CSC 2410 - Assignment 1

Due Jan 29, 2004

General rules : In solving this you may consult books and you may also consult with each other, but you must each write your on solution. In each problem list the books and people you consulted. This list will not affect your grade.

1. Let k, l be positive integers and let $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_l)$ be two sequences of non-negative integers. You need to decide whether there is a bipartite graph in which $a_1, a_2, \ldots, a_k$ are the degrees for one partite set, and $b_1, b_2, \ldots, b_l$ are the degrees for the other. Describe a polynomial algorithm to solve this problem. Show correctness of your algorithm and analyze its running time.

2. In class we showed that among the n-vertex simple graphs with no $r+1$ clique, the Turán graph $T_{n,r}$ has the most number of edges. Show that it is the *unique* graph having no $r+1$ clique with that many edges. You may assume for simplicity that n is divisible by r.

3. Let S and T be trees with leaves (i.e. vertices of degree 1) $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$, respectively. Suppose that $d_S(x_i, x_j) = d_T(y_i, y_j)$ for each pair i, j. Prove that S and T are isomorphic.

4. (a) Prove or disprove : If T is a minimum-weight spanning tree of a weighted graph G, and u, v are two vertices in G, then the path connecting u and v in T is the shortest path between u and v in G.

   (b) We have shown that minimum-weight spanning trees are exactly the ones that minimize the *product* of the weights of their edges (in addition to minimizing the sum). What about other operators? A *minimax* spanning tree is a spanning tree that minimizes the maximal edge weight. Show that every minimum-weight spanning tree is also a minimax spanning tree, and show an example that the converse is false.

5. Let G be a weighted graph with vertices $\{v_1, \ldots, v_n\}$. Here the weights might be negative but thare are no negative-cycles (i.e. cycles whose edge-weights sum to a negiative number). Denote by $A^{[k]}(i,j)$ the length of the shortest path from v_i to v_j that is limited to passing through no other vertices but $S_k = \{v_1, \ldots, v_k\}$. In other words paths that start with v_i end with v_j and whose internal vertices are in S_k. For $k > 0$ express $A^{[k]}(i,j)$ as a function of $A^{[k-1]}(i',j')$ (for appropriate choices of i' and j'). Use this relation to devise an efficient algorithm to compute the distances between all pairs of vertices in the graph.

6. Let $A = (A_1, \ldots, A_m)$ be a collections of subsets of a set Y. A system of distinct representatives (SDR) for A is a set of distinct elements $a_1, \ldots, a_m$ in Y such that $a_i \in A_i$. Prove that A has an SDR if and only if $|\bigcup_{i \in S} A_i| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$.