First-Order Theories of Approximate Space *

extended abstract

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Abstract

In this paper we present a language for expressing "approximate" spatial relations on points and regions in space. Space is assumed to be two dimensional, orthogonal and discrete. Initially, we present a first-order theory of one-dimensional approximate space. We axiomatize the theory of space in terms of points and the *haze* and *precedence* relations. We then study the models of this theory and we show that in the one dimensional case the theory has a unique model up to isomorphism. This result allows us to affirm that our theory is complete and decidable. Secondly, we propose a conservative two-dimensional extension of the theory which retains its completeness property. In the two-dimensional case, we define a complete set of topological and directional relations that are useful for practical reasoning about space. The derived relations are related to existent formalisms for spatial relations such as Egenhofer's 3-intersection model, for which they provide an alternative semantical account. The developed formalism is strictly qualitative with a built-in concept for imprecision. An axiomatization of varying the degree of detail is briefly discribed and discussed as a conclusion of the present work.

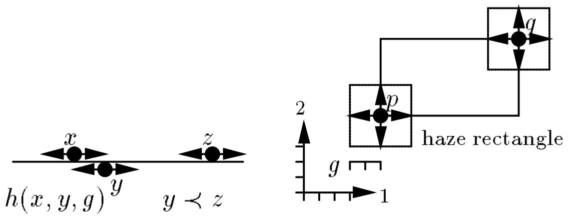
1 Introduction and Motivation

There are three issues that set the rationale for the present work: spatial ontologies, spatial imprecision and formal aspects of space representations:

- A major issue in the spatial reasoning research is the selection of an ontology for space, that is, a formal pattern for space, its objects and the defined spatial relations. The approaches found in the literature are charecterized by a wide diversity. AI reseachers debate whether a Euclidean or non-Euclidean ontology would suit the needs of spatial reasoning problems encountered in commonsense reasoning [Dav90]. Database and GIS researchers develop point-based on point-set based spatial data models, and spatial relations pertaining to logical, algebraic or fuzzy semantics [GB90].
- Spatial knowledge is approximate in its nature. Spatial relations are perceived in relation to the degree of detail that the world is viewed. The ability to conceptualize the world at different granularities is fundamental in human intelligence. It is important to incorporate the same ability in the representations of space as well. Earlier representation models for approximate spatial knowledge such as [MD84], [Dav86] represented imprecision using interval values on measurements. [Dut91] defines a framework for approximate spatio-temporal knowledge using fuzzy set-theory semantics. Finally, qualitative representations provide a promising response to this problem [Her92], [Ran92], [Ege91].
- The most successful computational paradigms base their success on their mathematical clarity. Relational databases are without any doubt the most obvious example. The study of formal aspects of spatial representation and reasoning is a fairly new area. A comprehensive theory for spatial representation and reasoning is expected to provide answers to questions such as (1) what is a minimal set of spatial relations in a given ontology that provides complete coverage of the qualitatively distinct relations in space; (2) are inferences, in some given representation, sound and/or complete; (3) are certain theories of space complete and/or decidable; etc. Some progress in this direction is reported in [Ran92], [Kau91], [EF91].

In the present study, we consider a *haze*-point ontology of space (see figure 1). Space is viewed as a totality of spatial objects connected by certain relations. Objects in space are either haze points or non-point objects. Haze points refer to points of space which are surrounded by a haze area, the smallest distin-

*AAAI-94 Workshop on Spatial and Temporal Reasoning, Seattle, Washington, August 1-4, 1994**



(a) 1-dim haze points (b) 2-dim haze points

Figure 1: The *haze*-point ontology

guishable quantity in the representation. Non-point objects are constructed by connecting such areas. A consequence of the haze area around points is that, at sufficiently small scales, objects are indistinguishable. Other authors use the terms "tolerance" or "noticable difference" [Rob73], [Kau91] for the same notion. A haze-point ontology provides for a formal treatment of granularity along the lines of [Hob85] because the definitions of spatial relations are parameterized by dergee of detail.

We assume two primitive relations, *haze* and *precedence*. At present, we also assume that space is two-dimensional, orthogonal and discrete. Furthermore, we hypothesize space as a mathematical structure over some domain (the integers) along with the primitive relations. We provide a formal axiomatization of space in a first-order language. Hence, the models and the completeness of the developed first-order theory of space can be studied. The research approach is particularly inspired by the fundamental work of Peter Ladkin for temporal points and intervals [Lad87].

The technical results presented in this paper are summarized as follows: Initially we develop a first-order theory of one-dimensional space, $H1$, with haze and precedence relations, and we show that it has a unique model up to isomorphism. This result allows us to affirm that $H1$ is complete and decidable. We propose a conservative two-dimensional extension, called *independent combination*, of $H1$ which retains its completeness property. From the two-dimensional point theory, $H2$, and by independent combination, we derive the theory of haze rectangles, HR. In HR, we define a complete and decidable set of topological and directional binary spatial relations. We also outline the effects of haze in reasoning with varying granularity.

The rest of this paper is organized as follows: Sections 2, 3 and 4, develop the theories $H1$, $H2$ and HR, respectively, along with an assessment of their models. Section 5, discusses the problem of varying the granularity and section 6 presents an application of our theoretical framework to the definition of binary spatial relations. Finally, section 7 concludes the paper with an outlook for future research.

2 One-dimensional Haze Space

Assume a sorted first-order language $\mathcal{L}$ with P (for point) and S (for scale) to be its sorts. The non-logical symbols of $\mathcal{L}$ include the predicate symbols $\prec$ and h of sorts $P \times P$ and $P \times P \times S$, respectively, the predicate symbol $<$ relating individuals of sort $S \times S$, a constant symbol g of sort S, and equality for both sorts. g is the measure of the scope for the h(aze) relation. In the following the letters x, y, z, u are taken as variables ranging over P. The axioms of the theory of one-dimensional space, $H1$, are stated as follows:

A1. $\forall x(h(x, x, g))$
A2. $\forall xy(h(x, y, g) \Rightarrow h(y, x, g))$
A3. $\forall xy(x \prec y \Rightarrow \neg y \prec x)$
A4. $\forall xyz(x \prec y \wedge y \prec z \Rightarrow x \prec z)$
A5. $\forall xyz(x \prec y \vee y \prec x \vee h(x, y, g)$
A6. $\forall x \exists y(x \prec y)$
A7. $\forall x \exists y(y \prec x)$
A8. $\forall xy(x \prec y \Rightarrow \exists z(x \prec z \wedge z \prec y \Rightarrow$
$\neg \exists u(x \prec u \wedge u \prec z)))$

In addition, the equality axioms for each sort and strict partial order axioms for $<$ in S, are assumed.

The symbols of $\mathcal{L}$ are interpreted with respect to a fixed structure **P1** which captures our assumptions, i.e., space is discrete, unbounded and with precedence and haze relations defined as above. The domain of P in **P1** is $\mathbb{Z}$. To the constant symbol g, **P1** assigns an integer constant g (called granularity) and to each of the predicate symbols $\prec$, h and $<$, **P1** assigns the relations p, r and $<_{\mathbb{Z}}$ respectively. Relations p and r are defined as follows:

$r = \{(x, y, \mathsf{g}) | x, y \in \mathbb{Z} \text{ and } -\mathsf{g} \leq_{\mathbb{Z}} x - y \leq_{\mathbb{Z}} \mathsf{g}\}$
$p = \{(x, y) | x, y \in \mathbb{Z} \text{ and } y - x >_{\mathbb{Z}} \mathsf{g}\}$

It is easy to show that for a given g, **P1** defines a class of models for $H1$. It suffices to show that all axioms are true in **P1**. This is easily verified for each one of them. For instance, the meaning of A5 is that for every two integers a, b, a resides either g number of steps in $\mathbb{Z}$ on the left or on the right of b, or it is within distance g from b independently of the direction. This is true for every a, b for a given g. In the same way we can verify the truth of all other axioms under **P1**.

2.1 Models of $H1$

We first introduce a few syntactic definitions that help us obtain some intuition about the structure of the models of $H1$. For notational convenience, we will omit from the discussion the third argument of the haze relation given also that until section 5 our discussion assumes constant g. Hence, we will use $h(x, y)$ instead of $h(x, y, \mathsf{g})$. We define a *neighbours* relation on P that relates all the points that are in the haze of the same point,

$$neighbours(x, y, z) \equiv h(x, y) \wedge h(x, z)$$

Lemma 1 *neighbours is an equivalence relation*
1. $neighbours(x, y, y)$*,*
2. $neighbours(x, y, z) \Rightarrow neighbours(x, z, y)$*,*
3. $neighbours(x, y, z) \wedge neighbours(x, z, u) \Rightarrow neighbours(x, y, u)$*.*

Proof: It follows directly from axioms A1, A2. □

We call the equivalent class that is associated to each haze point x, a neighbourhood and we by x^*, i.e., $x^* = \{(y, z) | neighbours(x, y, z)\}$. The set of equivalence classes on P is denoted by P^*.

We define an order relation between equivalence classes as follows:

1. $x^* \angle\, y^*$ if $\exists zu(neighbours(x, y, z) \wedge neighbours(y, x, u) \wedge h(x, y) \wedge z \prec u)$
2. $x^* \angle\, u^*$ if $x \prec u$

Lemma 2 *The* $\angle$ *relation on equivalence classes is a linear ordering.*

Proof: As a consequence of axioms A1-A5. □

We can now show that a haze point structure $H(S)$ defined on an abritrary linear order S, forms a model of $H1$. $H(S)$ is constructed by replacing the points of S by short intervals denoting the haze of each point. [1]

Theorem 1 *Given an arbitrary unbounded linear ordering* $<$ *on a set* S*, the haze structure of* S*,* $H(S)$*, forms a model of* $H1$ *under the definition of* h *and* $\prec$ *(axioms A1-A8). Furthermore, the ordering* $\angle$ *on the equivalence classes of neighbours in* $H(S)$ *is isomorphic to* $<$ *on* S*.*

Sketch of Proof: For some g, we construct $H(S)$ such that for each $a \in S$, $H(a) = \{a \pm i | -\mathsf{g} \leq i \leq \mathsf{g}\} \in H(S)$ is created. Then, it is easy to see that the reverse mapping shows S to be the set of equivalent classes associated with a haze structure $H(S)$, and that $H(S)$ is a model of $H1$. Each point of S corresponds to a haze in $H(S)$. By the construction, the mapping from equivalence classes from $H(S)$ to S is one-to-one and $\angle$ is preserved by $<$ in S, hence it is isomorphic.□

We now show that $H(S)$ is the only model of $H1$. It suffices to show that any other model of $H1$ is homomorphic to $H(S)$. First, we need to show that for each point x of P there is a unique equivalence class x^* with which it is associated.

Lemma 3 *For any* x *there is a unique equivalence class* x^**.*

Proof: It follows from the *neighbours* definition. □

We can now summarize and state the theorem which categorizes all models of $H1$. Given any model M of $H1$, we can form the set M^* of neighbourhoods under h, which is linearly ordered by $\angle$. Let $H(M^*)$ be the haze structure of M^*. All models of $H1$ are isomorphic to $H(M^*)$.

[1] Notice that, under this notation, **P1** would be $H(\mathbb{Z})$.

Theorem 2 $H(M^*)$ *is a homomorphic image of* M *and a model of* $H1$*. If* M *is also a model of* $H1$ *then* $H(M^*)$ *is isomorphic to* M*.*

Proof: Note that M in order to be a model of $H1$, is a structure of the form $H(S)$. M^* is the set of its equivalent classes w.r.t. the *neighbours* relation, and $H(M^*)$ is constructed in the known way. The mapping from haze points to equivalence classes preserves the structure, and, in addition, it is onto and one-to-one (by lemma 3), hence, isomorphic. □

So far, we showed that $H1$ has a unique model up to isomorphism. We will now answer the question whether $H1$ is complete or not. In addition, we need to include in the discussion axiom A8 which states the discretness property of haze space. We know that both $(\mathbb{Q}, <)$ and $(\mathbb{Z}, <)$ theories are complete [vB91]. By Vaught's test for completeness, we also know that any countable categorical theory is complete; and by Robinson's test for completeness for non-countable models [Bri77], the completeness condition requires the models to be elementary equivalent. Elementary equivalence is a coarser than isomophism relation between two models which are elementary embeddable, i.e, the first model's domain is a subset of the domain of the second and there is a mapping betwee them which preserves their structure and therefore the truth of the sentences of their common language [End72].

Due to our commitment to discreteness (first, we can interpret g and P over the same set i.e., the integers, and, second, this is a commonly taken assumuption in our intented application which is the spatial databases), we pick $(\mathbb{Z}, <)$ for embedding the models of $H1$. Then the theorem is stated as follows.

Theorem 3 (Completeness) *The theory axiomatized by A1-A8 is complete and all of its models are are elementary embeddable to* $H(\mathbb{Z})$*, and hence is* $Th(H(\mathbb{Z}))$*.*

Proof: As a consequence of Robinson's theory completeness test. □

Given also that a complete and axiomatizable theory is decidable we state the following collorary of theorem 3.

Corollary 1 *The theory of one-dimensional haze points axiomatized by A1-A8 is decidable.*

3 The Theory of Two-dimensional Haze Points

In this section we extend our language with an additional sort of individuals, namely, the sort of two-dimensional points. The set sorts in $\mathcal{L}$ now becomes $\{P_1, P_2, Q, S\}$, where P_1 and P_2 are two disjoint copies of P, and Q is the newly introduced sort. In fact, two copies of the $H1$ theory are employed and relation symbols $\prec$ and h appear with subscripts $_1$ and $_2$, too. We also introduce a new function symbol, the *pairing* function, form $P_1 \times P_2$ to Q.

$\langle x_1, x_2 \rangle$ it is a pairing function which returns the two-dimensional point formed by its one-dimensional coordinates

Having augmented our theory vocabulary with new symbols we have to extend the axioms appropriately. The new theory is denoted by $H2$.

B1. $\forall x/Q \; \exists x_1/P_1 \; x_2/P_2 \; (x = \langle x_1, x_2 \rangle)$

The next sentence extends the notion of haze to two-dimensional points:

B2. $\forall xy/Q \; \exists x_1 y_1/P_1 \; \exists x_2 y_2/P_2 \; (x = \langle x_1, x_2 \rangle \wedge y = \langle y_1, y_2 \rangle \wedge (h2(x,y) \Leftrightarrow h_1(x_1, y_1) \wedge h_2(x_2, y_2)))$

We now define a weak notion of colinearity along each of the distinct directions in two-dimensions, which, by convention, will be subscripted by $_h$ (horizontal) and $_v$ (vertical).

B3.1 $\forall xy/Q \; \exists x_1 y_1/P_1 \; \exists x_2 y_2/P_2 \; (x = \langle x_1, x_2 \rangle \wedge y = \langle y_1, y_2 \rangle \wedge (w\_colinear_h(x,y) \Leftrightarrow h_2(x_2, y_2)))$

B3.2 $\forall xy/Q \; \exists x_1 y_1/P_1 \; \exists x_2 y_2/P_2 \; (x = \langle x_1, x_2 \rangle \wedge y = \langle y_1, y_2 \rangle \wedge (w\_colinear_v(x,y) \Leftrightarrow h_1(x_1, y_1)))$

We also define the $hor(x)$, $vert(x)$ functions from sort Q to P_1 and P_2, respectively, to return a point's projection on the horizontal and vertical axis.

B4. $\forall x_1/P_1 \; \forall x_2/P_2 \; (hor(\langle x_1, x_2 \rangle) = x_1)$

B5. $\forall x_1/P_1 \; \forall x_2/P_2 \; (vert(\langle x_1, x_2 \rangle) = x_2)$

The point decomposition property which is expressed by the following sentence, is a valid sentence that follows from B1,B4,B5.

$$\forall x/Q \; (x = \langle hor(x), vert(x) \rangle)$$

Order (precedence) in the two-dimensional space is defined in terms of the $\prec_1, \prec_2$ relations for the one-dimensional case.

C1. $\forall xy/Q \; \exists x_1 y_1/P_1 \; \exists x_2 y_2/P_2 \; (x = \langle x_1, x_2 \rangle \wedge y = \langle y_1, y_2 \rangle \wedge (east(x,y) \Leftrightarrow x_1 \prec_1 y_1))$

C2. $\forall xy/Q \; \exists x_1 y_1/P_1 \; \exists x_2 y_2/P_2 \; (x = \langle x_1, x_2 \rangle \wedge y = \langle y_1, y_2 \rangle \wedge (west(x,y) \Leftrightarrow y_1 \prec_1 x_1))$

C3. $\forall xy/Q \; \exists x_1 y_1/P_1 \; \exists x_2 y_2/P_2 \; (x = \langle x_1, x_2 \rangle \wedge y = \langle y_1, y_2 \rangle \wedge (north(x,y) \Leftrightarrow y_2 \prec_2 x_2))$

C4. $\forall xy/Q \; \exists x_1 y_1/P_1 \; \exists x_2 y_2/P_2 \; (x = \langle x_1, x_2 \rangle \wedge y = \langle y_1, y_2 \rangle \wedge (south(x,y) \Leftrightarrow x_2 \prec_2 y_2))$

East, west, north and *south* are irreflexive, asymmetric and transitive, and, in addition, *east* is the inverse of *west* and *south* is the inverse of *north*. The following axiom states the totality property in the two-dimensional space, i.e., any two points are related with one of the nine disjunctive relationships.

C5.
$$\begin{aligned}
\forall xy/Q \; (h2(x,y) \vee & \quad (1)\\
(w_colinear_h(x,y) \wedge east(x,y)) \vee & \quad (2)\\
(w_colinear_h(x,y) \wedge west(x,y)) \vee & \quad (3)\\
(w_colinear_v(x,y) \wedge north(x,y)) \vee & \quad (4)\\
(w_colinear_v(x,y) \wedge south(x,y)) \vee & \quad (5)\\
(north(x,y) \wedge east(x,y)) \vee & \quad (6)\\
(south(x,y) \wedge east(x,y)) \vee & \quad (7)\\
(north(x,y) \wedge west(x,y)) \vee & \quad (8)\\
(south(x,y) \wedge west(x,y))) & \quad (9)
\end{aligned}$$

We will now state the orthogonality property.

C6.1 $\forall xy/Q \; (w\_colinear_h(x,y) \Rightarrow \forall z/Q \; (w\_colinear_v(x,z) \wedge \exists u/Q (w\_colinear_h(z,u) \wedge w\_colinear_v(y,u)))$

C6.2 $\forall xy/Q \; (w\_colinear_v(x,y) \Rightarrow \forall z/Q (w\_colinear_h(x,z) \wedge \exists u/Q (w\_colinear_v(z,u) \wedge w\_colinear_h(y,u)))$

It can be easily seen that theory $H2$ has a class of models, i.e., is consistent, which is based on the cartesian plane $\mathbb{Z} \times \mathbb{Z}$. Next section precisely characterizes these models.

3.1 Models of $H2$

We developed a two-dimensional theory for space, $H2$, based as a combination of two theories that correspond to each (independent) coordinate line. Such a combination views each dimension independently. Note that we require that the two-dimensional individuals to be pairs of one-dimensional individuals for which a separate structure is assumed. Cross-domain operators are defined only between two-dimensional individuals and are decomposable to primitive single domain operators. We will now analyze the models of $H2$ by extending our results for the models of $H1$.

Definition 1 (independent combination) *Let $Op(L)$ be the set of non-logical symbols of a first-order language L. Let C_1, C_2 be two theories referring to each different coordinate line with $Op(L_{C_1}) \cap Op(L_{C_2}) = \emptyset$. The independent combination of C_1, C_2 is the combined theory $C_1 \otimes C_2$ such that:*

- *the theory of $C_1 \otimes C_2$ consists of the union of the respective sets of symbols (since the both languages are copies of the same language, the formula-building rules are the same);*
- *if Σ_1, Σ_2 are axiom systems for C_1, C_2, respectively, then $\Sigma_1 \cup \Sigma_2$ is an axiom system for $C_1 \otimes C_2$;*
- *if structures $(\mathsf{P}_1, <_1, \mathsf{h}_1)$ and $(\mathsf{P}_2, <_2, \mathsf{h}_2)$ are models of C_1, C_2, then the structure $(\langle \mathsf{P}_1, \mathsf{P}_2 \rangle, \langle <_1, <_2 \rangle, \langle \mathsf{h}_1, \mathsf{h}_2 \rangle)$ is a model for $C_1 \otimes C_2$.*

Given an independent combination, every sentence of the combined language is decomposed into a $C1$ formula, a $C2$ formula and a *P(airs)*-part. The decomposition operator, denoted by HV, is a function on the sentences of $C_1 \otimes C_2$ such that:

$$HV(\forall x_1..x_i/Q \; \Phi)$$
is
$$\forall x_1..x_i/Q \; \exists x_{1_1}..x_{i_1}/P_1 \; \exists x_{1_2}..x_{i_2}/P_2 \; (x_1 = \langle x_{1_1}, x_{1_2} \rangle \wedge ... \wedge x_i = \langle x_{i_1}, x_{i_2} \rangle \wedge H(\Phi) \wedge V(\Phi)).$$

where $H(\phi)$ (resp. $V(\phi)$) involves only $C1$ (resp. $C2$) symbols, and $\bigwedge_i x_i = \langle x_{i_1}, x_{i_2} \rangle$ is the P-part and

$Q = P_1 \otimes P_2$ Subscripts $_1$, $_2$ denote $C1$, $C2$ symbols respectively.

The H, V formulas are obtained as follows:

1. atomic formulas

ϕ	$H(\phi)$	$V(\phi)$	$P(\Phi)$
$x_1 \prec_1 y_1$	$x_1 \prec_1 y_1$	T	ψ
$x_2 \prec_2 y_2$	T	$x_2 \prec_2 y_2$	ψ
$h_1(x_1, y_1)$	$h_1(x_1, y_1)$	T	ψ
$h_2(x_2, y_2)$	T	$h_2(x_2, y_2)$	ψ
$op(x, y)$	$H(\phi_1)$	$V(\phi_2)$	ψ

where ψ is $x = \langle x_1, x_2 \rangle \wedge\ y = \langle y_1, y_2 \rangle$, and op is one of the $\{h2,\ w\_colinear_h,\ w\_colinear\_v,\ east,\ west,\ north,\ south\}$ and the H and V parts are directly obtained.

2. non-atomic formulas

Φ	$H(\Phi)$	$V(\Phi)$	P(Φ)
$\neg\phi$	$\neg H(\phi)$	$\neg V(\phi)$	none
$\phi \wedge \psi$	$H(\phi) \wedge H(\psi)$	$V(\phi) \wedge V(\psi)$	none
$\forall x/Q(\phi(x))$	$\forall x_1/P_1(H(\phi(x_1))$	$\forall x_2/P_2(V(\phi(x_2))$	ψ

where ψ is $x = \langle x_1, x_2 \rangle$;
$\exists$, $\vee$, $\Rightarrow$ formulas are derived in the usual way.

Lemma 4 *Let $C_1 \otimes C_2$ be an independent combination of C_1 and C_2, and $M = M_1 \otimes M_2$ the combined structure of M_1 and M_2 as they are specified in definition 1. A $C_1 \otimes C_2$ sentence ϕ is true in M under some valuation v, if and only if $H(\phi)$ and $V(\phi)$ are true in M_1, M_2 with valuations v_1 and v_2 and $v = v_1 \circ v_2$, i.e.,*

$$M \vDash \phi[v] \text{ iff } M_1 \vDash H(\phi)[v_1],\ M_2 \vDash V(\phi)[v_2] \\ \text{and } v = \{\langle x_1, x_2 \rangle | x_1 \in v_1 \text{ and } x_2 \in v_2\}$$

Proof: By induction on the structure of Φ.$\Box$

Theorem 4 *Let $C1$ and $C1$ be two complete theories over the classes of models M_1 and M_2. Then, their independent combination $C1 \otimes C2$ is complete under the class of models $M1 \otimes M2$.*

Proof: By the Lemma 4. $\Box$

Note that theorem 4 conforms with the *independent combination* theorem by Finger [Fin93] which states that the independent combination of two sound, complete and decidable propositional logic systems carry the aforementioned properties. Finger, however, takes a proof theoretic approach whereas we follow a model theoretic one.

Corollary 2 *The theory of two-dimensional space haze points is complete.*

4 The Theory of Two-dimensional Rectangles

We will now extend $\mathcal{L}$ with an additional sort, R, for rectangles. A haze rectangle is constructed as a pair of two-dimensional haze points. The new symbols in the language include two predicate symbols for *inclusion*, and a pairing function symbol:

$in_0(p, a)$	point p is inside rectangle a
$in_1(a, b)$	rectangle a is inside rectangle b
$[p, q]$	is the pairing function which returns the rectangle which is formed by the two-dimensional points p, q.

The necessary axioms for the augmented theory are stated in the sequel. The new theory derived is denoted by HR. For notational convenience, in the following, the letters x, y, z, u will be taken as individual variables ranging over Q, and the letters a, b, c, d will be taken as individual variables ranging over R.

Similarly to the point pairing function $(\langle, \rangle)$, rectangle pairing possesses the axiom:

D1. $\forall a/R\ \exists xy/Q\ (a = [x, y]\ \Rightarrow$
$east(x, y)\ \wedge\ south(x, y))$

We define the first (resp. second) member of the rectangle pair to be the *low* (resp. *high*) rectangle point.

D2. $\forall xy/Q\ (low([x, y]) = x)$
D3. $\forall xy/Q\ (high([x, y]) = y)$

The rectangle decomposition property follows, as a valid sentence, from D1,D2,D3.

$$\forall a/R\ (a = [low(a), high(a)])$$

We abbreviate the formula $east(x, y) \wedge\ south(x, y)$, $x, y/Q$, by $x <_D y$, meaning that x "diagonally" precedes y. Then, we can state that all points included in a rectangle must be diagonally included by its low and high point. Also, any two rectangles are in inlucions relation, i.e., in_1, if and only if the inner rectangle's points are diagonally enclosed by the outer's extreme points.

D4. $\forall x/Q\ \forall a/R\ (in_0(x, a)\ \Leftrightarrow$
$low(a) <_D x <_D high(a))$
D5. $\forall ab/R\ (in_1(a, b)\ \Leftrightarrow\ low(b) <_D$
$low(a) <_D high(a) <_D high(b))$

In fact, D4,D5 define the two inclusion relations in terms of our ontological primitives, namely, points, haze and precedence. in_1 is shown that is a partial order relation (transitive, relfexive and antisymmetric), and in_0 is transitive over in_1.

The precedence relations between rectangles are also defined in terms of the corresponding relations in Q. As each rectangle is seen as a pair of two haze point objects, then precedence is defined as:

$$a\ op\ b \equiv high(a)\ op\ low(b)$$

where $op \in \{east, south, <_D\}$ as defined for two-dimensional haze points. Transitivity, irreflexivity and asymmetry of these operators in R is easily derived.

Concluding this section, we have seen how *inclusion*, a purely topological concept, is expressed in our ontology for the simplified case of rectangular regions. Theory HR is defined as an extention of $H2$ following the syntactic rules of independent combination. The following corollary is a consequence of the latter and theorem 4.

Corollary 3 *The theory of two-dimensional rectangles is complete.*

5 Granularity

In this section we study the effects of varying granularity. We first state the axioms that specify changes in granularity.

G1.($\rightarrow$) $\forall g_1 g_2/S\ (g_1 < g_2 \Rightarrow \forall xy/P\ (h(x,y,g_1) \Rightarrow h(x,y,g_2)))$ coarsening

G2.($\leftarrow$) $\forall g_1 g_2/S\ (g_2 < g_1 \Rightarrow \forall x/P \exists y/P\ (h(x,y,g_1) \wedge \neg h(x,y,g_2)))$ refinement

The idea is that when the granularity changes then the original theory, say T_0, has to change. In particular, G1 states that two points that are indistinguishable at g_1, they continue to be so at $g_2 > g_1$ ($\rightarrow$). G2 states that the transition from finer, g_1, to coarser granularity, $g_2 < g_1$, does not preserve indistinguishability ($\leftarrow$). Our future research goal is to investigate syntactic methods for computing the theories $T_{\leftarrow}$ and $T_{\rightarrow}$.

An early work towards a formal treatment of granularity in reasoning systems was presented by Hobbs [Hob85], where the use of a transitive indistinguishability relation was proposed to select a local theory out of a global one. One of Hobbs conclusions was that grain size should be an explicit argument of many predications. In our case, we chose the h relation to be the grain-dependent predicate and we develop a theory using it as a primitive. In the advantages of our solution is that local, and possibly minor, changes of the theory can reflect major modifications, because h influences the definition of many binary spatial relations. Our goal is to explore rewriting techniques for carrying out this task and provide soundness and completeness guarantees for theories of certain syntactic type.

Our work can be also compared to *temporal modules* and *time units* proposed by Wang and Jajodia [WJS93]. In their work, they deal with the problem of mismatches of temporal databases with different units of time. Their solution is to superimpose various units of time, finer or coarser, over the predifined unit that appears in the database. That method has a global effect in the sense that when the time unit changes the change affects the truth of all the domain facts. In our approach, which parameterizes with the granularity argument certain relations, changes to the grain affects the truth of only the neighbouring relations, where the notion of neighbouring relation is regarded as in [Fre92].

Finally, the rationale for granularity support in our model is not only to allow for imprecise descriptions but also to support *indeterminate* retrievals. In the latter case, an indeterminate query, underconstrained in the AIish terminology, is initially directed to the coarser possible theory and its answer is refined as the logical theory is refined.

6 An Application: Binary Spatial Relations

In this section we use the developed framework for the definition of *topological* and *directional relations*. In particular, we will show that we are able to define Egenhofer's topological [EF91], [Ege91] and Hernandez's directional relations [Her92] within the same framework.

Topological relations are defined between pairs of rectangles. In the cartesian plane, a rectangle is defined by two points, the left-bottom and the right-top point. When points are assumed to be haze points the extension of the rectangle may grow by size g at each side. This means that there is uncertainty about the exact position of the rectangle and the size of its area. In our simplified domain we can visualize a rectangle having a stripe of width $2g$ as boundary. The real shape can be placed anywhere in this area.

$\mathtt{disjoint}(a,b) \equiv \neg in_1(a,b) \vee \neg in_1(b,a)$
$\mathtt{tangent_l}(a,b) \equiv \exists p,q(in_0(p,a) \wedge in_0(q,b) \wedge h2(p,q))$
$\mathtt{overlap}(a,b) \equiv \exists c(in_1(c,a) \wedge in_1(c,b)$
$\mathtt{inside_t}(a,b) \equiv in_1(a,b) \wedge \forall c(\mathtt{tangent_l}(c,a) \wedge \mathtt{tangent_l}(c,b))$
$\mathtt{inside_i}(a,b) \equiv in_1(a,b) \wedge \neg\exists c(\mathtt{tangent_l}(c,a) \wedge \mathtt{tangent_l}(c,b)$
$\mathtt{contain_t}(a,b) \equiv \mathtt{inside_t^{-1}}(a,b)$
$\mathtt{contain_i}(a,b) \equiv \mathtt{inside_i^{-1}}(a,b)$
$\mathtt{equal}(a,b) \equiv in_1(a,b) \wedge in_1(b,a)$

The notion of tagency as defined here is "loose" tangency in the sense that two the rectangles share a point of their haze. If the size of the haze decreases tangent relationship changes to disjointness.

Directional relations are defined between pairs of points. The first point is the reference and the second is the referencing point. The characterization of direction is done by means of the precedence and haze relations. Thus, granularity plays a role here, as well. Axiom C5 defines nine disjoint positionings of a haze point y with respect to a haze point x. The eight of them are directional and listed bellow – in parentheses is the number that identifies the disjuncts of C5 and the `s_` prefix stands for "straight":

`s_east`(2), `s_west`(3), `s_north`(4), `s_south`(5), `north_east`(6), `south_east`(7), `north_west`(8), `south_west`(9)

The following propositions show that the above set of relations characterize all the qualitative distinct relative positionings of two object using the proposed ontology of space, i.e., are representationally complete.

Proposition 1 *The set of topological relations that are defined using the haze space ontology, i.e., using the h and $\prec$ primitives only, are mutually disjoint.*

It is easy to prove that:

$$\vDash_{HR} \forall ab(\mathtt{disjoint}(a,b) \vee \mathtt{tangent_l}(a,b) \vee \mathtt{overlap}(a,b) \vee \mathtt{inside_t}(a,b) \vee \mathtt{inside_i}(a,b) \vee \mathtt{equal}(a,b)$$

Proposition 2 *The set of directional relations for topologically disjoint objects that are defined using the haze space ontology, cover completely the two-dimensional space.*

7 Concluding Remarks

In this paper we focused on a formal treatement of imprecision in spatial representations and we showed our initial technical results which regard (a) the development of a mathematical basis for the definition of spatial relations, and (b) the formal handling of spatial uncertainty within the definition of spatial relations.

Among our future research plans we highlight the following: (a) devise syntactic methods for computing coarser or finer theories when the granularity changes, (b) formulate spatial reasoning in a varying granularity setting, (c) study the implications of the relations structure (neighbouring relations) in reasoning, (d) study soundness, completeness and complexity issues of spatial inference, and finally, (e) integrate the spatial reasoner with a general purpose reasoner.

Acknowledgements

I wish to thank Prof. John Mylopoulos for his guidance and support, and Prof. Janice Glasgow (Queen's University, Kingston Ontario) and Dr. Fangzhen Lin for their insightful comments on the earlier drafts of this paper. I am also grateful to my colleagues Dimitris Plexousakis and Dr. Manolis Koubarakis (Imperial College, London) for the fruitful discussions that we had and also for their encouragement and support. Financial support was provided by the Department of Computer Science, University of Toronto.

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