Towards a Constructive Version of Banaszczyk's Vector Balancing Theorem

Daniel Dadush [1] Shashwat Garg [2] Shachar Lovett [3]

Sasho Nikolov [4]

[1]CWI

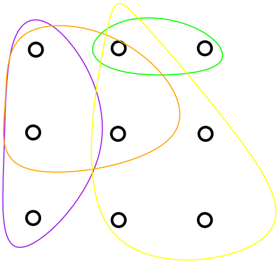
[2]TU Eindhoven

[3]UCSD

[4]U of Toronto

Discrepancy of Set Systems

Given: System of m subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$ of $[n] = \{1, ldots, n\}$.
Color each element of P red or blue, so that *each* set is as balanced as possible.

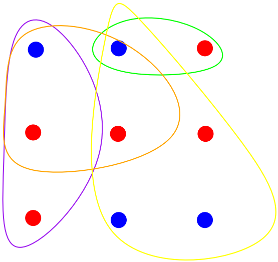


Discrepancy of a coloring: maximum imbalance (above: 1).
Discrepancy of $\mathcal{S}$: discrepancy of the best coloring.

$$\operatorname{disc} \mathcal{S} := \min_{\chi:[n]\to\{-1,1\}} \max_i \left| \sum_{j \in S_i} \chi(j) \right|$$

Discrepancy of Set Systems

Given: System of m subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$ of $[n] = \{1, ldots, n\}$.
Color each element of P red or blue, so that *each* set is as balanced as possible.



Discrepancy of a coloring: maximum imbalance (above: 1).
Discrepancy of $\mathcal{S}$: discrepancy of the best coloring.

$$\operatorname{disc} \mathcal{S} := \min_{\chi:[n]\to\{-1,1\}} \max_i \left| \sum_{j \in S_i} \chi(j) \right|$$

Beck-Fiala

Theorem ([Beck and Fiala, 1981])

Suppose each $i \in [n]$ *appears in at most* t *sets of* $\mathcal{S}$. *Then* $\operatorname{disc} \mathcal{S} \leq 2t - 1$.

Beck-Fiala Conjecture. $\operatorname{disc} \mathcal{S} = O(\sqrt{t})$.

Beck-Fiala

Theorem ([Beck and Fiala, 1981])

Suppose each $i \in [n]$ appears in at most t sets of $\mathcal{S}$. Then $\operatorname{disc} \mathcal{S} \leq 2t - 1$.

Beck-Fiala Conjecture. $\operatorname{disc} \mathcal{S} = O(\sqrt{t})$.

- Recently improved to $2t - \log^* t$ [Bukh, 2013]
- No better bound known in terms of t only!
- The proof of the theorem is *an (efficient) algorithm*!

Komlòs Conjecture

Komlòs Conjecture. For any vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ with $\max_i \|u_i\|_2 \leq 1$, there exist signs $\varepsilon_1, \ldots, \varepsilon_n$ for which

$$\left\| \sum_i \varepsilon_i u_i \right\|_\infty = O(1).$$

Komlòs Conjecture

Komlòs Conjecture. For any vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ with $\max_i \|u_i\|_2 \leq 1$, there exist signs $\varepsilon_1, \ldots, \varepsilon_n$ for which

$$\left\| \sum_i \varepsilon_i u_i \right\|_\infty = O(1).$$

- $O(1)$ is independent of m and n.
- Implies the Beck-Fiala Conjecture: Take u_j to be the j-th column of the incidence matrix of $\mathcal{S}$, scaled by $t^{-1/2}$.
  - j-th column of incidence matrix: indicator vector of $\{i : j \in S_i\}$.
  - $\sqrt{t}\|\sum_j \varepsilon_j u_j\|_\infty$ is the discrepancy of the coloring $\chi(j) = \varepsilon_j$.

Banaszczyk's Theorem

Theorem ([Banaszczyk, 1998])

Let X be a standard m-dimensional Gaussian, and let K be a convex body in $\mathbb{R}^m$ such that $\Pr[X \in K] \geq 1/2$.

Banaszczyk's Theorem

Theorem ([Banaszczyk, 1998])

Let X be a standard m-dimensional Gaussian, and let K be a convex body in $\mathbb{R}^m$ such that $\Pr[X \in K] \geq 1/2$.

For any vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ with $\max_i \|u_i\|_2 \leq 1/5$, there exist signs $\varepsilon_1, \ldots, \varepsilon_n$ for which

$$\sum_i \varepsilon_i u_i \in K.$$

Banaszczyk's Theorem

Theorem ([Banaszczyk, 1998])

Let X be a standard m-dimensional Gaussian, and let K be a convex body in $\mathbb{R}^m$ such that $\Pr[X \in K] \geq 1/2$.

For any vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ with $\max_i \|u_i\|_2 \leq 1/5$, there exist signs $\varepsilon_1, \ldots, \varepsilon_n$ for which

$$\sum_i \varepsilon_i u_i \in K.$$

- The proof is **not** an efficient algorithm!
- By taking $K = O(\sqrt{\log m}) \cdot [-1, 1]^m$, we get a bound of $O(\sqrt{\log m})$ for Komlòs and $O(\sqrt{t \log m})$ for Beck-Fiala.
  - Recent algorithmic proof of these bounds, but not the full theorem, in [Bansal, Dadush, and Garg, 2016].

Banaszczyk's Theorem

Theorem ([Banaszczyk, 1998])

Let X be a standard m-dimensional Gaussian, and let K be a convex body in $\mathbb{R}^m$ such that $\Pr[X \in K] \geq 1/2$.

For any vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ with $\max_i \|u_i\|_2 \leq 1/5$, there exist signs $\varepsilon_1, \ldots, \varepsilon_n$ for which

$$\sum_i \varepsilon_i u_i \in K.$$

- The proof is **not** an efficient algorithm!
- By taking $K = O(\sqrt{\log m}) \cdot [-1,1]^m$, we get a bound of $O(\sqrt{\log m})$ for Komlòs and $O(\sqrt{t \log m})$ for Beck-Fiala.
  - Recent algorithmic proof of these bounds, but not the full theorem, in [Bansal, Dadush, and Garg, 2016].
- Also used in approximation algorithm for hereditary discrepancy, bounds on discrepancy of boxes, vector-rearrangement problems.

Interlude: Subgaussian Random Variables

Definition

A real-valued random variable X is s-subgaussian if

$$\Pr[|X| \geq t] \leq 2\exp\left(-\frac{t^2}{2s^2}\right).$$

Interlude: Subgaussian Random Variables

Definition

A real-valued random variable X is s-subgaussian if

$$\Pr[|X| \geq t] \leq 2\exp\left(-\frac{t^2}{2s^2}\right).$$

A random variable $Y \in \mathbb{R}^m$ is s-subgaussian if for every unit vector $\theta \in \mathbb{S}^{m-1}$ the marginal $\langle \theta, Y \rangle$ is s-subgaussian.

Interlude: Subgaussian Random Variables

Definition

A real-valued random variable X is s-subgaussian if

$$\Pr[|X| \geq t] \leq 2\exp\left(-\frac{t^2}{2s^2}\right).$$

A random variable $Y \in \mathbb{R}^m$ is s-subgaussian if for every unit vector $\theta \in \mathbb{S}^{m-1}$ the marginal $\langle \theta, Y \rangle$ is s-subgaussian.

I.e., an s-subgaussian random variable shrinks about as fast as a Gaussian with variance s^2 *in every direction*.

The Main Equivalence

Theorem

Let $T = \{\sum_i \pm u_i\}$ *where the vectors* $u_1, \ldots, u_n$ *satisfy* $\max_i \|u_i\|_2 \leq 1/5$*. The following two are equivalent:*

1. *Banaszczyk's theorem restricted to convex bodies* K *symmetric around* 0*.*
2. *There exists an* $O(1)$*-subgaussian* Y *supported on* T*, where* $O(1)$ *is independent of* m*,* n*, or the vectors.*

The Main Equivalence

Theorem

Let $T = \{\sum_i \pm u_i\}$ *where the vectors* $u_1, \ldots, u_n$ *satisfy* $\max_i \|u_i\|_2 \leq 1/5$*.*
The following two are equivalent:

1. *Banaszczyk's theorem restricted to convex bodies* K *symmetric around* 0*.*
2. *There exists an* $O(1)$*-subgaussian* Y *supported on* T*, where* $O(1)$ *is independent of* m*,* n*, or the vectors.*

- 2. was not known before, and we know no direct proof.
- If we can sample Y efficiently, we would have an algorithmic version of Banaszczyk's theorem!
- Using a *random walk*, we can sample an $O(\sqrt{\log m})$-subgaussian Y: recovers Banaszczyk algorithmically for symmetric K, up to a factor of $O(\sqrt{\log m})$.

2. $\Rightarrow$ 1.

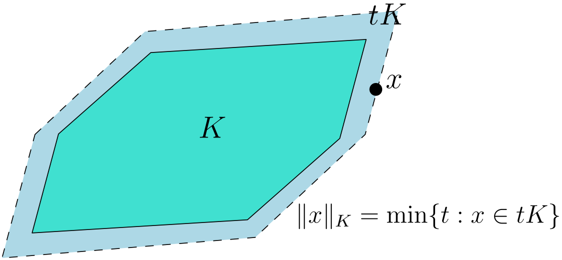
Theorem

Let X be a standard Gaussian in $\mathbb{R}^m$, and $K \subset \mathbb{R}^m$ be a symmetric convex body such that $\Pr[X \in K] \geq 1/2$. Then, for any s-subgaussian Y,

$$\Pr[Y \in O(s) \cdot K] \geq 1/2.$$

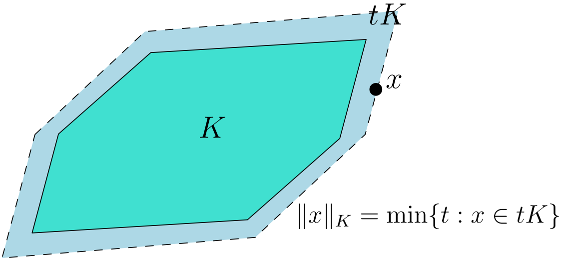
- *Universal sampler*: there is a *single* distribution on $\sum_i \pm u_i$ which works for *all* K.

Proof of Theorem



Need: $\mathbb{E}\|Y\|_K = O(s)$. Then done by Markov.

Proof of Theorem



Need: $\mathbb{E}\|Y\|_K = O(s)$. Then done by Markov.

i. [Borell, 1975] For any symmetric convex body K, and a standard Gaussian X, $\Pr[X \in K] \geq 1/2 \Rightarrow \mathbb{E}\|X\|_K = O(1)$.

ii. [Talagrand, 1987] For any s-subgaussian Y, and any symmetric convex body K, $\mathbb{E}\|Y\|_K = O(s) \cdot \mathbb{E}\|X\|_K$.

From i. and ii., we get $\mathbb{E}\|Y\|_K = O(s)$.

1. $\Rightarrow$ 2.

Define a *zero-sum game*:

- **Min** has strategies $T = \{\sum_i \pm u_i\}$.
- **Max** player has strategies $\{v \in \mathbb{R}^m\}$.
- The payoff of $y \in T$ and $v \in \mathbb{R}^m$ is $(e^{\langle y,v \rangle} + e^{-\langle y,v \rangle})/e^{\|v\|_2^2/2}$.

1. ⇒ 2.

Define a *zero-sum game*:

- **Min** has strategies $T = \{\sum_i \pm u_i\}$.
- **Max** player has strategies $\{v \in \mathbb{R}^m\}$.
- The payoff of $y \in T$ and $v \in \mathbb{R}^m$ is $(e^{\langle y,v\rangle} + e^{-\langle y,v\rangle})/e^{\|v\|_2^2/2}$.

Using Banaszczyk's theoremm, and the von Neumann min-max principle, we can bound the value of the game:

$$\min_{Y \text{ r.v. supp. on } T} \quad \max_{v\in\mathbb{R}^m} \mathbb{E}\left[\frac{e^{\langle Y,v\rangle} + e^{-\langle Y,v\rangle}}{e^{\|v\|_2^2/2}}\right] \leq 2.$$

Implies $\mathbb{E}[e^{|\langle Y,v\rangle|}] \leq 2e^{\|v\|_2^2/2}$. By Chernoff trick, Y is $O(1)$-subgaussian.

Asymmetric Bodies

Does an efficient sampler for $O(1)$-subgaussian Y imply algorithmic Banaszczyk for asymmetric K?

Asymmetric Bodies

Does an efficient sampler for $O(1)$-subgaussian Y imply algorithmic Banaszczyk for asymmetric K?

Bad News: Take $K = \{x \in \mathbb{R}^m : x_1 \leq 0\}$ and $Y = e_1$. Then:

- Y is $O(1)$-subgaussian
- $\Pr[X \in K] = 1/2$ for standard Gaussian X.
- For any $t > 0$, $Y \notin tK = K$.

Asymmetric Bodies

Does an efficient sampler for $O(1)$-subgaussian Y imply algorithmic Banaszczyk for asymmetric K?

Bad News: Take $K = \{x \in \mathbb{R}^m : x_1 \leq 0\}$ and $Y = e_1$. Then:

- Y is $O(1)$-subgaussian
- $\Pr[X \in K] = 1/2$ for standard Gaussian X.
- For any $t > 0$, $Y \notin tK = K$.

Good news: If K's *barycenter* $b(K) = \mathbb{E}[X \cdot 1\{X \in K\}]$ is at the origin, then $\Pr[Y \in O(1) \cdot (K \cap -K)] \geq 1/2$.

Asymmetric Bodies

Does an efficient sampler for $O(1)$-subgaussian Y imply algorithmic Banaszczyk for asymmetric K?

Bad News: Take $K = \{x \in \mathbb{R}^m : x_1 \leq 0\}$ and $Y = e_1$. Then:

- Y is $O(1)$-subgaussian
- $\Pr[X \in K] = 1/2$ for standard Gaussian X.
- For any $t > 0$, $Y \notin tK = K$.

Good news: If K's *barycenter* $b(K) = \mathbb{E}[X \cdot 1\{X \in K\}]$ is at the origin, then $\Pr[Y \in O(1) \cdot (K \cap -K)] \geq 1/2$.

We design a *recentering procedure* that

- Either finds signs $\varepsilon_1, \ldots, \varepsilon_n$ such that $\sum_i \varepsilon_i u_u \in K$,
- Or reduces to the case when $b(K) = 0$.

Open Problems

- Find a direct proof that there exists an $O(1)$-subgaussian Y supported on $\{\sum_i \pm u_i\}$.
- Find an efficient algorithm to sample Y.

Thank you!

Wojciech Banaszczyk. Balancing vectors and Gaussian measures of *n*-dimensional convex bodies. *Random Structures Algorithms*, 12(4): 351–360, 1998. ISSN 1042-9832.

Nikhil Bansal, Daniel Dadush, and Shashwat Garg. Algorithm for Komlós conjecture: Matching Banaszczyk's bound. To appear in FOCS 2016., 2016.

József Beck and Tibor Fiala. "Integer-making" theorems. *Discrete Appl. Math.*, 3(1):1–8, 1981. ISSN 0166-218X.

Christer Borell. The Brunn-Minkowski inequality in Gauss space. *Invent. Math.*, 30(2):207–216, 1975. ISSN 0020-9910.

Boris Bukh. An improvement of the Beck-Fiala theorem. *CoRR*, abs/1306.6081, 2013.

Michel Talagrand. Regularity of Gaussian processes. *Acta Math.*, 159 (1-2):99–149, 1987. ISSN 0001-5962. doi: 10.1007/BF02392556. URL `http://dx.doi.org/10.1007/BF02392556`.