CSC 2411H - Assignment 1

Due Jan 24, 2005

General rules : In solving this you may consult books and you may also consult with each other, but you must each write your own solution. In each problem list the people you consulted. This list will not affect your grade.

1. **(a) minimizing an objective functions which is a (finite) maximum** Consider the following problem.

$$\begin{array}{ll} & \min_x \max_{i \in I} x_i \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

   Where $x = (x_1, \ldots, x_n)^t$, and I is a fixed subset of $\{1, \ldots, n\}$. Show how to express the problem as an LP. You may add variables and linear constraints as long as there is an easy way to get a solution to the original problem from the newly formulated problem.

   (b) Let $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be a set of n points in $\mathbb{R}^2$ so that $x_i \neq x_j$ for $i \neq j$. Your goal is to find a line $ax + b$ so that the maximal *vertical* distance between the line and the points is minimized (the vertical distance of (x_i, y_i) to the line is $|ax_i + b - y_i|$). Write an LP for the problem.

2. **BFS in nonstandard form.** Analogously to the definition of BFS for LP in standard form, we may define the BFS for

$$P = \{x : Ax \leq b\}$$

   as the points y for which there are n linearly independent inequalities satisfied as equalities, where n is the number of variables.

(a) Show that in this setting it is also the case that a point is BFS iff it is extreme. Hint : for $x \in P$, let E be the set of indices corresponding the inequalities that are satisfied as equalities in x, and I the set of indices that correspond to the inequalities that are strict in x. Now $A_I x = b_I$ and $A_E x < b_E$. **Bonus** (but not much more difficult really): show that a point is BFS of P iff it is a vertex of P.

(b) The convex hull of a set of points $S \subset \mathbb{R}^n$ is the set of all convex combinations of points from S, in other words $\text{conv}(S) = \{\sum_i \lambda_i x_i | x_i \in S, \lambda_i \geq 0, \sum_i \lambda_i = 1\}$. Show that if P is the feasible set for an LP in standard form, namely $\{x \geq 0 \mid Ax = b\}$, and further assume that P is bounded, then P is the convex hull of all the Basic Feasible Solutions to the LP. Notice that is the same as saying that the bounded polytope $\{x \geq 0 \mid Ax = b\}$ is the convex hull of its vertices.

3. **(a) size of BFS** In class we stated without proof that the length of numbers involved in a BFS are polynomially related to the input size. More precisely, consider an LP in standard form with integer coefficients, n variables and m equalities, and let $L = mn + \lceil \log |P| \rceil$, where P is the product of the nonzero coefficients in A,b and c. Then any BFS is a rational number with both the absolute value and denominators of each of its coordinates is bounded by 2^L. Prove it. Hint : Let B be a square nonsingular matrix with small coefficients, how can the coordinates of $B^{-1}b$ be bounded?.

   (b) feasibility is hard Use (a) to show that the Feasibility problem is "as hard" as Optimality problem in LP. In other words, one can use a subroutine which gives return a feasible point for LP in standard form or reports the problem is infeasible, in order to solve the optimality problem for an LP in standard form.