1. a) Let $S(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$. We calculate $S(n)$ for some small values of $n \geq 1$.

   $S(1) = 1/2$
   $S(2) = 1/2 + 1/6 = 2/3$
   $S(3) = 2/3 + 1/12 = 3/4$
   $S(4) = 3/4 + 1/20 = 4/5$

   Conjecture: $S(n) = n/n + 1$ for $n \geq 1$.

   b) Prove conjecture by induction on n.

   Basis: $n = 1$. $\frac{1}{1\cdot 2} = S(1)$.

   Induction step:
   Assume $S(n) = \frac{n}{n+1}$. Show $S(n+1) = \frac{n+1}{n+2}$.

$$\begin{aligned} S(n+1) &= S(n) + \frac{1}{(n+1)(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \text{ by induction hypothesis} \\ &= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2} \end{aligned}$$

2. Rather than be somewhat pedantic, we will prove that S1, S2 and S3 are loop invariants by induction on i (rather than introduce a new variable t) noting (as can easily be proven by induciton on t) that on the t th time we enter the loop body we have $i = t$.

   a) Base case: $i = 1$.

   Clearly S3 holds since no instructons changing A have been executed. Furthermore, S1 and S2 are "trivially" true since $i = 1$; i.e. "if P then Q" is true whenever P is false.

   b) Induction step:

   We assume that the invariants hold on entering the loop for the ith time. If we exit the loop on the ith time (i.e. $i = n$) then there is nothing more to be proved. So we assume $i < n$ and we will enter the loop for the $i + 1$st time. For notational convenience, let $i' = i + 1$.

   Consider the ith iteration of the loop execution. By "max-inv", following the "for loop", we have:

$$A[\text{candidate}] \leq A[j] \quad \text{for all } j : i \leq j \leq n$$

   Note that the "for loop" does not change i or n or any value of A, so that the statements S1, S2 and S3 are all still true. The next three instructions interchange $A[\text{candidate}]$ and $A[i]$. This maintains S3 and also gives us:

$$(*) \quad A[i] \leq A[j] \quad \text{for all } j : i \leq j \leq n$$

Now S1 and $(*)$ yield:

(S1)′ $A[1] \leq A[2] \leq \cdots A[i] = A[i'-1]$

and $(*)$ yields:

(S2)′ $A[i'-1] \leq A[k]$ for all $k : i \leq k \leq n$

(S1)′ and (S2)′ are just a restatement of S1 and S2 for the $i+1$st iteration since $i' = i+1$ is the updated value of i on the next iteration. QED

3. a) The characteristic equation of this recurrence is

$$x^2 - 2x - 3 = (x+1)(x-3).$$

Hence the form of the solution is:

$$\alpha_1(-1)^n + \alpha_2 3^n.$$

We solve for α_1, α_2 by setting $n = 0$ and $n = 1$.

$$\alpha_1 + \alpha_2 = a_0 = 1$$

$$\alpha_1(-1) + \alpha_2 \cdot 3 = a_1 = 7.$$

We obtain $\alpha_1 = -1$, $\alpha_2 = 2$. Hence $a_n = (-1)^{n+1} + 2 \cdot 3^n$.

b) We can prove $a_n = (-1)^{n+1} + 2 \cdot 3^n$ for $n \geq 0$ by induction on n. (Here we pretend that we do not know how this formula was derived.)

Base cases: $n = 0$; $(-1)^1 + 2 \cdot 3^0 = 1 = a_0$
$n = 1$; $(-1)^2 + 2 \cdot 3 = 7 = a_1$

Induction step: let $n \geq 2$
Assume formula is true for all $k < n$, prove true for n.

$$\begin{aligned} a_n &= 2a_{n-1} + 3a_{n-2} \\ &= 2 \cdot [(-1)^n + 2 \cdot (3)^{n-1}] + 3[(-1)^{n-1} + 2 \cdot (3)^{n-2}] \\ &= 2(-1)(-1)^{n-1} + 3(-1)^{n-1} + 4 \cdot 3 \cdot 3^{n-2} + 6 \cdot 3^{n-2} \\ &= 1 \cdot (-1)^{n-1} + 18 \cdot 3^{n-2} \\ &= (-1)^{n+1} + 2 \cdot 3^n \end{aligned}$$