CSC 2411H - Assignment 2

Due March 2, 2009

1. In class we saw that a linear program P has an optimal solution if and only if a certain linear program that includes the dual program is feasible. Write down an LP that is feasible if and only if P is unbounded. Assume P is given in standard form.

2. You have an inventory of m different kinds of raw material. Specifically, you have b_j units of raw material of kind j. They are n different kinds of products that you can make from these raw material. Each unit of product i takes a_{i1} unit of raw material 1, a_{i2} units of raw material 2 etc. to create and can be sold at the price c_i dollars. It is possible to sell fractional amounts of any product. Your goal is to produce the most profitable set of products with your available material.

   (a) Formulate the above optimization problem as a linear program:

   (b) Write the dual of this program and give a natural interpretation of it (tell a story) as an optimization problem.

   (c) Use strong duality to connect the optimum of these two programs.

3. Let $p_1, p_2, \ldots, p_r$ and $q_1, q_2, \ldots, q_s$ be points in $\mathbb{R}^n$, and let $P = \text{conv}(p_1, p_2, \ldots, p_r)$ and $Q = \text{conv}(q_1, q_2, \ldots, q_r)$ be their convex hulls (recall that $\text{conv}(s_1, \ldots, s_n)$ is the set $\{\sum_i \lambda_i s_i | \lambda_i \geq 0; \sum_i \lambda_i = 1\}$.)

   We say that a hyperplane H is a strict separating hyperplane between P and Q if $P \subset H^- \setminus H$ and $Q \subset H^+ \setminus H$, where H^- and H^+ are the two halfspaces associated with H. Show that P and Q have a strict separating hyperplane if and only if they are disjoint.

4. Theorem: if $K_1, K_2, \ldots, K_t$ are convex bodies in $\mathbb{R}^n$ such that every $n+1$ of them intersect, then all of them intersect ($\cap_i K_i \neq \emptyset$).

   (a) Prove the theorem for the special case where the K_i are halfspaces.

   (b) Prove the theorem for the (less) special case where the K_i are polyhedra.

5. Assume that you are given a linear program in standard form with n variables and m constraints, such that m is very small (a constant, say). In this question we are considering algorithms that solve the linear program in time linear in n (but not necessarily linear in m). For example a running time of $O(n2^m)$ will be acceptable but a running time of $O(n^2 m)$ will not be. You may give randomized algorithms, in which case you should consider the *expected* running time of the algorithm.

Remember that a linear program in standard form is

$$
\begin{aligned}
\text{Minimize} \quad & c \cdot x \\
\text{subject to} \quad & \\
& Ax = b \\
& x \geq 0.
\end{aligned}
$$

(a) Devise an algorithm that finds the optimum *value* of the objective function, i. e., the minimum of $c \cdot x$. Your algorithm should run in time linear in n.

(b) (Bonus) Devise an algorithm that finds an optimal solution x. Your algorithm should run in time linear in n.