On the relation between Interpreted Systems and Kripke Models

A. Lomuscio
Email: A.R.Lomuscio@cs.bham.ac.uk
Phone: +44 121 4143734
School of Computer Science
University of Birmingham
Birmingham B15 2TT, U.K.

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Abstract

We compare a simplified notion of *interpreted systems* and *Kripke models* as semantic structures for reasoning about knowledge. We find that interpreted systems are more specific than Kripke models and satisfy the formulas $\phi \Leftrightarrow D_A\phi$ and $\neg K_i \neg K_j \phi \Rightarrow K_j \neg K_i \neg \phi$, for any $i \neq j$. We show that frames generated by interpreted systems satisfy some properties which are not widely known. Our method is to define a map from the class of systems to the class of Kripke frames, and another in the opposite direction, and study their properties and compositions. We also prove completeness for an extension of $S5$ with respect to the images of the systems.

1 Introduction

Interpreted systems have first been proposed by Fagin, Halpern, Moses and Vardi [HF85] to model distributed systems. The growing interest in Multi-Agent Systems (MAS) and their specifications has brought forward the concept of interpreted system as useful formal tool to model key characteristics of the agents, such as the evolution of their knowledge, communication, etc. This work has culminated in the publication of [FHMV95] in which the authors use the notion of interpreted system to explore systematically fundamental classes of MAS (such as synchronous, asynchronous, with perfect recall ability, etc.) by the use of interpreted systems.

As is well known, these issues have been for a long time objects of study among modal logicians as well, using *Kripke models*. Over the last 40 years, many techniques have been developed, such as completeness proofs via canonical models, decidability via the finite model property [HC96], and more recently, techniques for combining logics [KW91, Gab96].

It seems to be interesting to bridge the gap between the two approaches. On one hand, this would clarify the extent to which the heritage of modal logics techniques can be applied to MAS defined on interpreted systems; on the other hand it might allow to define key MAS (such as MAS with perfect recall) in terms of Kripke models.

It is intuitively clear that interpreted systems represent a special class of Kripke models. Our goal is to make this intuition formal, and to clarify the relationship between the two semantic structures.

We will operate in a simplified setting: we do not consider the environment and we restrict our attention to the static case (i.e. no temporal evolution). Since we do not consider runs over global states, in the following we use the term 'system' to identify what in [FHMV95] is called 'set of global states'.

Preliminaries. We assume a modal propositional language, defined in the usual way from a set of propositional variables by the use of classical connectives, the operators K_i and D_B. The index i varies over a set $A = \{1, \ldots, n\}$, representing the agents of the system and B varies over subsets of A. The modal operator K_i represents the knowledge of the agent i, while D_B represents the distributed knowledge among the group B.

If W is a set, id_W is the identity relation on W. If $\sim$ is an equivalence relation on W and $w \in W$, then $W/\sim$ is the set of equivalence classes, and $[w]_\sim$ is the equivalence class containing w.

2 Mappings between systems and frames

Although systems are intuitively a special class of Kripke frames, it is clear that they are not simply a subset. In order to clarify the relationship, we have to use the construction given implicitly in [FHMV95] for obtaining a frame from a system. Our framework will be the following:

- We define the class of systems $\mathcal{S}$, and the class of Kripke frames $\mathcal{F}$.
- We define two maps, $\mathcal{S} \xrightarrow{f} \mathcal{F}$ (based on [FHMV95]) and $\mathcal{F} \xrightarrow{g} \mathcal{S}$.

We prove that f does not generate the whole class of frames, even up to the natural notion of isomorphism. We also show that the frame images of all systems validate the schema $\phi \Leftrightarrow D_A\phi$, and other interesting formulas.

Systems and frames are always defined over a set A of n agents, which we assume as given.

Definition 2.1 (Systems) *A set of global states, or system, is a Cartesian product $S = L_1 \times \cdots \times L_n$, where $L_1, \ldots, L_n$ are non-empty sets. The set L_i represents the local states possible for agent i. Elements of a local state L will be indicated with $l_1, l_2, \ldots$ The class of systems is denoted by $\mathcal{S}$.*

Definition 2.2 (Frames) *A frame is a tuple $F = (W, \sim_1, \ldots, \sim_n)$, where W is a non-empty set and for every i in A, $\sim_i$ is an equivalence relation over W^2. Elements of W are called worlds and are denoted as: $w_1, w_2, \ldots$ $\mathcal{F}$ denotes the class of frames.*

Every system generates ([FHMV95]) a frame:

Definition 2.3 (Systems to frames) *$f : \mathcal{S} \rightarrow \mathcal{F}$ is the function that maps a system S onto a Kripke frame in the following way:*
If $S = L_1 \times \cdots \times L_n$, $f(S) = (L_1 \times \cdots \times L_n, \sim_1, \ldots, \sim_n)$, where $\sim_i$ is defined as: $(l_1, \ldots, l_n) \sim_i (l'_1, \ldots, l'_n)$ if and only if $l_i = l'_i$.

Lemma 2.1 *If S is a system, and $f(S) = (W, \sim_1, \ldots, \sim_n)$ is the frame defined from it by Definition 2.3, then*

1. $\bigcap_{i\in A} \sim_i = id_W$;

2. *For any* $w_1, \dots, w_n$ *in* W *there exists a* $\overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \dots, n$.

This shows that the Kripke frames that are usually built by the means of the standard technique ([FHMV95]) from the systems are only a subset of all the possible reflexive, symmetric and transitive Kripke frames.

One of the differences between $f(\mathcal{S})$ and $\mathcal{F}$ is that some formulas that are not valid on $\mathcal{F}$ are valid on $f(\mathcal{S})$.

Lemma 2.2 *Consider a frame* $F = (W, \sim_1, \dots, \sim_n)$.
$\bigcap_{i\in A} \sim_i = id_W$ *if and only if* $F \models \phi \Leftrightarrow D_A\phi$.

Corollary 2.1 *If* S *is a system,* $f(S) \models \phi \Leftrightarrow D_A\phi$.

Condition 2. of Lemma 2.1 forces the frames generated from systems to satisfy the following formula.

Lemma 2.3 *If* S *is a system,* $f(S) \models \neg K_i \neg K_j \phi \Rightarrow K_j \neg K_i \neg \phi$, *where* $i \neq j$.

It can be argued that this formula is not desirable even in ideal models of knowledge like $S5$.

We now define a mapping in the other direction; in fact, it is also possible to generate a system from a frame:

Definition 2.4 (Frames to systems) $g : \mathcal{F} \to \mathcal{S}$ *is the function that maps a frame* $F = (W, \sim_1, \dots, \sim_n)$ *onto the system* $g(F) = W/\sim_1 \times \cdots \times W/\sim_n$.

We now introduce the relation of isomorphism on $\mathcal{F}$ and $\mathcal{S}$. Intuitively we want to reason about frames and systems that validate the same set of formulas. Many notions (such as p-morphisms for frames) may be appropriate for doing so, but for our aims we need a strong equivalence between the structures.

Consider two MAS. If we can draw a bijection between the agents of the MAS such that the local states of the corresponding agents are themselves in a bijection, then in a way we can think that one MAS can simulate the other, and so the two MAS can be thought as being equivalent. We formalise this as follows:

Definition 2.5 (Isomorphism of systems) *Two systems* $S = L_1 \times \cdots \times L_n$, $S' = L'_1 \times \cdots \times L'_n$ *are isomorphic* ($S \cong_{\mathcal{S}} S'$) *if and only if* $\mid L_i \mid = \mid L'_i \mid$ *for* $i = 1, \dots, n$.

To reason about equivalent frames we take the standard notion of isomorphism.

Definition 2.6 (Isomorphism of frames) *Two frames* $F = (W, \sim_1, \dots, \sim_n)$, $F' = (W', \sim'_1, \dots, \sim'_n)$ *are isomorphic* ($F \cong_{\mathcal{F}} F'$) *if and only if:*

- *There exists a bijection* $b : W \to W'$,

- *For all* $s, t \in W$, *and all* $i \in A$, $s \sim_i t$ *if and only if* $b(s) \sim'_i b(t)$.

Lemma 2.4 *If* $S \cong_{\mathcal{S}} S'$, *then* $f(S) \cong_{\mathcal{F}} f(S')$.

Lemma 2.5 *If* $F \cong_{\mathcal{F}} F'$, *then* $g(F) \cong_{\mathcal{S}} g(F')$.

3 Back and forth between systems and frames

We now investigate the extent to which the composition of f with g (or g with f) results in a system (frame) which is isomorphic to the one we started with.

Given a system $S = L_1 \times \cdots \times L_n$, consider the image under f of $S, f(S)$. Let $S' = (L_1 \times \cdots \times L_n)/{\sim_1} \times \cdots \times (L_1 \times \cdots \times L_n)/{\sim_n}$ be the image under g of $f(S)$. We want to investigate the relationship between S and S'.

Theorem 3.1 *For any system S in $\mathcal{S}$, $S \cong_{\mathcal{S}} g \circ f(S)$).*

In other words, if we start from a system S, build the corresponding Kripke frame $f(S)$, it is still possible to extract all the information from the frame by applying the function g that produces another system S', which is in a bijection with the original S.

We now investigate the other side of the relation. Consider a frame F and its image under g, $g(F)$. If we take the image under f of $g(F)$, that frame will satisfy the property stated by Lemma 2.1 and therefore will not in general be isomorphic to F. Note also that, by Lemma 2.2, since $\phi \Leftrightarrow D_A\phi$ will be valid on $f \circ g(F)$, this frame is not even a p-morphic image of F.

What we can prove is the following:

Lemma 3.1 *If F is a frame such that there exists a system S, with $F \cong_{\mathcal{F}} f(S)$, then $F \cong_{\mathcal{F}} f \circ g(F)$.*

If we consider a frame $F = (W, \sim_1, \ldots, \sim_n)$ such that $\bigcap_{i \in A} \sim_i = id_W$, $f \circ g(F)$ will not in general be isomorphic to F. As an example, consider $F = (\{w_1, w_2\}, \{(w_1, w_1), (w_2, w_2), (w_1, w_2), (w_2, w_1)\}, \{(w_1, w_1), (w_2, w_2)\})$. We need to restrict our attention to *both* the properties inherited from the mapping from systems.

Theorem 3.2 *If $F = (W, \sim_1, \ldots, \sim_n)$ is a frame such that:*

- $\bigcap_i \sim_i = id_W$,
- $\forall w_1, \ldots, w_n, \exists \overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$;

then $F \cong_{\mathcal{F}} f \circ g(F)$.

Theorem 3.2 and Lemma 2.1 allows us to characterise the frames that are images of some system:

Theorem 3.3 *Let $F = (W, \sim_1, \ldots, \sim_n)$ be a frame. The following are equivalent:*

1. $\bigcap_i \sim_i = id_W$ *and* $\forall w_1, \ldots, w_n, \exists \overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$;
2. *there exists an S, such that $F \cong_{\mathcal{F}} f(S)$.*

Theorem 3.3 completely characterises the frames that are commonly used to give a semantics to MAS systems described in terms of interpreted systems.

4 Axiomatisation of the images of the systems

Theorem 3.3 offers a precise characterisation of the images of the frames. Thanks to this mapping we can axiomatise the systems by looking at the axiomatisation of their equivalent counterparts in the class of Kripke frames, namely equivalence frames that satisfy properties 1. and 2. of Theorem 3.3.

In the following we shall focus on a language that does not include a modal operator for distributed knowledge. Since the operator of distributed knowledge cannot be rewritten in terms of standard knowledge operators, and property 2. of Theorem 3.3 corresponds (by Lemma 2.2) to the formula $D_A\phi \Leftrightarrow \phi$, in the following limited syntax we cannot distinguish between extensions of $S5_n$ that satisfy property 1. and the ones which do not.

We call n-directedness property 2. of the frames.

Definition 4.1 (n-Directedness) *A frame $F = (W, R_i)$ is n-Directed if and only if $\forall w_1, \ldots,$ $w_n, \exists \overline{w}$ such that $\overline{w} R_i w_i$, $i = 1, \ldots, n$.*

Aim of this section is to introduce a logic which is sound and complete with respect to directed Kripke frames whose relations are equivalence relations. Here we investigate only the case that arises when $n = 3$, but the results can be generalised to arbitrary n.

We notice that this logic has to be a proper extension of $S5$. To see this, consider the following $S5$ counter-model for $\neg K_1 \neg K_2 p \Rightarrow K_2 \neg K_1 \neg p$ (that we proved to be valid on $f(\mathcal{S})$), in which we do not indicate the reflexive relations:

$$M = (\{w_1, w_2, w_3\}, \{w_1 R_1 w_2, w_1 R_2 w_3\}, \pi(p) = \{w_1, w_2\}).$$

We also note the following important fact:

Lemma 4.1 *No modal formula corresponds to n-Directedness.*

Proof: Suppose the opposite and assume there is a formula ϕ that corresponds to n-directedness. Consider two disjoint frames, $F_1 = (W_1, \sim_i)$ and $F_2 = (W_2, \sim'_i)$, $W_1 \bigcap W_2 = \emptyset$, such that both F_1 and F_2 are n-directed. Since by assumption $F_1 \models \phi$ and $F_2 \models \phi$, then $F_1 \bigcup F_2 \models \phi$. But, then ϕ is valid on a frame, which, in general, has not n-directedness, which is opposite to what we assumed at the beginning. ■

In the following we will be using the standard modal logic syntax with $\Box_i$ representing K_i and $\Diamond_i$ representing $\neg K_i \neg$. We repeat the definitions of *directedness* and *Weak-Directedness* for the case $n = 3$.

Definition 4.2 (3-Directedness) *A frame is 3-Directed (3D) if and only if $\forall w_1, w_2, w_3$, $\exists \overline{w}$ such that $w_i R_i \overline{w}$, for $i = 1, 2, 3$.*

To study completeness, we need another to reason about another property.

Definition 4.3 (3-Weak-Directedness) *A frame is 3-Weakly-Directed (3WD) if and only if $\forall w, w_1, w_2, w_3$, such that $w R_1 w_1, w R_2 w_2, w R_3 w_3$, $\exists \overline{w_1}, \overline{w_2}$ such that $w_1 R_2 \overline{w_1}, w_2 R_3 \overline{w_1}$, $w_3 R_1 \overline{w_1}$ and $w_1 R_3 \overline{w_2}, w_2 R_1 \overline{w_2}, w_3 R_2 \overline{w_2}$.*

It is immediate to note that:

Lemma 4.2 *If a frame is 3D then it is 3WD.*

Consider the following axiom schema, that we call **3WD**:

$$(\mathbf{3WD})(\Diamond_1\Box_2 p \wedge \Diamond_2\Box_3 q \Rightarrow \Box_3\Diamond_1(p \wedge q)) \wedge (\Diamond_1\Box_3 r \wedge \Diamond_2\Box_1 s \Rightarrow \Box_3\Diamond_2(r \wedge s))$$

The axiom **3WD** corresponds to the property *3WD* of the frames:

Lemma 4.3 *$F \models (\mathbf{3WD})$ if and only if F is 3WD.*
Proof: From left to right. Suppose F satisfies **3WD**, consider four points w, w_1, w_2, w_3, and assume $wR_iw_i, i = 1, 2, 3$ on F. We want to prove the existence of two points that conform with Definition 4.3. We carry out the proof only for the first point, the rest in analogous by the use of the second conjunct of the axiom. Take a valuation π, such that $\pi(p) = \{v : w_1R_2v\}, \pi(q) = \{u : w_2R_3u\}$, and consider the model $M = (F, \pi)$, built on F, by taking π. So we have $M \models_w \Diamond_1\Box_2 p \wedge \Diamond_2\Box_3 q$. So, since **3WD** holds, we have $M \models_w \Box_3\Diamond_1(p \wedge q)$. Therefore there is a point $\overline{w}$ accessible from w_3 by means of relation 1, such that $M \models_{\overline{w}} p \wedge q$. But for definition of π, this means that $w_1R_2\overline{w}$, and $w_2R_3\overline{w}$.
From right to left. Assume the frame F is 3WD, $M = (F, \pi)$, and suppose $M \models_w \Diamond_1\Box_2 p \wedge \Diamond_2\Box_3 q$. This means that there are two points w_1 and w_2 such that wR_1w_1, wR_2w_2, and $M \models_{w_1} \Box_2 p$ and $M \models_{w_2} \Box_3 q$. Consider any point w_3, such that wR_3w_3, we need to prove that $M \models_{w_3} \Diamond_1(p \wedge q)$. But, since F is 3WD there is a point $\overline{w}$, such that $w_1R_2\overline{w}$, $w_2R_3\overline{w}$, and $w_3R_1\overline{w}$. So $M \models_{\overline{w}} p \wedge q$ and therefore $M \models_{w_3} \Diamond_1(p \wedge q)$, which is what we needed to prove. ■

We now prove a completeness result for the logic $S5_3 + \mathbf{3WD}$, which is composed by $S5_3$ plus the axiom **3WD**.

Theorem 4.1 *The logic $S5_3$ + **3WD** is sound and complete with respect to the class of reflexive, symmetric, transitive and 3WD frames.*
Proof: Soundness was proved in Lemma 4.3.
For completeness, it is known (for example see [HC96]) that the frame of the canonical model of $S5_3$ is in the class of reflexive, symmetric and transitive frames. We prove that the frame of the canonical model for $S5_3 + \mathbf{3WD}$, $M_C^{S5_3+\mathbf{3WD}}$, is also 3WD. Consider four maximal consistent sets for $S5_3 + \mathbf{3WD}$, that is four points in the canonical model, w, w_1, w_2, w_3, such that $wR_1w_1, wR_2w_2, wR_3w_3$, we want to prove the existence of a point $\overline{w_1}$, such that $w_1R_2^{S5_3+\mathbf{3WD}}\overline{w_1}, w_2R_3^{S5_3+\mathbf{3WD}}\overline{w_1}, w_3R_1^{S5_3+\mathbf{3WD}}\overline{w_1}$. By definition of the accessibility relation of the canonical model and the maximal extension theorem, we only need to prove that the set

$$\Gamma = \{\alpha : \Box_2\alpha \in w_1\} \bigcup \{\beta : \Box_3\beta \in w_2\} \bigcup \{\gamma : \Box_1\gamma \in w_2\}$$

is consistent. In fact, if so, there is a maximal extension that, by construction is a point that guarantees the frame of the canonical to be 3WD. The proof is by contradiction, so suppose such a set is not consistent. Then, there are $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_p$, with $\Box_2\alpha_1, \ldots, \Box_2\alpha_m \in w_1$, $\Box_3\beta_1, \ldots, \Box_3\beta_n \in w_2$, $\Box_1\gamma_1, \ldots, \Box_1\gamma_p \in w_3$, such that

$$\vdash_{S5_3+\mathbf{3WD}} \neg(\alpha_1 \wedge \cdots \wedge \alpha_m \wedge \beta_1 \wedge \cdots \wedge \beta_n \wedge \gamma_1 \wedge \cdots \wedge \gamma_p).$$

Let us now call $\alpha = \wedge_{i=1}^m \alpha_i$, $\beta = \wedge_{j=1}^n \beta_j$, $\gamma = \wedge_{k=1}^p \gamma_k$, it results

$$\vdash_{S5_3+\mathbf{3WD}} (\alpha \wedge \beta) \Rightarrow \neg\gamma$$

that, necessitating over $\Box_1$ becomes

$$\vdash_{S5_3+\mathbf{3WD}} \Diamond_1(\alpha \wedge \beta) \Rightarrow \neg\Box_1\gamma \tag{1}$$

Observe now, that since $\Box_2\alpha_1, \ldots, \Box_2\alpha_m$ are in w_1, then also $\Box_2(\alpha_1 \wedge \cdots \wedge \alpha_m)$, is in w_1, i.e. $\Box_2\alpha \in w_1$. Equivalently $\Box_3\beta \in w_2$, and $\Box_1\gamma \in w_3$. Then it results $\Diamond_1\Box_2\alpha \in w$ and $\Diamond_2\Box_3\beta \in w$, so since w is $S5_3 + \mathbf{3WD}$-maximal consistent, then $\Box_3\Diamond_1(\alpha \wedge \beta) \in w$. So, $\Diamond_1(\alpha \wedge \beta) \in w_3$, so by 1 $\neg\Box_1\gamma \in w_3$, which implies w_3 to be inconsistent itself. Therefore Γ cannot be inconsistent and there must exist a point $\overline{w_1}$ in the canonical model such that $w_1R_2\overline{w_1}, w_2R_3\overline{w_1}, w_3R_1\overline{w_1}$. A similar proof can be repeated to prove the existence in the canonical model of a point $\overline{w_2}$. This involves showing that the set

$$\Delta = \{\delta : \Box_3\delta \in w_1\} \bigcup \{\epsilon : \Box_1\epsilon \in w_2\} \bigcup \{\eta : \Box_2\eta \in w_3\}$$

cannot be inconsistent by using the second conjunct of **3WD**. Therefore the frame of the canonical model is 3WD, and so $S5_3 + \mathbf{3WD}$ is sound and complete with respect to the class of frames that are reflexive, symmetric, transitive and 3WD. ■

We now strengthen the result proven above and we show that $S5_3 + \mathbf{3WD}$ is sound and complete with respect to reflexive, symmetric, transitive and $3D$ frames. This means that the syntax we are using is not powerful enough to distinguish between the two class of frames. To prove completeness with respect to 3D we need the following:

Lemma 4.4 *If a frame is reflexive, symmetric, transitive and 3WD, then it is also 2WD.*

Lemma 4.5 *If a frame is reflexive, symmetric, transitive, 3WD and connected, then it is also 3D.*

We can now prove the following:

Theorem 4.2 *The logic* $S5_3 + \mathbf{3WD}$ *is sound and complete with respect to the class of reflexive, symmetric, transitive and 3D frames.*
Proof: Since $S5_3 + \mathbf{3WD}$ is sound with respect to frames that are reflexive, symmetric, transitive and 3WD, to prove soundness, it is sufficient to show that if a frame is 3D, then it is 3WD. But, this was noted in Lemma 4.2.
To prove completeness, it sufficient to show that if a formula ϕ is not a theorem of $S5_3 + \mathbf{3WD}$ then it is not valid on a reflexive, symmetric, transitive, and 3D frame. So, suppose $\not\vdash_{S5_3+\mathbf{3WD}} \phi$, then it has to exist a point $\overline{w}$ in the canonical model $M^{S5_3+\mathbf{3WD}}$, such that $M^{S5_3+\mathbf{3WD}} \not\models_{\overline{w}} \phi$. Consider now the generated model M^* of $M^{S5_3+\mathbf{3WD}}$, built from the point $\overline{w}$, by considering the union of the accessibility relations. We have $M^* \not\models_{\overline{w}} \phi$ and, by Theorem 4.1 the frame of M^* is reflexive, symmetric, transitive and 3WD. But, by Lemma 4.5 the frame is also 3D, and so ϕ is not valid on M^*, and so ϕ is not valid on the class of reflexive, symmetric, transitive and 3D frames. Therefore $S5_3 + \mathbf{3WD}$ is complete with respect to this class of frames. ■

5 Conclusions and further work

Interpreted systems are a useful formalism for representing MAS knowledge. In this note we have taken a simplified definition of Interpreted System, that does not include a state for the environment and does not take temporal evolution into account. We have shown how these systems generate a special class of Kripke structures that satisfy not only the axioms characterising $S5$, but also $\phi \Leftrightarrow D_A\phi$ and $\neg K_i \neg K_j \phi \Rightarrow K_j \neg K_i \neg \phi$, for any $i \neq j$.

We have completely characterised the Kripke structures that result from this mapping; we have shown that a mapping from frames to systems can also be defined and we have investigated the compositions of these functions. We have also defined a logic which is sound and complete (for the case $n = 3$) with respect to the images of the systems. Although we have not presented the general case here, it is possible to provide a sound and complete axiomatisation for the general case as well. At present we are finishing to investigate this matter.

The correspondence we have investigated gives rise to some applications. First it may allow us to transfer some standard completeness techniques to logics for agents with a semantics grounded on systems.

Secondly, it may help in the attempt to apply combining logics techniques for modal logics (for example [KW91]) to the case of complex MAS defined in terms of systems. The idea is that complex MAS specifications would benefit from an approach focused on the identification of classes of interactions between basic and well-understood modal logics with respect to the transfer of important properties such as completeness (see [LR97] for details).

We plan to extend this work by taking into account the temporal evolution (by considering runs over global states) and the representation of the environment.

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A Proofs of theorems

Lemma 2.1 *If S is a system, and $f(S) = (W, \sim_1, \ldots, \sim_n)$ is the frame defined from it by Definition 2.3, then*

1. $\bigcap_{i \in A} \sim_i = id_W$*;*
2. *For any* $w_1, \ldots, w_n$ *in* W *there exists a* $\overline{w}$ *such that* $\overline{w} \sim_i w_i$*,* $i = 1, \ldots, n$*.*

Proof: For 1, Consider any two elements $w = (l_1, \ldots, l_n), w' = (l'_1, \ldots, l'_n)$ in W such that $w(\bigcap_{i \in A} \sim_i)w'$. Then for all i in A, $(l_1, \ldots, l_n) \sim_i (l'_1, \ldots, l'_n)$. Therefore by definition, for all i in A, $l_i = l'_i$, that is $w = w'$.
For 2, consider any $w_1 = (l_1, \ldots, l_n), \ldots, w_n = (m_1, \ldots, m_n)$. Let $\overline{w} = (l_1, \ldots, m_n)$. By definition 2.3, the element $\overline{w}$ is in W and for each i, $\overline{w} \sim_i w_i$. ∎

Lemma 2.2 *Consider a frame* $F = (W, \sim_1, \ldots, \sim_n)$*.* $\bigcap_{i \in A} \sim_i = id_W$ *if and only if* $F \models \phi \Leftrightarrow D_A\phi$*.*
Proof. Left to right. Let M be a model based on F such that $M \models_w \phi$. Since $\bigcap_{i \in A} \sim_i = id_W$, then $M \models_w D_A\phi$. Analogously, suppose $M \models_w D_A\phi$. Since $w(\bigcap_{i \in A} \sim_i)w'$ implies $w = w'$, then $M \models_w \phi$.
Right to left. Suppose $F \models \phi \Leftrightarrow D_A\phi$ and for all i $w_1 \sim_i w_2$. Take a valuation π such that $\pi(p) = \{w_1\}$. Since $F, \pi \models_{w_1} p \Leftrightarrow D_A p$ and $F, \pi \models_{w_1} p$, we have $F, \pi \models_{w_1} D_A p$ and so $F, \pi \models_{w_2} p$. But since $\pi(p) = \{w_1\}$, it must be that $w_1 = w_2$. ∎

Lemma 2.3 *If S is a system,* $f(S) \models \neg K_i \neg K_j \phi \Rightarrow K_j \neg K_i \neg \phi$*, where* $i \neq j$*.*
Proof: For a contradiction suppose that $f(S) \not\models \neg K_i \neg K_j \phi \Rightarrow K_j \neg K_i \neg \phi$. Then there exists a point w and a valuation π such that $(F, \pi) \models_w \neg K_i \neg K_j \phi \wedge \neg K_j \neg K_i \neg \phi$. Therefore there must exist two points w_1 and w_2 such that $w \sim_i w_1$ and $w \sim_j w_2$ and $(F, \pi) \models_{w_1} K_j \phi$ and $(F, \pi) \models_{w_2} K_i \neg \phi$. But by property 2. of Lemma 2.1 there exists a point $\overline{w}$ such that $\overline{w} \sim_j w_1$ and $\overline{w} \sim_i w_2$. Since $(F, \pi) \models_{w_1} K_j \phi$ and the relations are symmetric, we have $(F, \pi) \models_{\overline{w}} \phi$, but this contradicts $(F, \pi) \models_{w_2} K_i \neg \phi$ that requires $\overline{w}$ to satisfy $\neg \phi$. ∎

Lemma 2.4 *If* $S \cong_{\mathcal{S}} S'$*, then* $f(S) \cong_{\mathcal{F}} f(S')$*.*
Proof: Let $S = L_1 \times \cdots \times L_n$, and $S' = L'_1 \times \cdots \times L'_n$. Since $S \cong_{\mathcal{S}} S'$ there is a family of bijections $b_i : L_i \to L'_i$. Consider $b = b_1 \times \cdots \times b_n$. The function b is a bijection, and therefore the universes of the frames $f(S)$ and $f(S')$ are in a bijection.
Consider now $s = (l_1, \ldots, l_i, \ldots, l_n), s' = (l'_1, \ldots, l'_i, \ldots, l'_n)$ such that $s, s' \in S$, and $s \sim_i s'$ on $f(S)$. Consider $b(s) = (b_1(l_1), \ldots, b_i(l_i), \ldots, b_n(l_n))$ and $b(s') = (b_1(l'_1), \ldots, b_i(l'_i), \ldots, b_n(l'_n))$. Since, by definition, $l_i = l'_i$, then $b_i(l_i) = b_i(l'_i)$ and therefore $b(s) \sim'_i b(s')$.
Let now be $b(s) \sim'_i b(s')$. Then, by definition $b_i(l_i) = b_i(l'_i)$ and then $l_i = l'_i$, that implies $s \sim_i s'$. ∎

Lemma 2.5 *If* $F \cong_{\mathcal{F}} F'$*, then* $g(F) \cong_{\mathcal{S}} g(F')$*.*
Proof: Consider two isomorphic frames $F = (W, \sim_1, \ldots, \sim_n), F' = (W', \sim'_1, \ldots, \sim'_n)$ such that $b : W \to W'$ is a bijection. We want to prove that there is a family of bijections c_i between the components of $g(F) = W/\sim_1 \times \cdots \times W/\sim_n$ and $g(F') = W'/\sim'_1 \times \cdots \times W'/\sim'_n$.
Let $c_i : W/\sim_i \to W'/\sim'_i$ such that $c_i([w]_{\sim_i}) = [b(w)]_{\sim'_i}$.
The function c_i is well defined. In fact, let $[w]_{\sim_i} = [w']_{\sim_i}$, with $w, w' \in W$. Then $c_i([w]_{\sim_i}) = [b(w)]_{\sim'_i} = [b(w')]_{\sim'_i} = c_i([w']_{\sim_i})$.

The function c_i is injective. $c_i([w]_{\sim_i}) = c_i([w']_{\sim_i})$, then $[b(w)]_{\sim'_i} = [b(w')]_{\sim'_i}$, that is $b(w) \sim_i b(w')$, $w \sim_i w'$ and then $[w]_{\sim_i} = [w]_{\sim'_i}$.
The function c_i is surjective. Consider $[w']_{\sim'_i}$, such that $w' \in W'$ and let $w \in W$ be such that $b(w) = w'$. Then $c_i([w]_{\sim_i}) = [w']_{\sim'_i}$. ∎

Theorem 3.1 *For any system S in $\mathcal{S}$, $S \cong_{\mathcal{S}} g \circ f(S)$).*
Proof: We prove that $b_i : L_i \to (L_1 \times \cdots \times L_n)/\sim_i$, defined as $b_i(l_i) = [(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i}$, where $l_j, i \neq j$, is any element in L_j, is a bijection.
The function b_i is well defined. In fact, let $l_i = l'_i$. Then $b_i(l_i) = [(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i}$ and $b_i(l'_i) = [(l'_1, \ldots, l'_i, \ldots, l'_n)]_{\sim_i}$. But $(l_1, \ldots, l_i, \ldots, l_n) \sim_i (l'_1, \ldots, l'_i, \ldots, l'_n)$ and therefore $b_i(l_i) = b_i(l'_i)$.
The function b_i is an injection. In fact, let $b_i(l_i) = b_i(l'_i)$, then $[(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i} = [(l'_1, \ldots, l'_i, \ldots, l'_n)]_{\sim_i}$, that implies $l_i = l'_i$.
The function b_i is a surjection. In fact, consider any $[(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i} \in (L_1 \times \cdots \times L_n)/\sim_i$. $b_i(l_i) = [(l'_1, \ldots, l_i, \ldots, l'_n)]_{\sim_i} = [(l_1, \ldots, l_i, \ldots, l_n)]_{\sim_i}$. ∎

Theorem 3.2 *If $F = (W, \sim_1, \ldots, \sim_n)$ is a frame such that:*

- $\bigcap_i \sim_i = id_W$,
- $\forall w_1, \ldots, w_n, \exists \overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$ *;*

then $F \cong_{\mathcal{F}} f \circ g(F)$.
Proof: Consider the frame $f \circ g(F) = (W/\sim_1 \times \cdots \times W/\sim_n, \sim'_1, \ldots, \sim'_n)$ built according to Definition 2.4 and Definition 2.3. Let now h be a mapping $h : W \to W/\sim_1 \times \cdots \times W/\sim_n$, defined by $h(w) = ([w]_{\sim_1}, \ldots, [w]_{\sim_n})$. We prove that h is a bijection.
Injective: suppose $h(w_1) = h(w_2)$, i.e. $([w_1]_{\sim_1}, \ldots, [w_1]_{\sim_n}) = ([w_2]_{\sim_1}, \ldots, [w_2]_{\sim_n})$. Therefore, for all i, $[w_1] \sim_i [w_2]$, but since $\bigcap_i \sim_i = id_W$, it must be $w_1 = w_2$.
Surjective: consider any element $([w_1]_{\sim_1}, \ldots, [w_n]_{\sim_n})$ in $W/\sim_1 \times \cdots \times W/\sim_n$. By Hypothesis on F, there exists a world $\overline{w}$ in W, such that $[\overline{w}]_{\sim_i} = [w_i]_{\sim_i}$, for each $i = 1, \ldots, n$. Therefore $([w_1]_{\sim_1}, \ldots, [w_n]_{\sim_n}) = ([\overline{w}]_{\sim_1}, \ldots, [\overline{w}]_{\sim_n}) = h(\overline{w})$.
Now we prove that $w_1 \sim_i w_2$ in F if and only if $h(w_1) \sim'_i h(w_2)$ in $f \circ g(F)$. Suppose $w_1 \sim_i w_2$, that is $[w_1]_{\sim_i} = [w_2]_{\sim_i}$; by definition of $\sim_i$, this is equivalent to $([w_1]_{\sim_1}, \ldots, [w_1]_{\sim_n}) \sim'_i ([w_2]_{\sim_1}, \ldots, [w_2]_{\sim_n})$.
This proves that F and $f \circ g(F)$ are isomorphic. ∎

Theorem 3.3 *Let $F = (W, \sim_1, \ldots, \sim_n)$ be a frame. The following are equivalent:*

1. $\bigcap_i \sim_i = id_W$ *and* $\forall w_1, \ldots, w_n, \exists \overline{w}$ *such that* $\overline{w} \sim_i w_i$, $i = 1, \ldots, n$;
2. *there exists an S, such that $F \cong_{\mathcal{F}} f(S)$*

Proof: 1 implies 2: Under these conditions by Theorem 3.2, $F \cong_{\mathcal{F}} f \circ g(F)$. That is: $S = g(F)$.
2 implies 1: By Lemma 2.1 the frame $f(S)$ has the properties expressed by proposition 1. But F is isomorphic to $f(S)$ and therefore it has those properties as well. ∎

Lemma 4.4 *If a frame is reflexive, symmetric, transitive and 3WD, then it is also 2WD.*
Proof: Consider three points, w, w_1, w_2, such that wR_1w_1, wR_2w_2, we want to prove that there exists a point $\overline{w}$, such that $w_1R_2\overline{w}, w_2R_1\overline{w}$. Consider wR_1w_1, since the relations are reflexive, we also have wR_2w, wR_3w. So, since the frame is 3WD, there exists a point

w_3, such that $w_1R_2w_3, wR_1w_3, wR_3w_3$. Analogously there exists a point w_4, such that $wR_2w_4, wR_3w_4, w_2R_1w_4$. So, by transitivity we have $w_3R_3w_4$. Consider now $w_3R_3w_4$ and apply 3WD again by considering the reflexive relations on w_3. There must exist a point $\overline{w}$, such that $w_3R_2\overline{w}, w_3R_3\overline{w}, w_4R_1\overline{w}$. Therefore, by transitivity, the frame is 2WD. ∎

Lemma 4.5 *If a frame is reflexive, symmetric, transitive, 3WD and connected, then it is also 3D.*

Proof: Consider three points, w_1, w_2, w_3, we want to prove that there are $3! = 6$ points that satisfy Definition 4.3.

a) Since the relations are symmetric and transitive, and the frame is connected, it does not break the generality of the problem to assume that there are two chains connecting w_1 to w_2 and w_1 to w_3, in which every step is labelled with a number different from the previous and the next.

b) We now prove that every chain between two points can be reduced to a chain of two steps at maximum in which the first step can be chosen arbitrarily. We do it for the chain connecting w_1 to w_2. Assume we have the chain $w_1R_{x_1}w', w'R_{x_2}w'', \ldots, w^nR_{x_n}w_2$, in which $x_j \neq x_{j-1} \neq x_{j+1}$, we prove we can reduce this chain to w_1R_1x, xR_xw_2, in which R_x can be either R_1, R_2 or R_3. Consider $w_1R_{x_1}w'$, and the reflexive relations on w_1; by 3WD, we can find a point w_a such that $w_1R_1w_a, w'R_aw_a$. Consider now $w_aR_aw', w'R_{x2}w''$, where $a \neq 1$. We apply 3WD again by considering a reflexive relation on w', so there must be a w_b, such that $w_aR_1w_b, w_bR_bw''$. By transitivity we have $w_1R_1w_b$. We can now proceed through the whole chain in the same way obtaining w_1R_1x, xR_xw_2, where x is the last point in the construction and R_x is either R_1, R_2, or R_3. The same can be said for the chain connecting w_1 to w_3; so we have w_1R_1y, yR_yw_3.

c) We now prove the existence of a point that connects w_1, w_2, and y by three different relations. Assume $x \neq 1$; we have $xR_1w_1, xR_xw_2, xR_1y, yR_yw_3$. Consider x, w_1, w_2, by 3WD, there must exist a point x' such that $x'R_1x$, and related to w_1 and w_2 by different relations, either R_2 or R_3; by transitivity we also have $x'R_1y$. Consider now $x = 1$, we have w_1R_1y, and, by 3WD, there exists a point x'', such that $w_1R_2x'', yR_1x'', yR_3x''$. By transitivity, then w_2R_1x''.

d) We are now in a situation in which there is a point, we call z, that is related by three different relations to w_1, w_2, y. Now if we cannot apply transitivity for from z to w_3, still by 3WD (considering z, w_2, y) there is a point z' such that yR_yz', and w_2, w_1 related to z' by the other two relations. Whatever the configuration of these we have that z' connects w_1, w_2, w_3 by three different relations.

d) It does not break the generalities to consider the situation now to be: $w_1R_1z, w_2R_2z, w_3R_3z$. Since the frame is 3WD by Hypothesis, we also have that there exist two points that connect w_1, w_2, w_3 in two different permutations. To finish the proof we need to show that there are points connecting w_1, w_2, w_3 by taking fixed-points of the permutations. It is sufficient to show that given $w_1R_1z, w_2R_2z, w_3R_3z$, there is a point $\overline{z}$ such that $w_1R_1\overline{z}, w_2R_3\overline{z}, w_3R_1\overline{z}$. To see this, by Lemma 4.4 the frame is also 2WD and so, considering z, w_2, w_3, there is a point z' such that w_2R_3z', w_3R_2z'. Consider now z, w_1, w_2, by 3WD there exists a z'', such that w_1R_2z'', zR_1z'', w_2R_3z''. By transitivity w_1R_1z'', and $z'R_3z''$. Consider now $z'R_3z''$, $z'R_2z'$, $z'R_1z'$, by 3WD and symmetry, there exists a point $\overline{z}$, such that $z''R_1\overline{z}$, $z'R_2\overline{z}$, $z'R_3\overline{z}$. By transitivity we have the result.

Therefore, we can produce all the combinations of relations, that implies that if the connected frame is 3WD, then it is also 3D. ∎