A Non-Hamiltonicity Proof System

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Abstract

To date, the field of proof complexity contains only one major example of a graph theoretic proof system, the Hajós Calculus. With the goal of further diversifying the field of proof complexity, we describe the 'Non-Hamiltonicity Proof System' (NHPS), for which we prove soundness, completeness, exponential lower bounds on necessary proof length, as well as a simulation by Tree Resolution.

1 Introduction

In [PU95], Pitassi and Urquhart prove that the Hajós Calculus proof system for non-3-colourability is as powerful as any Extended Frege proof system. This surprising result remains the only major graph theoretic contribution to the field of proof complexity. The purpose of this work is to further diversify the area by exploring a proof system for another graph theoretic co$\mathcal{NP}$-Complete problem. In this paper we introduce the Non-Hamiltonicity Proof System (NHPS). A Hamiltonian Cycle is a simple cycle which passes through every vertex in a graph exactly once. Graphs containing Hamiltonian Cycles are termed to be 'Hamiltonian', and as its name suggests, the NHPS is a proof system for verifying any graph's non-Hamiltonicity. We provide proofs of soundness, completeness, as well as exponential lower bounds on the necessary lengths of proofs in this system.

2 Terminology

Since this work falls in the area of overlap between the fields of graph theory and proof complexity, we assume that the reader is familiar with elementary graph-theoretic terminology, the basic ideas behind $\mathcal{NP}$-Completeness, as well as the basics of proof complexity. Consistent with the standard definition, we use the notation $G = (V, E)$ to refer to a graph. In addition, we use n to denote $|V|$, and m to denote $|E|$.

The NHPS deals with standard undirected graphs that have no self loops nor any multiple edges, as well as 'marked' graphs. A marked graph $G = (V, E, M)$ is simply an undirected graph where $M \subseteq E$ is a (possibly empty) subset of 'special' edges with markings on them. An example of a marked graph is shown below in Figure 1.1. The standard interpretation of marked edges in a graph G is that anyone searching for a Hamiltonian Cycle in G must include the marked edges. Therefore, an otherwise Hamiltonian graph can be turned into a non-Hamiltonian marked graph by marking some set of edges which disqualifies all Hamiltonian Cycles. We say that a marked edge is 'forced' if it is necessarily part of every Hamiltonian cycle occurring in a graph. The marked edges in Figure 1.1 are forced because they are incident on a degree 2 vertex. If three or more marked edges are incident on a vertex, we call that vertex a 'marked hub'. Finally, a cycle with c edges where $c < n$ and all c edges are marked is a 'marked subcycle'. It is important to distinguish between 'marked hubs' and 'forced hubs', and similarly important to distinguish

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between 'marked subcycles' and 'forced subcycles'. The NHPS deals exclusively with the forced versions. Examples of these obstructions to Hamiltonicity are shown below in Figures 1.2 and 1.3, respectively. Note that every marked edge in a non-Hamiltonian graph is vacuously forced because it appears in all 0 of the possible Hamiltonian Cycles.

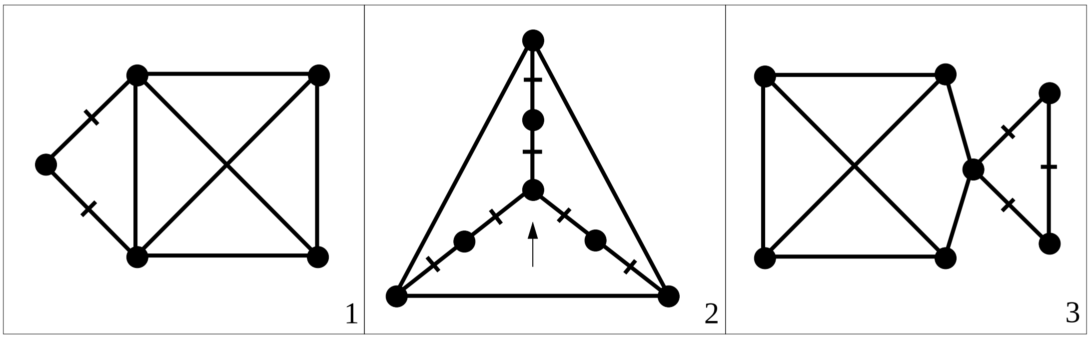

Figure 1: Examples of marked graphs containing forced edges, a forced hub, and a forced subcycle.

In addition, we say that a marked graph $G = (V, E, M)$ is 'obviously non-Hamiltonian' if any of the following conditions hold:

1. G contains a forced hub.
2. G contains a forced subcycle.

3 Description of the NHPS

The NHPS is quite simple and has the following characteristics:

1. Every NHPS proof is a tree.
2. Every tree node contains either a marked graph or an unmarked graph.
3. Tree edges are directed towards the root and represent applications of inference rules. It is possible for multiple edges to correspond with one application of a rule.
4. The root contains an unmarked graph that is to be proven as non-Hamiltonian.
5. Each leaf contains an axiom. The set of axioms consists of all unmarked graphs containing at least one vertex v such that $degree(v) \leq 1$.
6. All tree nodes at even depths contain unmarked graphs, whereas all tree nodes at odd depths contain marked graphs.
7. Every subtree rooted at an unmarked graph is an NHPS proof.
8. Rules of inference:
   (a) *Rule for forcing edges:* Let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, ... $G_k = (V, E_k)$ be unmarked graphs with NHPS proofs of non-Hamiltonicity. If there exist two sets E and M subject to the following conditions:
      i. For each $G_i = (V, E_i)$, there exists exactly one edge e_i such that $E = E_i \cup \{e_i\}$.

ii. $M = \bigcup_{i=1}^{k} \{e_i\}$

Then we may infer $G = (V, E, M)$.

(b) *Rule for removing markings from all forced edges:* If a graph with forced edges is obviously non-Hamiltonian, then the corresponding unmarked graph is also non-Hamiltonian.

4 Example

Intuitively, the NHPS is used to show that any unmarked non-Hamiltonian graph that is not an axiom contains edges, that if marked, would constitute a forced hub or subcycle, thereby proving that it is indeed non-Hamiltonian. The first inference rule is based on the fact that any edge added to a non-Hamiltonian graph, if marked, is forced [Her04]. The second inference rule states that any graph containing a forced hub or forced subcycle has a corresponding unmarked graph which is also non-Hamiltonian. Of course, the second rule presupposes that all marked edges in hubs and subcycles are in fact forced, something that must be proven via NHPS proof sub-trees.

Figure 2 below illustrates an example of a NHPS proof of non-Hamiltonicity for the graph in the root. It is not biconnected, and is therefore not Hamiltonian. Note that all of the graphs in the leaves of the proof tree are axioms.

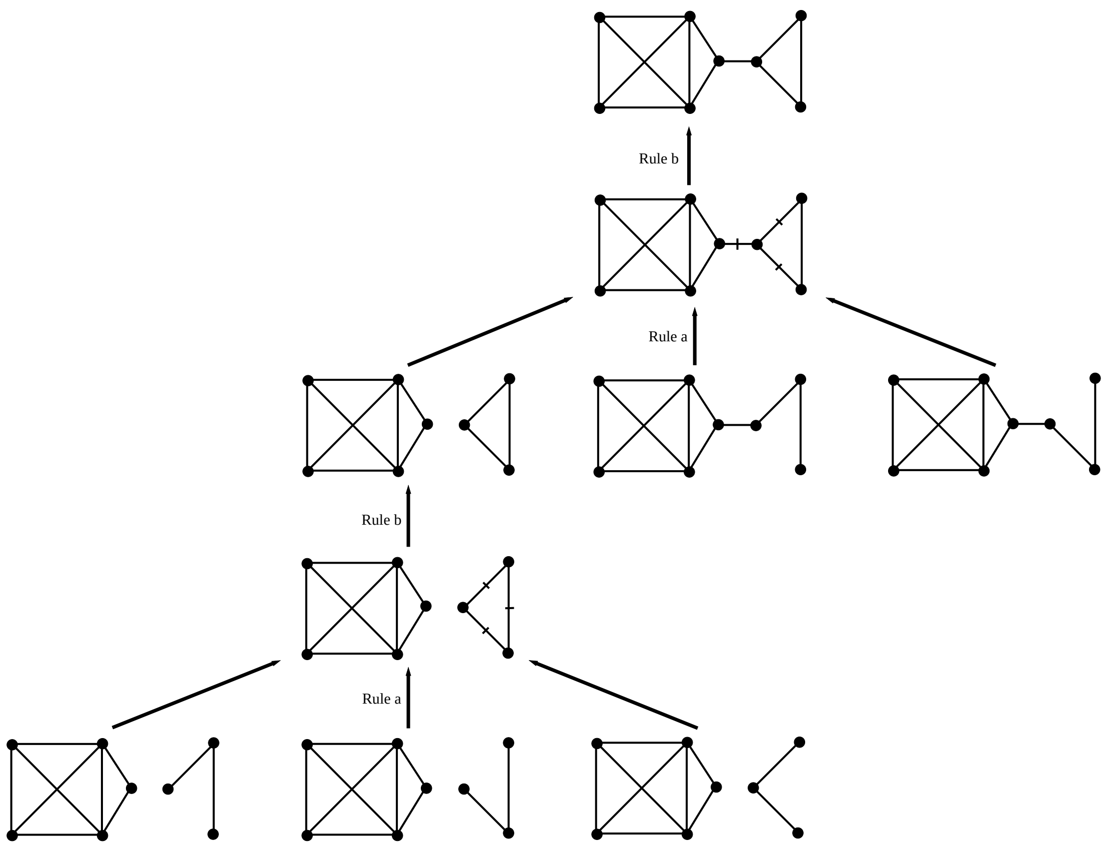

Figure 2: A NHPS Proof Of Non-Hamiltonicity

5 Soundness

Lemma 5.1. *All marked edges introduced by Inference Rule a are forced.*

Proof: Suppose that G_1, G_2, ... G_k are all unmarked, non-Hamiltonian graphs and that for each $G_i = (V, E_i)$, there exists exactly one edge e_i such that $E = E_i \cup \{e_i\}$. Let $G_i{}'$ be G_i with the edge e_i added and marked. Since any single edge added to any G_i must be in every possible Hamiltonian Cycle of G_i, we know that each marked edge e_i in each $G_i{}'$ is forced. However, for each $G_i{}'$, $V_i{}' = V$, and $E_i{}' = E$. Therefore, each forced edge in each $G_i{}'$ must also be forced in G, as required. □

Theorem 5.2. *The* NHPS *is Sound.*

Proof: From Lemma 5.1, we know that Inference Rule a is sound; that is, all edges marked by the proof system are forced. It follows that Inference Rule b is therefore also sound, because all forced hubs and subcycles are clearly obstructions to Hamiltonicity, and cannot contain edges which are marked but not forced. Since its axioms are clearly non-Hamiltonian and all of its inference rules are sound, the NHPS is sound, as required. □

6 Completeness

Lemma 6.1. *For any non-Hamiltonian graph $G = (V, E)$, at least one of the following holds:*

1. *G contains a vertex v such that $degree(v) \leq 1$*
2. *G contains a vertex u such that $degree(u) \geq 3$*
3. *G contains a cycle with c vertices, where $c < n$*

Proof: Suppose there exists a non-Hamiltonian graph $G = (V, E)$ which fails all three conditions. By failing conditions 1 and 2, every vertex of G must have a degree of 2. By also failing condition 3, we know that G must be a cycle containing n vertices. In other words, G is a Hamiltonian cycle, and is therefore Hamiltonian, a contradiction. □

Theorem 6.2. *The* NHPS *is Complete.*

Proof: The proof of completeness proceeds by induction on the number of edges in non-Hamiltonian graphs. We show that every non-Hamiltonian graph has a NHPS proof.

Basis: Consider any non-Hamiltonian graph that has no edges. It contains a vertex of degree 0, and is therefore a NHPS axiom, which is a NHPS proof, as required.

Induction Hypothesis: Suppose that each non-Hamiltonian graph with m edges has a NHPS proof.

Induction Step: Let $G = (V, E)$ be any arbitrary non-Hamiltonian graph with $m + 1$ edges. We know from Lemma 6.1 that at least one of the following cases holds:

1. G contains a vertex v such that $degree(v) \leq 1$
2. G contains a vertex u such that $degree(u) \geq 3$
3. G contains a cycle with c vertices, where $c < n$

If case 1 holds, then G is an axiom, so it has a NHPS proof. If case 2 holds, then G contains some vertex that would be a hub if its edges were forced. If case 3 holds, then G contains some proper subset of edges that if marked would constitute a forced subcycle. Every edge in a non-Hamiltonian graph is vacuously forced, so regardless of whether case 2 or case 3 is chosen, we need simply show that the edges corresponding to the relevant obstruction are forced, thereby showing G to be non-Hamiltonian. Let us use G_M to refer to G with these edges marked. Let us use G_1, G_2, ..., G_k each to refer to G with exactly one of these edges removed. Since G is non-Hamiltonian, and removing edges preserves non-Hamiltonicity, each of G_1, G_2, ..., G_k is also non-Hamiltonian. Therefore, by the induction hypothesis, each of G_1, G_2, ..., G_k has a NHPS proof of non-Hamiltonicity. By placing G, G_M, and G_1, G_2, ..., G_k inside proof-tree nodes, and creating an edge corresponding with Inference Rule b from G_M to G, edges corresponding with Inference Rule a from G_1, G_2, ..., G_k to G_M, and nesting their NHPS proofs below, we can create an NHPS proof of non-Hamiltonicity for G. Therefore, in any case, G has a NHPS proof of non-Hamiltonicity.

Therefore, by induction, every non-Hamiltonian graph has a NHPS proof of non-Hamiltonicity, as required. □

7 NHPS Simplification

It is worth noting that the marked edges used by the NHPS are not actually necessary, and can be left out. They clarified the justification for why the proof system is correct, but with that understood, it is easy to see that they are dispensable. More specifically, we can combine rules of inference a and b into a single rule as follows:

Let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, ... $G_k = (V, E_k)$ be unmarked graphs with NHPS proofs of non-Hamiltonicity. If the following hold:

1. For each $G_i = (V, E_i)$, there exists exactly one edge e_i such that $E = E_i \cup \{e_i\}$.
2. $\bigcup_{i=1}^{k} \{e_i\}$ constitutes either a hub or a forced subcycle.

Then we may infer $G = (V, E)$.

Eliminating marked edges from the system clearly simplifies it.

8 Exponential Lower Bounds

This proof system is weak in the sense that some infinitely large families of graphs require exponentially long NHPS proofs of non-Hamiltonicity. Three such families are shown below in Figure 3.

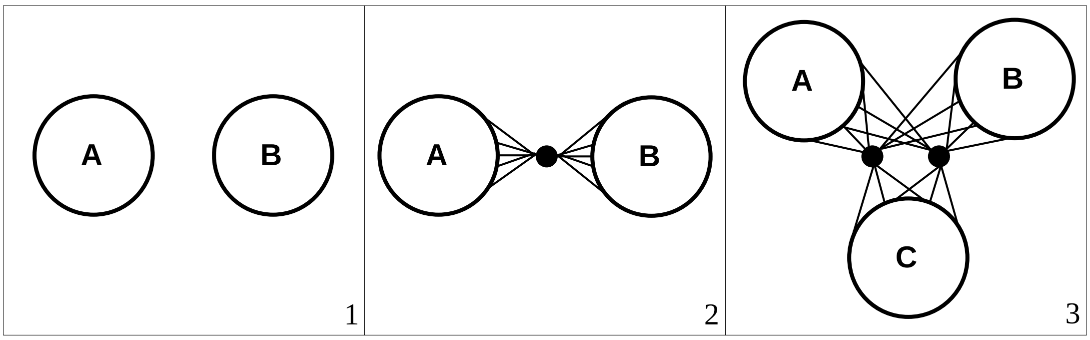

Figure 3: Graph Families Requiring Exponentially Long NHPS Proofs

The circles labelled 'A', 'B', and 'C' represent complete subgraphs, each containing $O(n)$ vertices. The family represented by Figure 3.1 contains all disconnected graphs containing a constant number of complete components such that each component contains $O(n)$ vertices. The family represented by Figure 3.2 contains all graphs consisting of a constant number of size $O(n)$ complete components separated by a single cut vertex which is adjacent to all other vertices. Finally, the family represented by Figure 3.3 contains all graphs containing a constant number $q \geq k+1$ of size $O(n)$ complete components which are held together by k vertices that are incident on all other vertices. A graph is said to be 1-tough if it contains no subset of k vertices, that, if removed, would disconnect the graph into $k+1$ or more components. All of the graphs in these families are non-Hamiltonian because none of them are 1-tough, which is a necessary condition for Hamiltonicity [Her04]. In fact, family 3 subsumes families 1 and 2, but the first two families also happen to violate biconnectivity, making it worthwhile to study them separately.

Theorem 8.1. *The lengths of* NHPS *proofs have* $\Omega(3^n)$ *size-lower bounds.*

Proof: Consider any of the graph families shown in Figure 3. Let G be any arbitrary graph chosen from any of these families. Since the minimum-degree vertex in G has $\Omega(n)$ edges incident on it, creating an axiom containing a degree 0 or degree 1 vertex requires a removal of $\Omega(n)$ edges. The number of edges in the nodes of any NHPS proof decreases linearly with the depth of those nodes, so the leaves in a NHPS proof of G's non-Hamiltonicity must all occur at a depth of $\Omega(n)$ or greater. Furthermore, every application of Inference Rule a must cause the proof tree to branch by at least 3, since forced hubs and subcycles both contain at least 3 edges. Every tree node at an odd depth is a marked graph, and requires an application of Rule a in order to justify its marked edges. In other words, every tree node at an odd depth branches into at least three more nodes. Since the depth of the tree is $\Omega(n)$, this branching behavior will be repeated recursively for every node at $\Omega(n)$ depths. This corresponds to a minimum proof tree which contains $3^{\Omega(n)} = \Omega(3^n)$ nodes. Since G was arbitrary, the growth rate for NHPS proof lengths for each of the families shown in Figure 3 has an exponential lower bound, as required. □

9 Relationship Between NHPS and Other Proof Systems

As might be expected from the lower bounds above, NHPS is a fairly weak proof system. As explained in [HH06], because a reduction is always required, and since a reduction can either introduce or reduce complexity, it does not make sense to talk about two proof systems over different languages p-simulating each other. Instead, we need the more general notion of effective p-simulation:

Definition 9.1. *Let* $f_1 : S_1^* \rightarrow L$ *and* $f_2 : S_2^* \rightarrow L$ *be proof systems. If there exists a* k *and a polytime reduction* $r : L_1 \rightarrow L_2$ *such that* $y \in L_1$ *if and only if* $r(y) \in L_2$ *and for all* $x_1 \in S_1^*$ *there exists an* $x_2 \in S_2^*$ *such that* $r(f_1(x_1)) = f_2(x_2)$ *and* $|x_2| \leq |x_1|^k$*, then we say that* f_2 *effectively polynomially-simulates* f_1*.*

If there also exists a polytime computable function $t : S_1^* \rightarrow S_2^*$ *such that for all* $x \in S_1^* r(f_1(x)) = f_2(t(x))$*, then* f_2 *effectively p-simulates* f_1*.*

9.1 Effective P-Simulation By Tree Resolution

This definition allows us to compare the strength of NHPS to other proof systems. It turns out that NHPS can be effectively p-simulated by Tree Resolution (T-RES). The reduction which makes this simulation possible is given in [HH06]. Intuitively, the reduction takes an input graph G and produces a formula that enforces a mapping from the vertices in G to the positions of a Hamiltonian Cycle. The mapping must be a bijection, and can include clauses that enforce the mapping to be Total, 1-1, a Function, and Onto. In addition, there are clauses which enforce the edge structure of G. The output formula has variables of the form $m_{i,j}$ which are interpreted as meaning that vertex i in G is mapped to position j in the Hamiltonian Cycle. Let the resulting formulas of this reduction be called $H(G)$. Additional subscripts are added to this notation to indicate which clauses were used to enforce the bijection. For example, if the reduction used clauses from the total and 1-1 groups, then the resulting formula is labeled as $H(G)_{T,1}$. We are interested in the version of the reduction which uses all of the clauses.

Theorem 9.2. *Under the reduction described in [HH06] which uses all of the clause groups,* T-RES *effectively p-simulates* NHPS*.*

Proof: The proof relies on the Prover / Delayer Game upper bounds described in [HU06]. (Not finished yet.)

□

9.2 Effective Separation Between T-RES and NHPS

Beyond the effective p-simulation of NHPS by T-RES, there in fact exists an effective exponential separation between them. Recall that that the NHPS has exponential lower bounds for the disconnected graphs shown in Figure 3 above. More formally, let $G_{\frac{n}{2},\frac{n}{2}}$ be the graph consisting of two disjoint cliques of size $\frac{n}{2}$. If we take these graphs and apply the reduction from [HH06] which uses the T, O, 1, and F clauses, the resulting formulas have polynomial T-RES upper bounds:

Theorem 9.3 ([HU06]). T-RES *proofs for the unsatisfiability of $H(G_{\frac{n}{2},\frac{n}{2}})_{T,O,1,F}$ formulas have $O(n^3)$ size upper bounds, where n is the number of distinct variables contained in the formulas.*

Since the NHPS has exponential lower bounds for the $G_{\frac{n}{2},\frac{n}{2}}$ graphs, but T-RES has polynomially-bounded proofs for the $H(G_{\frac{n}{2},\frac{n}{2}})_{T,O,1,F}$, we have an effective separation between the two proof systems.

10 Future Research

There are a few main avenues that this research might take:

10.1 Lower Bounds For A Stronger NHPS

The NHPS as described above is the most simplistic and weak version; it has only one simple axiom scheme, and both hubs and forced subcycles are required for completeness. The lower bounds are correspondingly simplistic. A natural course of research would be to strengthen the NHPS and develop more interesting and sophisticated lower-bounds arguments.

There are many ways in which the NHPS could be strengthened. Note that these methods are all disjoint; they are orthogonal in the sense that they are all so different that they do not interfere with one another. In other words, any arbitrary number of them could be combined in order to maximize the strength of the system.

10.1.1 Add More Axioms

The most obvious way to defeat the lower bounds arguments so that none of the three families of graphs shown in Figure 3 require long proofs is by adding more axiom schemes. Checking for connectivity can be done in polynomial time, and more importantly, disconnected graphs have polynomially-sized certificates, so there is no reason why disconnectivity cannot be an axiom. The inclusion of such an axiom scheme would mean that graphs from the family shown in Figure 3.1 would no longer require exponentially large NHPS proofs, but would rather only require a single proof tree node in order to show that they are not Hamiltonian. Defeating the lower bounds for the family of graphs shown in Figure 3.2 would require a similar axiom scheme, this time for non-biconnectivity, which also has a polynomially-sized certificate. Finally, the family of graphs shown in Figure 3.3 would require an axiom scheme for non-1-toughness. Recognizing 1-tough graphs is co$\mathcal{NP}$-Hard [BHS90], so in practice applying such an axiom may be computationally intractable, but recognizing non-1-tough graphs is in $\mathcal{NP}$, and therefore has a short certificate, allowing us to include this axiom scheme as well. Since the third family of graphs subsumes the other two, including just the non-1-toughness axiom would suffice to defeat the lower bounds for all three families. A natural course of research would be to find lower bounds for the NHPS after it has been strengthened with the non-1-toughness axiom.

10.1.2 Add More Obstructions To Hamiltonicity

In its simplest form, the NHPS's definition of 'obviously non-Hamiltonian' contains hubs and forced subcycles as its only obstructions to Hamiltonicity. Another way to strengthen the proof system is by adding more obstructions to this list. Two more examples of obstructions to Hamiltonicity are 'barricades' and 'odd-forced-cuts' [Her04]. A barricade is a forced edge between the two vertices of any 2-vertex-cut, and is shown below in Figure 4.1. An odd-forced-cut is a cut across an odd number of edges, all of them forced. Figure 4.2 below illustrates a graph containing such cuts.

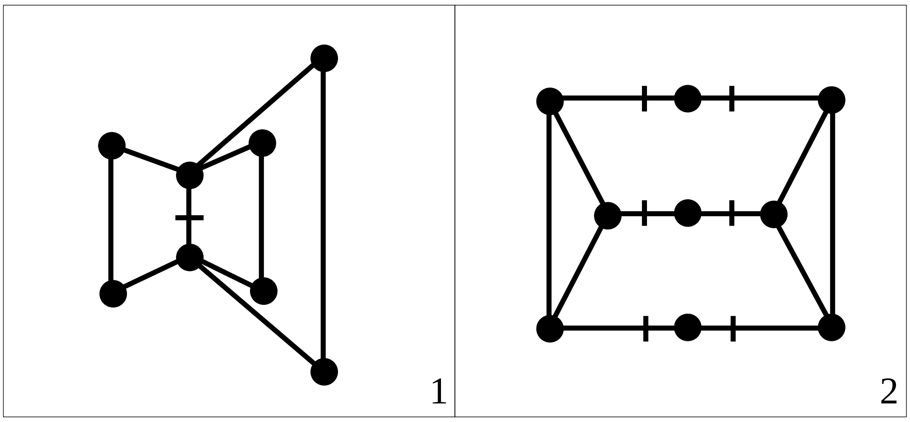

Figure 4: A graph containing a barricade (left), and a graph containing 8 odd-forced-cuts (right)

Adding these obstructions to the NHPS would not help to defeat the established lower bounds, but might be combined with some other way of strengthening the system to yield an interesting result.

10.1.3 Restrict The Input Class

Strengthening the NHPS does not necessarily involve editing its rules or structure. Instead, we could restrict its input to classes of graphs for which the Hamiltonian Cycle problem nonetheless remains $\mathcal{NP}$-Complete. The Hamiltonian Cycle problem has been well-studied, and there are many such classes. For example, it remains $\mathcal{NP}$-Complete even when restricted to the following inputs:

1. Bipartite Graphs [Kri75]
2. Planar, Cubic, 3-Connected Graphs with no face having fewer than 5 sides [GJT76]
3. Planar, Cubic, 2-Connected, Bipartite Graphs [ANS80]
4. Cubic, 3-Connected, Bipartite Graphs [ANS80]
5. Maximal Planar Graphs (Triangulations) [Chv85, Wig82]
6. 4-Connected, 4-Regular Graphs [Ros03]

As might be expected, more restrictive classes strengthen the NHPS more. For example, if we limit our inputs to planar, cubic, 3-connected graphs with no face having fewer than 5 sides, then the graph families shown in Figure 3 are immediately excluded from consideration because they are neither planar nor cubic.

Restricting ourselves to cubic graphs is desirable when strengthening the NHPS because it ensures that each vertex has a constant degree. Since the NHPS depends on edge removals in order to expose degree-1 vertices (axioms), we are guaranteed that each degree-1 vertex can add a depth of at most 2 to the proof tree. Contrast this with the graph families in Figure 3, where each vertex requires a depth of $\Omega(n)$. Even the weakest form of the NHPS appears to be quite useful when constrained to this input class.

Restricting ourselves to planar graphs is further desirable because it allows us to include an axiom scheme for Grinberg's Theorem [BM76]. Grinberg's Theorem only applies to planar graphs, and its usefulness for proving graphs to be non-Hamiltonian should not be understated.

Theorem 10.1 (Grinberg's Theorem). *Every planar graph $G = (V, E)$ containing a Hamiltonian Cycle C satisfies the following equation:*

$$\sum_{i=3}^{n}(i-2)(f_i' - f_i'') = 0, \tag{1}$$

where f_i' represents the number of faces of degree i on the interior of C, and f_i'' represents the number of faces of degree i on the exterior of C.

If the Grinberg equation associated with a planar graph is inconsistent, then the graph is non-Hamiltonian. Proofs showing that equations are inconsistent can be very short, thus allowing us to include Grinberg's condition as an axiom scheme if we desire.

10.1.4 Allow DAG-Like Proofs

In its simplest form, the NHPS is Tree-like. A DAG-like NHPS proof is essentially a NHPS proof in which we allow proof nodes to be re-used an arbitrary number of times. This type of re-use requires us to use unlabelled graphs, or provide some kind of scheme for certifying isomorphism, neither of which are problems. For example, Figure 5 below shows a DAG-Like proof for the non-Hamiltonian graph consisting of two K_4 components.

The graph in the root is a member of one of the families that yielded our exponential lower bound. Unlike with tree-like proofs in which every successive level of the tree sees at least a tripling of the number of nodes from the level before, the ability to re-use nodes shortens our proof considerably. The argument used to establish our lower bounds therefore no longer holds.

Establishing lower bounds for the DAG-like NHPS raises some interesting questions. For example, does the DAG-like NHPS have polynomially-sized proofs for all non-biconnected graphs?

References

[ANS80] T. Akiyama, T. Nishizeki, and N. Saito. NP-Completeness of the Hamiltonian Cycle Problem for Bipartite Graphs. *Journal of Information Processing*, Vol. 3 No. 2:73–76, 1980.

[BHS90] D. Bauer, S.L. Hakimi, and E. Schmeichel. Recognizing Tough Graphs is NP-Hard. *Discrete Appl. Math.*, Vol. 28:191–195, 1990.

[BM76] J.A. Bondy and U.S.R Murty. *Graph Theory With Applications*. Macmillan, London, 1976.

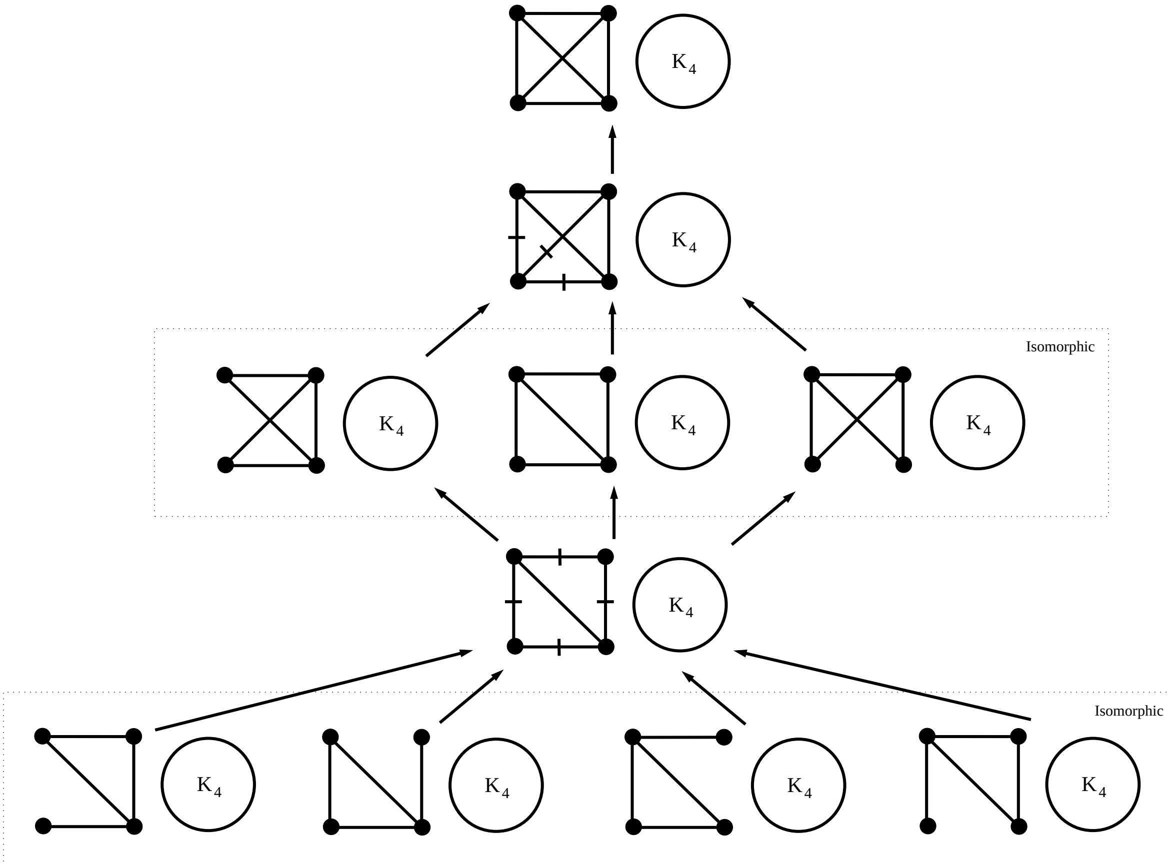

Figure 5: A DAG-Like NHPS Proof

[Chv85] V. Chvátal. Hamiltonian cycles. In E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoys, editors, *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, pages 403–429. John Wiley & Sons Ltd., New York, 1985.

[GJT76] M.R. Garey, D.S. Johnson, and R.E. Tarjan. The Planar Hamiltonian Circuit Problem is NP-Complete. *SIAM Journal of Computing*, Vol. 5 No. 4:704–714, 1976.

[Her04] A. Hertel. Hamiltonian Cycles In Sparse Graphs. Master's thesis, University of Toronto, 2004.

[HH06] A. Hertel and P. Hertel. Formalizing Dangerous Reductions. 2006.

[HU06] A. Hertel and A. Urquhart. Prover / Delayer Game Upper Bounds For Tree Resolution. 2006.

[Kri75] M.S. Krishnamoorthy. An NP-Hard Problem in Bipartite Graphs. *SIGACT News*, Vol. 7, No. 1:26, 1975.

[PU95] T. Pitassi and A. Urquhart. The Complexity of the Hajós Calculus. *SIAM J. Disc. Math.*, Vol. 8, No. 3:464 – 483, 1995.

[Ros03] M. Rosenfeld. Personal Communication, November 2003.

[Wig82] A. Wigderson. The Complexity Of The Hamiltonian Circuit Problem For Maximal Planar Graphs. *Technical Report, Princeton University, Dept. of EECS*, Vol. 298, February, 1982.