CSC2414 - Metric Embeddings*
Lecture 2: Expander graphs

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Summary: In this tutorial we discuss expander graphs, Cheeger constant and its relation to eigenvalues of the Laplacian.

1 Introduction

Expander graphs are of great importance in computer science, combinatorics, and many other areas of mathematics. To define the notion of a family of expanders first we need some definitions: Consider a graph $G(V, E)$ with possibly loops and multiple edges on n vertices.

- **Edge Boundary:** For a set $S \subseteq V$, define $\partial(S) = E(S, S^c)$, where $S^c = V \setminus S$.
- **Cheeger Constant:** Cheeger constant, $h(G)$, is defined as

$$h(G) = \min_{S \subseteq V} \frac{|\partial(S)|}{\min(|S|, |S^c|)}.$$

Remark 1.1. In some texts $\partial(S)$ denotes the *vertex boundary* of S which is $\{u : d(u, S) = 1\}$.

A sequence of distinct graphs $\{G_i\}_{i=1}^{\infty}$ is called a *family of expander graphs* if there exists a constant $\epsilon > 0$ such that $h(G_i) > \epsilon$ for every $i \geq 1$. For example $\{K_{i+1}\}_{i=1}^{\infty}$ is a family of expanders, where K_i is the complete graph on i vertices. To overrule such trivial examples we need another assumption and that is every G_i is d-regular, where d is a fixed constant.

Exercise 1.2. For $d < 3$, show that there is no family of d-regular expander graphs.

Before studying the properties of the family of d-regular expander graphs, we need to show that they do exist.

Theorem 1.3. *For $d \geq 3$ there exists a family of d-regular expander graphs.*

* Lecture Notes for a course given by Avner Magen, Dept. of Computer Sciecne, University of Toronto.

Proof. The proof is very simple. We show that there exists $\epsilon > 0$ such that for every even $n > 0$, if we pick a random d-regular graph G, then with positive probability $h(G) > \epsilon$.

Consider a perfect matching on dn vertices. Partition these vertices into n sets $S_1, \ldots, S_n$, each of size d. Identify the vertices in S_i to one single vertex i without eliminating any edge. We obtain a d-regular graph G with n vertices labelled as $\{1, \ldots, n\}$ with possibly loops and multiple edges. Let $S \subseteq \{1, \ldots, n\}$ be such that $|S| \leq n/2$. To bound $\partial(S)$ from below we need to bound the number of edges inside S from above. Let us bound the probability that at least $|S|(d/2 - \epsilon)$ edges are inside S. Note that we can assume $S = \{1, \ldots, |S|\}$ (why?), and then any matching edge that lies inside $S_1 \cup \ldots \cup S_{|S|}$ will be inside S in the graph G. So we need to bound from above the probability that if we choose $|S|d$ vertices from the original dn vertices we pick at least $|S|(d/2 - \epsilon)$ edges (why?). This probability is bounded by

$$\frac{\binom{dn/2}{|S|(d/2-\epsilon)}\binom{dn-|S|(d-2\epsilon)}{2\epsilon|S|}}{\binom{dn}{d|S|}}.$$

Tedious but straightforward calculation shows that for sufficiently small $\epsilon > 0$,

$$\sum_{s=1}^{n/2} \binom{n}{s} \frac{\binom{dn/2}{s(d/2-\epsilon)}\binom{dn-|S|(d-2\epsilon)}{2\epsilon s}}{\binom{dn}{sd}} < 1.$$

Note that this is the probability that $h(G) < \epsilon$. $\square$

Although the above theorem shows that the proof of existence is easy, explicit construction turns out to be very difficult. There are up to date only few methods of explicit constructions of families of expanders. The first such family is constructed by the famous mathematician Margulis in 1973[Mar73]:

For every integer m consider G_m the graph whose vertices is $\mathbb{Z}_m \times \mathbb{Z}_m$. The eight neighbors of a vertex $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are the followings:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x \qquad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} x \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} x \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where all operations are in $\mathbb{Z}_m \times \mathbb{Z}_m$. Although this construction is very simple, the proof that this provides a family of expanders is difficult and it is based on the works of Kazhdan (another great mathematician) in representation theory of semi-simple Lie groups (see [Lub94]).

2 The Cheeger inequality

Although Cheeger constant is a nice combinatorial notion, it is not very convenient to work with it directly. For example it is NP-hard to compute the Cheeger constant of a given graph. So it is desirable to find a more convenient notion that approximates it. To this end we need to define the Laplacian of a graph.

Definition 2.1. The Laplacian of a d-regular graph G is the matrix $L_G := dI - A_G$, where A_G is the adjacency matrix of G.

Since L_G is symmetric it is diagnoisable, and thus has n (not necessarily distinct) eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. It is easy to see that L_G is positive semi-definite, i.e. its eigenvalues are nonegative.

Exercise 2.2. Show that L_G is positive semi-definite and 0 is one of its eigenvalues.

The above exercise shows that the eigenvalues of L_G are $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The second eigenvalue, λ_2 captures many interesting properties of the graph including an approximation of the Cheeger constant.

Before proving this, let us recall some basic facts from linear algebra. Suppose that $v_1, \ldots, v_n$ are orthonormal eigenvectors corresponded to $\lambda_1, \ldots, \lambda_n$, respectively, i.e.

- $Lv_i = \lambda_i v_i$.
- $\langle v_i, v_j \rangle = 0$ for $i \neq j$.
- $\langle v_i, v_i \rangle = 1$.

Exercise 2.3. Show that for every vector $x \in \mathbb{R}^n$, we have $x = \sum_{i=1}^n a_i v_i$, where $a_i = \langle x, v_i \rangle$. Moreover

$$\|x\|_2^2 = \sum_{i=1}^n a_i^2.$$

Note that one can take $v_1 = \frac{1}{\sqrt{n}}(1, \ldots, 1)$ as the corresponding eigenvector of $\lambda_1 = 0$. Consider the set of all vectors x such that $\langle x, v_1 \rangle = 0$ or equivalently $\sum x_i = 0$. What is the minimum value of $\frac{x^t Lx}{\|x\|_2^2}$ over all such vectors x. We claim that

$$\lambda_2 = \min_{x : \langle x, v_1 \rangle = 0} \frac{x^t Lx}{\|x\|_2^2}. \tag{1}$$

To prove this first note that for $x := v_2$,

$$\frac{x^t Lx}{\|v_2\|_2^2} = \frac{v_2^t \lambda_2 v_2}{\|v_2\|_2^2} = \lambda.$$

On the other hand take x to be any vector that satisfies $\langle x, v_1 \rangle = 0$. Then $x = \sum_{i=2}^n a_i v_i$ and so

$$\frac{x^t Lx}{\|x\|_2^2} = \frac{\sum_{i=2}^n \lambda_i a_i^2}{\sum_{i=2}^n a_i^2} \geq \lambda_2.$$

Theorem 2.4. *For any d-regular graph G,*

$$\frac{h(G)^2}{2} \leq \lambda_2 \leq 2h(G).$$

Proof. We only prove the first inequality. First note that

$$x^t L x = d x^t x - 2 \sum_{ij \in E} x_i x_j = \sum_{ij \in E} (x_i - x_j)^2.$$

Now suppose $h(G)$ is acheived by the cut (S, S^c), where $|S| \leq |S^c|$. Consider the vector x defined as

$$x_i = \begin{cases} \frac{1}{|S|} & i \in S \\ \frac{-1}{|S^c|} & i \notin S \end{cases}$$

Then

$$x^t L x = \sum_{ij \in E} (x_i - x_j)^2 = |\partial S| \left(\frac{1}{|S|} + \frac{1}{|S^c|} \right)^2 = |\partial S| \left(\frac{n}{|S||S^c|} \right)^2,$$

and

$$\|x\|_2^2 = |S| \frac{1}{|S|^2} + |S^c| \frac{1}{|S^c|^2} = \frac{n}{|S||S^c|}.$$

So

$$\lambda_2 \leq \frac{x L x^t}{x^t x} = \frac{|\partial S| n}{|S||S^c|} \leq 2h(G),$$

as we assumed $|S| \leq |S^c|$.

$\square$

As we mentioned above instead of Cheeger constant is not very convenient to work with, and usually the second eigenvalue of the Laplacian is being used to measure how good an expander is. The following Theorem of Alon and Boppana shows that for constant d the best we can hope for is $d - 2\sqrt{d-1}$.

Theorem 2.5. *For every d-regular graph:*

$$\lambda \geq d - 2\sqrt{d-1} + o(1).$$

Lubotzky-Phillip-Sarnak [LPS88] and Margulis [Mar88] independently constructed families of expanders (so called Ramanujan expanders because the proof of the above theorem is directly connected to the Ramanujan conjectures[1]) for which $\lambda_2 \geq d - 2\sqrt{d-1}$.

[1]now, theorems.

References

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[Lub94] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures*, volume 125 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994. With an appendix by Jonathan D. Rogawski.

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