CSC 2410 - Assignment 2

Due Mar 1, 2004

General rules : In solving this you may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the books and people you consulted. This list will not affect your grade.

1. Show that for any bipartite graph G on partite sets X, Y ($|X|, |Y| > 1$), the following three statements are equivalent:
   (a) G is connected and each edge of G is contained in a perfect matching.
   (b) For any $s \in X$ and $t \in Y$, $G - s - t$ has a perfect matching.
   (c) $|X| = |Y|$ and for each $\emptyset \neq S \subset X$, $|N(S)| > |S|$.
2. Let G be a graph with $2n$ vertices, all have degree at least n. Show that G has a perfect matching.
3. Let G be a k-edge-connected graph and let $F_1, \ldots, F_m$ be k-element cutsets of edges. Prove that $G - (F_1 \cup \ldots \cup F_m)$ has at most $2m$ connected components.
4. Use network flows to prove Menger's theorem for nonadjacent vertices in (undirected) graphs: The minimum number of vertices whose removal disconnects nonneighbours x and y equals the maximal number of internally vertex-disjoint paths between x and y.
5. Let N be a network with edge capacities and node conservation constraints plus lower bound constraints l_e on the flow in edges, meaning that $f_e \geq l_e$ is required. Suppose you are given an initial feasible flow. Explain how can the Ford-Fulkerson algorithm be modified to search for a maximum flow in this network.

6. Let G be a directed graph on vertices $\{v_1, v_2, \ldots, v_n\}$ with no (directed) cycles. We are interested to find a minimum collection of vertex-disjoint (directed) paths that cover all the vertices of the graph. The paths can start and end at any vertex, and their length can be any (e.g. they can be of length 0, namely contain a single vertex).

   (a) Describe an efficient algorithm solving the problem.
   Hint : construct a network N on vertices $\{s,t\} \cup \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\}$ and edges from s to all x_i, from all of y_i to t and from x_i to y_j if $v_i \to v_j$ is an edge in G. All edges have capacity 1. Relate the maximum-flow in N to the problem.

   (b) Is the condition that G does not contain cycles needed for the correctness of the algorithm? Explain!