CSC 2410 - Assignment 3

Due April 15, 2004

General rules : You may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the books and people you consulted. This list will not affect your grade.

1. Show that if G is bipartite then $\chi(\overline{G}) = w(\overline{G})$ ($\overline{G}$ is the complement graph to G, ie uv is an edge in $\overline{G}$ iff uv is not an edge in G). You may not use the Perfect Graph Theorem.

2. In class we saw Wigderson's algorithm to find a $O(\sqrt{n})$ colouring of a 3-colourable graph (pick vertices of degree at least $\sqrt{n}$ and colour their neighbourhood with 2 new colours; finish by greedily $\sqrt{n}$ colouring the remaining graph when $\Delta \leq \sqrt{n}$.) Use the same ideas to get an efficient algorithm to colour a k-colourable graph with "only" $O(n^{\frac{k-2}{k-1}})$ colours.

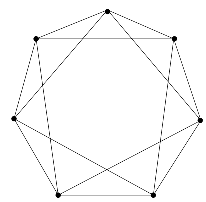
3. A graph is outerplanar if it has an embedding with every vertex on the boundary of the unbounded face.

   Without using the four colour theorem, prove that every outerplanar graph is 3-colourable. Apply this to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with n sides, then it is possible to place $\lfloor \frac{n}{3} \rfloor$ guards such that every point of the interior is visible to some guard. Show that is tight, i.e. for an infinite sequence of values for n construct a polygon that requires $\lfloor \frac{n}{3} \rfloor$ guards.

4. A graph H is a *minor* of a graph G if a copy of H can be obtained from G by deleting and/or contracting edges of G. Wagner's theorem is the same as Kuratowski's theorem except **minors** replace **subdivisions**. That is a graph is planar iff it does not have K_5 or $K_{3,3}$ as minors.

   (a) Show Wagner's theorem. You may use Kuratowski's theorem.

   (b) Show that the graph below is not planar by either using Kuratowski's theorem or Wagner's theorem.



5. Let G be a graph with no isolated vertices. Prove that if $L(G)$ is connected and regular, then either G is regular or G is a bipartite graph in which vertices of the same partite set have the same degree.

6. The kth power of a graph G is the graph G^k with vertex set $V(G)$ and edge set $\{uv : d_G(u,v) \leq k\}$. Of course G^k is a supergraph of G and the bigger k is we expect it to be more likely that G^k is Hamiltonian (having a Hamiltonian cycle). Here we show that for connected graphs G, G^2 may not be Hamiltonian but G^3 always is.

   (a) Suppose $G - x$ has at least three nontrivial (ie of more than one vertex) componenets in each of which x has exactly one neighbour. Prove that G^2 is not Hamiltonian. Hint : consider the second graph in Example 7.2.5 in West.

   (b) Prove that if G is connected with at least three vertices then G^3 is Hamiltonian. Hint : show it for trees; in fact show the stronger claim that for each edge xy in the tree T there is a Hamiltonian cycle in T^3 using the edge xy.

7. The MAX-CUT in a graph is the maximal number of cut edges in a partition of the vertices to two parts. Unlike MIN-CUT which is easy to compute, finding the MAX-CUT is NP-hard. We show how we can c-approximate it, that is give a partition whose cut has at least MAX-CUT$/c$ edges.

   (a) Using a random partition of the vertices show how you can get a ranomized algorithm that acheives 2-approximation with high probabiliy.

   (b) Use a random equipartition of the vertices (i.e. partitions of equal size) to improve the previous part: if G has m edges and n (say an even number) vertices, show how to get a $\frac{n-1}{n/2}$-approximation with high probability.