Filter Banks and Basis Functions

Filter Banks: Linear filtering of $\vec{s}$ is provided by

$$\vec{a} = F^T \vec{s}, \quad \text{with } F \in \Re^{n^2 \times m}.$$

Here:

- The sampled $n \times n$ image $s(\vec{x})$ is taken to be equivalent to the n^2-dimensional column vector, $\vec{s}$, with the elements arranged in the standard raster order.
- One column of F, say $\vec{f_k}$, corresponds to one location for some filter kernel.
- The k^{th} element, a_k, is the *filter response*, $\vec{f_k} \cdot \vec{s}$.
- m is the number of filter response samples. Typically $m \geq n^2$.

Basis Functions: An expansion of $\vec{s}$ in the basis $\{b_k\}_{k=1}^m$ is given by

$$\vec{s} = B\vec{a}, \quad \text{with } B \in \Re^{n^2 \times m}.$$

- The k^{th} column of B is the *basis function (or vector)* $\vec{b}_k$.
- Typically $m \geq n^2$, although 'view-based' models use $m < n^2$.

Nature's Choice

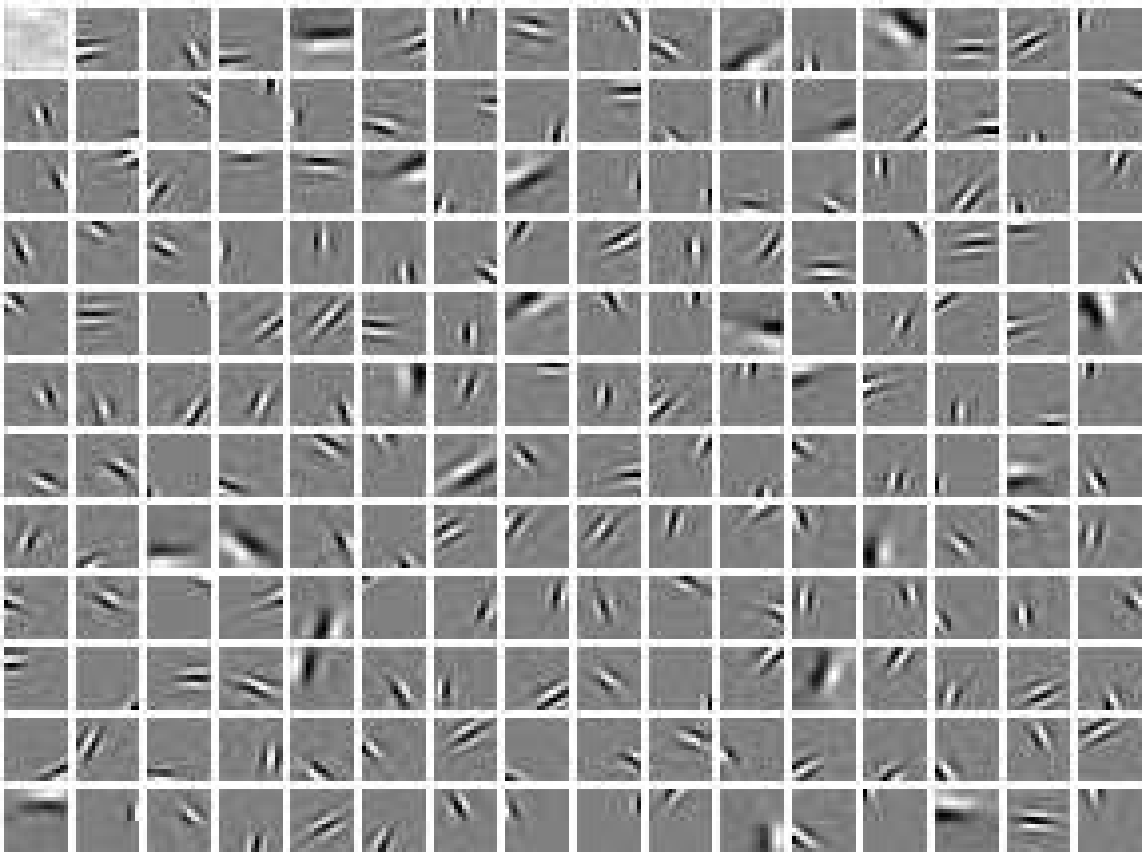
Filter bank:

- Self-similar copies of a small set of 'mother wavelets'.
- Filters of various scales, positions, and orientations.
- Mother wavelets are roughly the real/imaginary part of a Gabor filter:

$$Gabox(\vec{x}; \vec{k}, \sigma, \phi) = e^{i(\vec{k}\cdot\vec{x}+\phi)} e^{-(x^2+y^2)/(2\sigma^2)}$$

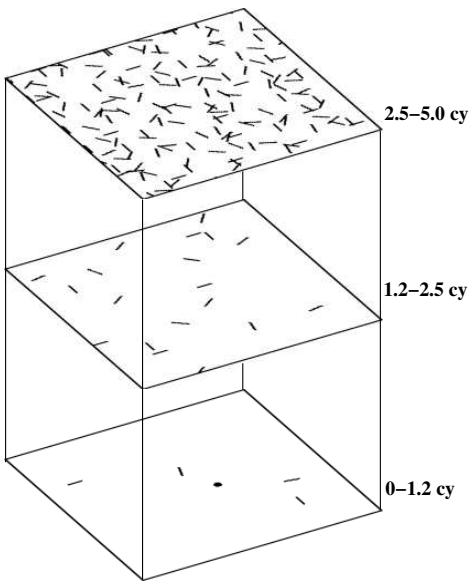
  i.e. a sinsusoidal grating times a Gaussian window.
- The grating has approximately 1.5 cycles across the 2σ window (the filter has roughly a one octave bandwidth).
- An independent component analysis of the ensemble of natural images leads to the above properties [Olshausen and Field, '96].
- Filters appear in quadrature pairs (roughly, even and odd symmetries, but see below).

a.

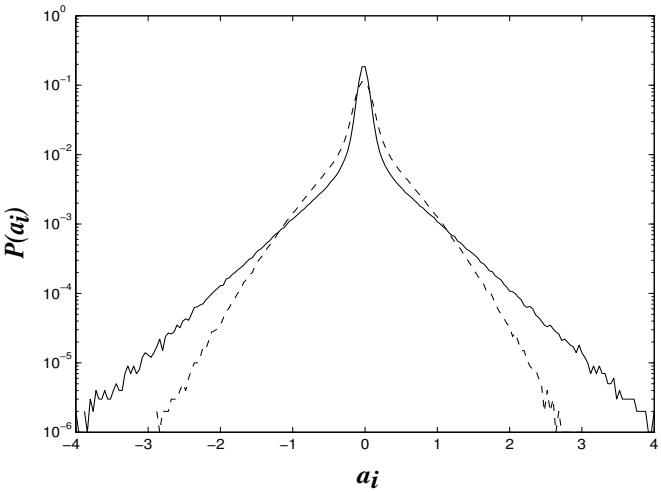


b.

c.



d.



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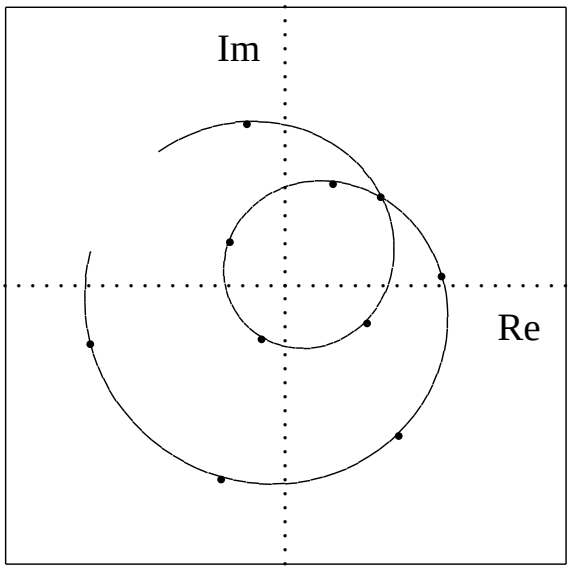
Quadrature Pairs

Filtering: Let $g(x) = G * I(x)$ and $h(x) = H * I(x)$, with $G(x)$ and $H(x)$ a quadrature pair (i.e. $\mathcal{G}(k) = \text{sign}(k)\mathcal{H}(k)$).



Complex Polar Representation:

$$\begin{aligned}(G(x) + iH(x)) * I =& g(x) + ih(x) \\ =& e^{i\phi(x)}A(x).\end{aligned}$$

For a bandpass $G(x)$ and $H(x)$ which form a quadrature pair:

- **Local Energy:** $A^2(x)$ is a low-pass (slowly varying) signal.
- **Local Phase:** $\phi(x) \approx \phi_0 + k_0(x - x_0)$ provides a roughly linear progression.

Example: G2 and H2

An approximate quadrature pair is given by [Freeman et al, '91]:

$$G2(x, y; \sigma) = [0.9213((x/\sigma)^2 - 1)]e^{-(x^2+y^2)/(2\sigma^2)},$$

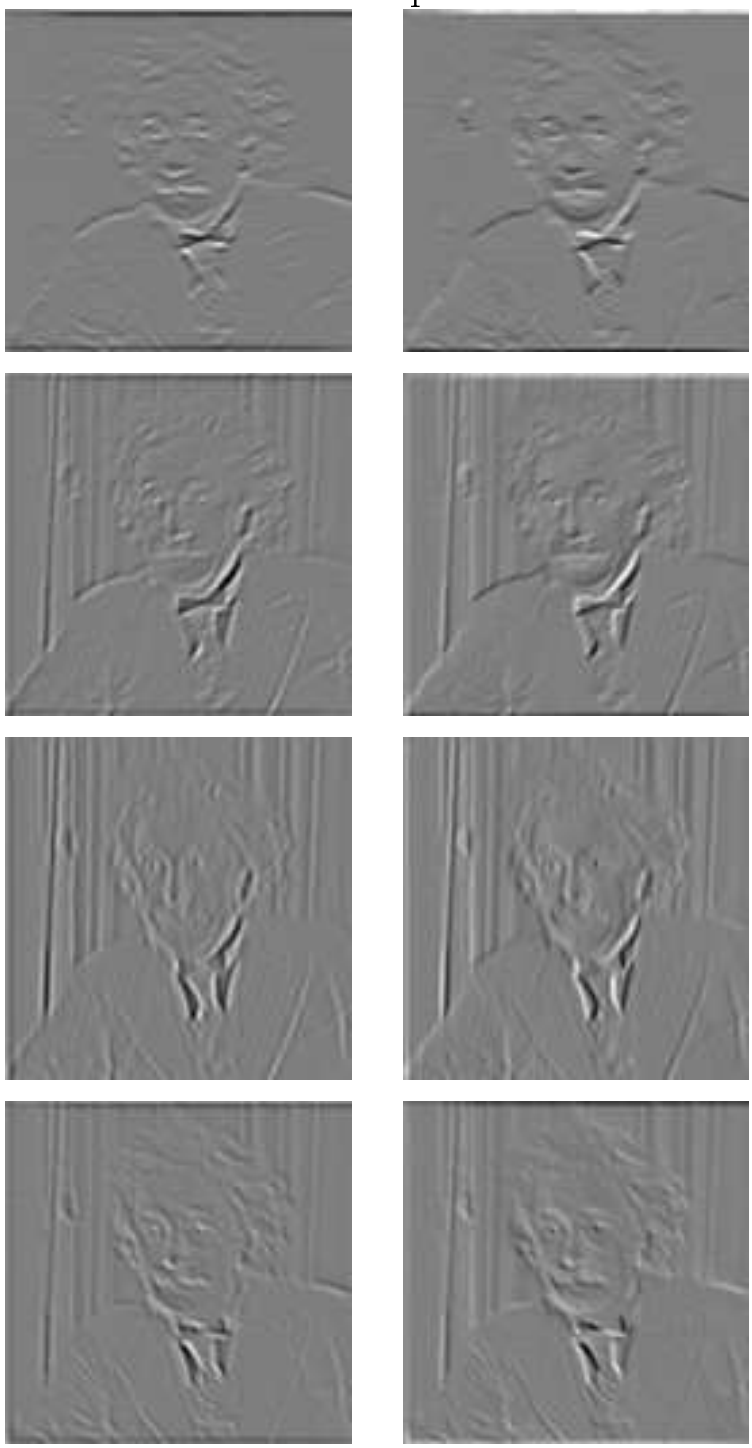
$$H2(x, y; \sigma) = [(x/\sigma)(0.3458(x/\sigma)^2 - 1.559)]e^{-(x^2+y^2)/(2\sigma^2)},$$

Here $G2(x, y; \sigma) \propto \frac{\partial^2}{\partial x^2} e^{-(x^2+y^2)/(2\sigma^2)}$.

An filter bank using G2 and H2.

- Define the scale using the wavelength, λ, for the most sensitive frequency in the fourier transform. Use $\sigma = \lambda/4.2$
- Use 4 scales, with wavelengths (say) $\lambda = 4, 8, 16$, and 32 pixels.
- Spatially rotate $G2(x, y)$ and $H2(x, y)$ by 0, 45, 90, 135 degrees.
- Sample each pair at the rate $\lambda/4$ in each direction.
- Efficient implementation uses x, y-separable filters.
- The orientation of the filters is continuously steerable from 7 real-valued basis filters (or 4 sampled orientations).

G2 $\lambda = 8.0$ pixels H2



128×128 images

G2 $\lambda = 16.0$ pixels H2

64×64 images

Amp $\lambda = 4.0$ pixels Phase

$\lambda = 8.0$ pixels

$\lambda = 16.0$ pixels

$\lambda = 32.0$ pixels

References:

1. C.S. Burrus, R.A. Gopinath, and H. Guo, Introduction to Wavelets and Wavelet Transforms, Prentice Hall.

2. W T Freeman and E H Adelson. The design and use of steerable filters. IEEE Pat. Anal. Mach. Intell., 13(9), 1991, pp.891-906.

3. Olshausen BA, Field DJ (1996). Emergence of simple-cell receptive field properties by learning a sparse code for natural images. Nature, 381: 607-609.
   `URL: http:///redwood.ucdavis.edu/bruno/papers.html`

4. E P Simoncelli, W T Freeman, E H Adelson, and D J Heeger. Shiftable multi-scale transforms, IEEE Trans. Information Theory, 38(2), 1992, pp.587-607.