(0, 1,35,45)	1	01001001100110000101101101210000
52	(2, 2, 8,48)	(19,44, 0,33)	(0, 1,35,57)	1	10111011200000100200000000112000
53	(31,24,30, 7)	(37,21,21, 8)	(0, 1,35,39)	1	00201000111100011111000000111000
54	(37,34,40,17)	(6, 3,44, 7)	(0, 1,35,29)	1	11000000101011001011002111010000

Table 3: *Conics and invariants for maximal 16-arcs in* PG$(2,64)$*, part 2.*

5 Conclusions and open problems

In Section 3 we have used certain polynomials $\{p,q\}$ to map additive subgroups A of GF$(2,2^m)$ to subsets of conics on a common nucleus corresponding to maximal arcs in PG$(2,2^m)$. Since a $\{p,q\}$-map is hereditary, subgroups of A are mapped to maximal subarcs of the arc generated by A, inducing a projective space on its conics. A constant $\{p,q\}$-map generates the well known Denniston maximal arcs.

No	(k',l',m',n')	(k,l,m,n)	(a,b,c,d)	$\|G\|$	8-subarcs(first 7 0's omitted)
55	(41,18,24, 1)	(9,14,32,31)	(0, 1,35,27)	1	0110000011000003100120011110000
56	(60,54,34,35)	(42, 9,53,15)	(0, 1,40, 9)	1	2200100100010010001001200101001
57	(55, 2,45,46)	(50,48,11,42)	(0, 1,40,10)	1	0220110000200020002000000200001
58	(42,46,26,27)	(16,52,39, 4)	(0, 1,40,15)	1	0010002000000002101001001121200
59	(11,50,30,31)	(50,48,11,42)	(0, 1,40, 7)	1	1001000212100011200100000100100
60	(17, 9,52,53)	(38,32,42, 3)	(0, 1,40,30)	1	1011110000100100002000020210100
61	(26,32,12,13)	(52,28,30,49)	(0, 1,40,27)	1	0010000210200000111100100110200
62	(50,47,27,28)	(38,32,42, 3)	(0, 1,40, 2)	1	0010000300000000111100001110400
63	(28, 3,46,47)	(2, 7,35,23)	(0, 1,40,14)	1	1000100000000101102000220112000
64	(47,42,22,23)	(12,34,51,53)	(0, 1,40,16)	1	1100000011000010102000311110000
65	(8,10,53,54)	(31,54,38,32)	(0, 1,40,21)	1	0000000010100021011110012101010
66	(7,51,31,32)	(3,14, 0,59)	(0, 1,40,36)	1	1002100000000101100000211111010
67	(42,56,45,59)	(0,58,11,59)	(0, 1,56,14)	1	1010100011200101020110000002000
68	(31, 6,58, 9)	(12,16,34,33)	(0, 1,56,35)	1	1001100000112001101000110002010 0
69	(41, 0,52, 3)	(19,11,12, 9)	(0, 1,56,54)	1	1000101110101011110100000011100
70	(9,17, 6,20)	(21,51,51,38)	(0, 1,56,35)	1	0000100201000111020100100100201
71	(43,11, 0,14)	(19,11,12, 9)	(0, 1,56,23)	1	1010100101200010020000001102001
72	(49,10,62,13)	(35,46,18,36)	(0, 1,56,61)	1	0010110000101200001100010110111
73	(42, 2,16,37)	(2,34,13,13)	(0, 1, 5, 9)	1	1210000200010001011010000111100
74	(24,48,62,20)	(50,30,53,48)	(0, 1, 5,49)	1	1101000200100110011100012001000
75	(58,38,52,10)	(10, 0,16, 1)	(0, 1, 5,34)	1	0201000110100101011100101020000
76	(1,50, 1,22)	(29,17,25,28)	(0, 1, 5,47)	1	0200000200000111011110101010100
77	(61,54, 5,26)	(60,35,41,18)	(0, 1, 5,15)	1	0210000300100000100110001021001
78	(26, 9,34,42)	(8,41,12,50)	(0, 1,10,13)	1	2001000001000012100110021002000
79	(60,31,56, 1)	(37,38, 8,38)	(0, 1,10,20)	1	1110000010011111010000010002101
80	(53,53,15,23)	(31,58,61,35)	(0, 1,10,21)	1	0121000120100000011100110011000
81	(61,12,37,45)	(42, ,48,54)	(0, 1,10,12)	1	1000000010110100100010111201011
82	(27,20,45,53)	(47,43,44,41)	(0, 1,10,62)	1	2010001120000000102100120001000
83	(48,42, 5,36)	(40,18,60,60)	(0, 1,22,11)	1	0000000000131110110100000011300
84	(26,56,17,58)	(59,13, 2,16)	(0, 1,57,42)	1	1001001110001100010100012011010
85	(39,10,43,10)	(4,36,21,41)	(0, 1,29, 3)	1	0201001010011032010000100010000
86	(59,50,20,50)	(34,10,18,21)	(0, 1,29,46)	1	0101000000110021100200201000011
87	(38,16,49,16)	(36,42,59,61)	(0, 1,29,23)	1	2201011000101110001100000010100
88	(45,62, 5,45)	(62, 1,54,14)	(0, 1,35,34)	1	1000000100000201011110012001110
89	(17,37,43,20)	(30,51,40,54)	(0, 1,35,47)	1	0200001010000121001001200011001
90	(47,42,22,23)	(36,51,28, 0)	(0, 1,40,54)	1	1200000000000120110110020000210 0
91	(50,49,53,12)	(53,30,31,28)	(0, 1,17,20)	1	1000001001101100110000201111100
92	(17,14,47,14)	(40,35,36,33)	(0, 1,42,51)	1	0220011000101110001110000010100
93	(19, 3,30,34)	(10,43,26, 5)	(0, 1,38,27)	1	0010011000002110000010010300210
94	(50,12,18,58)	(1,22,15,58)	(0, 1,35,19)	1	1000001101100011200100100011101

Table 4: *Conics and invariants for maximal 16-arcs in* PG$(2,64)$*, part 3.*

Based on the behavior of $\{p,q\}$-maps in PG$(2,2^m)$ for $4 \leq m \leq 8$, we conjecture that the number of maximal arcs in PG$(2,2^m)$ is growing exponentially with m. In PG$(2,128)$ we have generated thousands of degree 4 and 8 maximal arcs by extending the 3 Denniston maximal arcs of degree 4.

We conclude by posing a few question concerning the new family of maximal arcs.

1. Which of the new maximal arcs of degree q in PG$(2,q^2)$ are linear (i.e. arise from a Thas like construction in higher dimensional projective spaces)? It is known [7] that all Denniston maximal arcs are linear.

2. Is it true that if both an arc and its dual is generated by a $\{p,q\}$- map

then it is a Denniston maximal arc?

3. What is the largest d of a non-Denniston arc of degree d generated by a $\{p, q\}$-map in $\mathrm{PG}(2, 2^m)$? If $d = 2^{m-1}$ (as in $\mathrm{PG}(2, 16)$ and $\mathrm{PG}(2, 32)$) what hyperovals are generated by their duals?

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