1

Show that for any bipartite graph G on partite sets X, Y ($|X|, |Y| > 1$), the following three statements are equivalent:

(a) G is connected and each edge of G is contained in a perfect matching.

(b) For any $s \in X$ and $t \in Y$, $G - s - t$ has a perfect matching.

(c) $|X| = |Y|$ and for each $\emptyset \neq S \subset X$, $|N(S)| > |S|$.

(b) $\Rightarrow$ (a)
We shall show that G is connected. Suppose, by way of contradiction, that G is not connected and let $\{C_1, \ldots, C_l\}$ be its components. First, $|C_i| \geq 2$ for each $1 \leq i \leq l$. Otherwise, let $C_i = \{x\}$, and we pick $s \in X$ and $t \in Y$ such that $x \neq s$ and $x \neq t$, then $G - s - t$ has no perfect matching. Next, each C_i has the same number of vertices in X and in Y. Otherwise, consider another component C_j, pick $s \in X$ and $t \in Y$ from C_j (note $|C_j| \geq 2$ by the previous argument), then $G - s - t$ has no perfect matching. Now, we pick $s \in X$ in C_i and $t \in Y$ in C_j, then $G - s - t$ has no perfect matching since both $C_i - s$ and $C_j - t$ have an odd number of vertices, a contradiction. Therefore, G is connected.

Now for each edge $xy \in E(G)$, by (b), $G - x - y$ has a perfect matching M. Clearly $M \cup \{xy\}$ is a perfect matching in G, and thus (a) holds.

(c) $\Rightarrow$ (b)
For any $s \in X$ and $t \in Y$, let $X' = X - s$ and $Y' = Y - t$. Consider an arbitrary subset $S \subseteq X'$. Since $S \subset X$, by (c), $|N_G(S)| > |S|$ and thus $|N_{G-s-t}(S)| \geq |S|$. By Hall's Theorem, $G - s - t$ has a matching M that saturates X'. Since $|X| = |Y|$, $|X'| = |Y'|$ and therefore M is a perfect matching in $G - s - t$, and this proves (b).

(a) $\Rightarrow$ (c)
First, since G has a perfect matching by (a), $|X| = |Y|$. Consider an arbitrary set $\emptyset \neq S \subset X$. Since G is connected and $S \subset X$, there is an edge $st \in E(G)$ such that $t \in N(S)$ and $s \in X - S$. By (a), $G - st$ has a perfect matching. This implies that $|N(S) - t| \geq |S|$ and therefore $|N(S)| > |S|$, this proves (c).

2

Let G be a graph with $2n$ vertices, all have degree at least n. Show that G has a perfect matching.

Suppose, by way of contradiction, that G does not have a perfect matching. Let M be a maximum matching with $m < n$ edges and let x, y be two vertices that are not incident to any edge in M. Consider an edge $ab \in M$, if $xa, yb \in E(G)$, then $M - \{ab\} + \{xa, yb\}$ is a matching of size $m+1$. So, by our choice of M, there are at most 2 edges between $\{x, y\}$ and any edge ab in M; otherwise the above structure exists. Clearly, $xy \notin E(G)$; otherwise $M \cup \{xy\}$ is a matching of size $m+1$. Also, vertices in $N_G(x)$ and $N_G(y)$ are all in M; otherwise we can obtain a larger matching. Therefore, the number of edges in G with an endpoint in x or y is at most $2m < 2n$. This contradicts the fact that both x and y are of degree at least n. Therefore, G has a perfect matching.

3

Let G be a k-edge-connected graph and let $F_1, \ldots, F_m$ be k-element cutset of edges. Prove that $G - \{F_1, \ldots, F_m\}$ has at most $2m$ connected components.

Suppose, by way of contradiction, that $G - \{F_1, \ldots, F_m\}$ has connected components $\{C_1, \ldots, C_l\}$ where $l \geq 2m + 1$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$. Since G is k-edge-connected, there are at least k edges in $\mathcal{F}$ with one endpoint in C_i for $1 \leq i \leq l$. Since one edge has two endpoints, there are at least $\lceil lk/2 \rceil \geq mk + \lceil k/2 \rceil$ edges in $\mathcal{F}$. On the other hand, each F_i has k edges and thus $\mathcal{F}$ has at most mk edges, a contradiction. Therefore, $l \leq 2m$ and this completes the proof.

4

Use network flows to prove Menger's Theorem for nonadjacent vertices in (undirected) graphs: The minimum number of vertices whose removal disconnects nonneighbours x and y equals the maximum number of internally vertex-disjoint paths between x and y.

Let G be the given graph and x, y are nonneighbours in G. We construct a directed network N from G as follows:

1. For each vertex v in G, we create v^- and v^+ in N;
2. for each vertex v in G, we create a directed *special* edge v^-v^+ with capacity 1 in N;
3. for each undirected edge uv in G, we create two directed edges v^+u^- and u^+v^- with infinite capacity in N, where n is the number of vertices in G.

Notice that by our construction, there is an undirected path $xv_1 \ldots v_ly$ in G if and only if there is a directed path $x^+v_1^-v_1^+ \ldots v_l^-v_l^+y^-$ in N. So $\{u_1, \ldots, u_k\}$ is a x, y-vertex-cut in G if and only if $\{u_1^-u_1^+, \ldots, u_k^-u_k^+\}$ is a x^+y^--edge-cut in N. Since all other edges have infinite capacity, a minimum x^+y^--edge-cut in N only contains special edges. Therefore, the size of a minimum x^+y^--edge-cut in N is equal to the size of a minimum xy-vertex-cut in G.

Consider a maximum x^+y^--flow in N. By Ford-Fulkerson algorithm, we can assume that this is a integral flow. Since special edge has capacity 1, no two unit of flow can share a special edge. Therefore, each unit of flow corresponds to an edge-disjoint x^+y^- path in N and thus an internally vertex-disjoint xy-path in G. So, we conclude that the value of a x^+y^- maximum flow is at most the maximum number of internally vertex-disjoint xy-paths in G.

Finally, it is straightforward that the minimum size of a xy-vertex-cut in G is a upper bound on the maximum number of internally vertex-disjoint xy-paths in G. So, applying the max-flow min-cut theorem on x^+y^- in N and combining our previous arguments, we have our desired result.

5

Let N be a network with edge capacities and node conservation constraints plus lower bound constraints l_e on the flow in edges, meaning that $f_e \geq l_e$ is required. Suppose you are given an initial feasible flow. Explain how can the Ford-Fulkerson algorithm be modified to search for a maximum flow in this network.

In West's book, page 179, is the original Ford-Fulkerson algorithm. To adapt it to solve our problem, we just need to replace the sentence: "For each entering edge uv with $f(uv) > 0$ and $u \notin R$, add u to R" by the following sentence: "For each entering edge $e = uv$ with $f(uv) > l_e$ and $u \notin R$, add u to R".

To prove its correctness, we follow the proof in West's book page 180. Suppose we can find an augmentating path, then we can always increase the flow. So suppose no augmentating path can be found, we shall show that we have already found the maximum flow in N. In this case, consider the source-sink cut $[S, T]$. Every edge from S to T has no excess capacity and every edge e from T to S has flow equal to l_e. Clearly, $\sum_{e:S\to T} c_e - \sum_{e:T\to S} l_e$ is an upper bound on the maximum flow from s to t. And since the net flow out of S have already achieved this value, we have found the maximum flow in N.

6

Let G be a directed graph on vertices $\{v_1, v_2, \ldots, v_n\}$ with no (directed) cycles. We are interested to find a minimum collection of vertex-disjoint (directed) paths that cover all the vertices of the graph. The paths can start and end at any vertex, and their length can be any.
(a) Describe an efficient algorithm solving this problem.
Hint: construct a network N on vertices $\{s,t\} \cup \{x_1, x_2, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ and edges from s to all x_i, from all of y_i to t and from x_i to y_j if $v_i \to v_j$ is an edge in G. All edges have capacity 1. Related the maximum flow in N to the problem.
(b) Is the condition that G does not contain cycles needed for the correctness of the algorithm? Explain!

(a) We shall show that N has a st-flow of value $n - f$ if and only if there are f vertex-disjoint paths in G that cover all the vertices in G. Suppose there are f vertex-disjoint paths $\mathcal{P} = \{P_1, \ldots, P_f\}$ in G that cover all the vertices in G. For each path $P_j = v_1 v_2 \ldots v_l$ with $l \geq 2$, we push a unit flow through the path $s \to x_i \to y_{i+1} \to t$ for $1 \leq i \leq l-1$. Clearly, they are $l-1$ internally vertex-disjoint st-paths and thus we increase the flow by $l-1$ by using P_j. Since paths in $\mathcal{P}$ are vertex-disjoint that cover all the vertices in G, N has a st-flow of value $n - f$.

Suppose N has a st-flow of value $n - f$. By Ford-Fulkerson algorithm, we can assume this is an integral flow. We construct a path cover of size f by adding all the edges $v_i v_j$ to the path cover if $x_i y_j$ has flow 1. Since N has a st-flow of value $n - f$, we add $n - f$ edges to the path cover. Notice that each x_i has in-flow at most 1, so each v_i has out-degree at most 1 in the path cover; similarly, each y_i has out-flow at most 1, so each v_i has in-degree at most 1 in the path cover. Combining with the fact that there is no directed cycle in G (so k vertices can have at most $k - 1$ edges), there are f paths in the path cover. Therefore, finding a minimum path cover in G is reduced to finding a maximum flow in N. (In fact, this problem is just a maximum matching problem and so a faster algorithm exists).

(b) Yes, the requirement is necessary. Consider a directed graph that comes from an undirected graph (i.e. replace each undirected edge by two directed edges). By our previous construction, note that if G has a perfect matching, then a set of disjoint directed cycles of length 2 corresponds to a maximum flow in N. But obviously, this doesn't correspond to a minimum path cover. Perhaps more convincingly, we observe that the minimum path cover problem in general directed graphs is NP-complete (why?).