CSC236H: Introduction to the Theory of Computation

Homework 3

Due on Thursday March 4, 2010 (in class)

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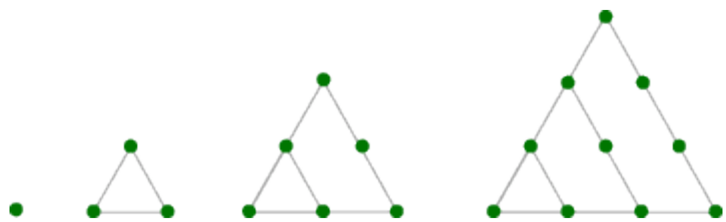
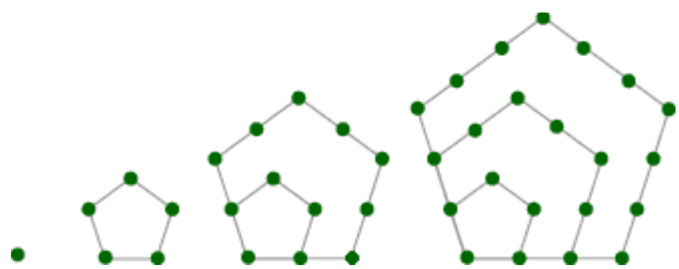
1. Early members of the Pythagorean Society defined figurate numbers to be the number of dots in certain geometrical configurations.

   (a) The first four triangular numbers are 1, 3, 6, and 10.

   Find a recursive expression for the nth triangular number. Guess (you may use substitution, but note that the answer is very easy, and you have seen it before) the closed form for this function and prove your answer to be correct.

   (a) The first four pentagonal numbers are 1, 5, 12, and 22.

   Find a recursive expression for the nth pentagonal number. Guess the closed form for this function and prove your answer to be correct.

2. Define the function f as follows:

   - $f(1) = 1$
   - $f(2) = 5$
   - $f(n+1) = 5f(n) - 6f(n-1)$

   (a) Compute $f(3)$ and $f(4)$.

   (b) Use strong induction to prove that $f(n) = 3^n - 2^n$ for every positive integer n.

3. Define a set $M \subseteq \mathbb{Z}^2$ as follows

   (1) $(3, 2) \in M$

   (2) If $(x, y) \in M$, then $(3x - 2y, x) \in M$

   Use structural induction to prove that elements of M always have the form $(2^{k+1} + 1, 2^k + 1)$, where k is a natural number. (The point of this problem is to learn how to use structural induction, so you may not rephrase this into a normal proof by induction on k.)

4. The Fibonacci trees T_k are a special sort of binary trees that are defined as follows.

   Base: T_1 and T_2 are binary trees with only a single vertex.

   Induction: For any $n \geq 3$, T_n consists of a root node with T_{n-1} as its left subtree and T_{n-2} as its right subtree.

   Use structural induction to prove that the height of T_n is $n-2$, for any $n \geq 2$. (Again, use structural induction rather than looking for an explicit induction variable n.)