Crossing the Bridge at Night

Günter Rote
Freie Universität Berlin, Institut für Informatik
Takustraße 9, D-14195 Berlin, Germany
rote@inf.fu-berlin.de

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Abstract

We solve the general case of the bridge-crossing puzzle.

1 The Puzzle

> Four people begin on the same side of a bridge. You must help them across to the other side. It is night. There is one flashlight. A maximum of two people can cross at a time. Any party who crosses, either one or two people, must have the flashlight to see. The flashlight must be walked back and forth, it cannot be thrown, etc. Each person walks at a different speed. A pair must walk together at the rate of the slower person's pace, based on this information: Person 1 takes $t_1 = 1$ minutes to cross, and the other persons take $t_2 = 2$ minutes, $t_3 = 5$ minutes, and $t_4 = 10$ minutes to cross, respectively.

The most obvious solution is to let the fastest person (person 1) accompany each other person over the bridge and return alone with the flashlight. We write this schedule as

$$+\{1,2\} - 1 + \{1,3\} - 1 + \{1,4\},$$

denoting forward and backward movement by $+$ and $-$, respectively. The total duration of this solution is $t_2 + t_1 + t_3 + t_1 + t_4 = 2t_1 + t_2 + t_3 + t_4 = 19$ minutes.

The interesting twist of the puzzle is that the obvious solution is not optimal. A second thought reveals that it might pay off to let the two slow persons (3 and 4) cross the bridge together, to avoid having both terms t_3 and t_4 in the total time. However, starting with

$$+\{3,4\} - 3 + \cdots \qquad \text{or} \qquad +\{3,4\} - 4 + \cdots$$

incurs the penalty of having person 3 or person 4 cross at least three times in total. The correct solution in this case is to let persons 3 and 4 cross in the middle:

$$+\{1,2\} - 1 + \{3,4\} - 2 + \{1,2\},$$

with a total time of $t_2 + t_1 + t_4 + t_2 + t_2 = t_1 + 3t_2 + t_4 = 17$.

I will present the solution for an arbitrary number $N \geq 2$ of people and arbitrary crossing times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_N$.

Theorem 1. *The minimum time to cross the bridge is*

$$\min\{C_0, C_1, \ldots, C_{\lfloor N/2 \rfloor - 1}\},$$

with

$$C_k = (N - 2 - k)t_1 + (2k + 1)t_2 + \sum_{i=3}^{N} t_i - \sum_{i=1}^{k} t_{N+1-2i}. \tag{1}$$

For example, when $N = 6$, this amounts to

$$\min\{4t_1 + t_2 + t_3 + t_4 + t_5 + t_6,\ 3t_1 + 3t_2 + t_3 + t_4 + t_6,\ 2t_1 + 4t_2 + t_4 + t_6\}.$$

The difference between C_{k-1} and C_k is $2t_2 - t_1 - t_{N-2k+1}$. Thus, the optimal value of k can be determined easily by locating the value $2t_2 - t_1$ in the sorted list of t_i's.

2 Previous Results

This problem has been around in many incarnations and with various anecdotes attached to it. On the World-Wide Web one can find dozens of versions under names like the Bridge-Crossing Puzzle, the Bridge Puzzle, the Four Men Puzzle, the Flashlight Puzzle, or the Bridge and Torch Problem.

Torsten Sillke[1] has explored the history of the problem and collected his findings and references on his web page [7]. The oldest reference is apparently a puzzle book by Levmore and Cook from 1981 [6].

Moshe Sniedovich has used the problem in order to illustrate the dynamic programming paradigm for his students. He deals also with the case when more than two persons at a time can cross the bridge. His web page[2] [8] discusses the problem from the viewpoint of operations research. It includes an on-line interactive module programmed in JavaScript for visualizing solutions and computing the best solution by dynamic programming over the set of all 2^N possible "states" of the problem. A state is characterized by the subset of people that are still on the original shore.

Calude and Calude [1] have recently treated the problem, but their claimed solution (for $N \geq 4$) is $\min\{C_0, C_1\}$, in the notation of Theorem 1. I leave it to the eager reader to find the error in [1], or rather, to look for the proof.

3 The Optimal Solution

Let us first state the formal requirements of a solution which is presented as an "alternating sum of sets"

$$+A_1 - A_2 + A_3 - \cdots + A_k.$$

[1] `http://www.mathematik.uni-bielefeld.de/~sillke/`

[2] `http://www.tutor.ms.unimelb.edu.au/bridge/`

Such a sequence represents a feasible schedule if and only if the following conditions hold.

- Each A_i is a nonempty subset of $\{1, \ldots, n\}$.
- For each person $a = 1, \ldots, n$, the occurrences of a in the sequence are alternatingly in a set prefixed by $+$ and a set prefixed by $-$, beginning and ending with $+$.
- The capacity constraint: $|A_i| \leq 2$, for all i.

For simplicity, we will assume that all times are distinct and positive:

$$0 < t_1 < t_2 < \cdots < t_N$$

This will simplify the phrasing of our statements because we can argue about *the* optimal solution and *the* sorted sequence of persons. The proof can be carried over to non-distinct times by a continuity argument.

Lemma 1. *In an optimal solution, two persons will alway cross the bridge in the forward direction, and single persons will return. Thus, a solution consists of $N-1$ forward moves and $N-2$ backward moves.*

Proof. This lemma is very intuitive and I encourage the reader to skip the proof, which works by an easy exchange argument. Sniedovich [8] has proved (in a more general setting) the stronger statement that one can choose the fastest person on the other shore as the person returning the flashlight to the origin.

Consider the first instant where the solution deviates from the pattern $+\{x, x\} - x + \{x, x\} - x + \{x, x\} - \cdots$.

Case 1. The deviation is of the form $+a$. This cannot occur in first step, because otherwise the solution would have to begin with $+a - a + \cdots$, and these two steps are clearly redundant.

So let us consider the move immediately before the offending move: $\cdots - b + a \cdots$. The case $a = b$ can be excluded. The last previous step in which a or b was moved is of the form $+\{b, c\}$ or $-a$. In either case, we can transform the solution to a faster solution as follows:

$$\begin{aligned} \cdots + \{b, c\} - \cdots - b + a \cdots \quad &\Longrightarrow \quad \cdots + \{a, c\} - \cdots \ \emptyset\ \emptyset\ \cdots, \\ \cdots - a + \cdots - b + a \cdots \quad &\Longrightarrow \quad \cdots - b + \cdots \ \emptyset\ \emptyset\ \cdots, \end{aligned}$$

with $\emptyset\ \emptyset$ indicating the two moves $-b + a$ that were canceled.

Case 2. If the deviation is of the form $-\{a, b\}$, consider the last previous step in which a was moved. W. l. o. g., let this be a move $+\{a, x\}$ (where $x = b$ is permitted). We can that cancel a from both moves without increasing the total time:

$$\cdots + \{a, x\} - \cdots - \{a, b\} + \cdots \quad \Longrightarrow \quad \cdots + x - \cdots - b + \cdots,$$

but the latter solution cannot be optimal, by the analysis of Case 1. □

I will now model the problem as problem on a graph with the persons $V = \{1, \ldots, n\}$ as vertices. For each pair $+\{i, j\}$ that crosses the bridge in the forward direction, we create an edge $\{i, j\}$ with a cost of $\max\{t_i, t_j\}$. Thus, a solution is represented as a multigraph $G = (V, E)$. Since each person must move forward at least once, the edge set must cover all vertices:

$$\text{The degree } d_i \text{ of every vertex } i \text{ is at least 1.} \tag{2}$$

Lemma 1 gives the following condition:

$$\text{The number of edges is } N - 1. \tag{3}$$

The degree d_i of a vertex is the number of times person i moves forward. Thus, it must move backwards $d_i - 1$ times, causing a cost of $(d_i - 1)t_i$. Thus, the overall cost is

$$\sum_{i=1}^{N} (d_i - 1)t_i + \sum_{ij \in E} \max\{t_i, t_j\}. \tag{4}$$

In the summation $\sum_{ij \in E}$, edge weights must of course be taken according to multiplicity. If we add the constant $\sum_{i=1}^{N} t_i$, we can, instead of minimizing (4), minimize the expression

$$\sum_{i=1}^{N} d_i t_i + \sum_{ij \in E} \max\{t_i, t_j\}.$$

Each edge ij contributes 1 to the degrees of i and j, Thus we can redistribute the "degree costs" $\sum_{i=1}^{N} d_i t_i$ to the edges, and the problem can therefore be written as follows:

$$\text{Minimize } \sum_{ij \in E} c_{ij}$$

with

$$c_{ij} := t_i + t_j + \max\{t_i, t_j\},$$

subject to constraints (2–3).

This problem is a special kind of weighted degree-constrained subgraph problem, augmented by a cardinality constraint (3). By standard techniques, it can be reduced to a weighted perfect matching problem on an auxiliary graph of $O(N^2)$ vertices and therefore be solved in polynomial time. (There are also more direct methods for degree-constrained subgraph problems, see [5, Section 11], [4], or [3, Section 5.5].) Due to the special structure of the cost coefficients c_{ij}, it is however possible to solve the problem explicitly.

Every solution of the crossing problem gives rise to an edge set E, but it is not obvious that every multigraph that satisfies (2–3) can be realized by a schedule. This is indeed the case, but we will first work out the optimal graph E, and for this graph, we will construct the schedule for the crossing problem.

Lemma 2. *An optimal solution E has the following properties:*

(i) (*Non-crossing property of disjoint edges.*) *If two edges of E are incident to four vertices $i < j < k < l$, then these edges must be $\{i, j\}$ and $\{k, l\}$.*

(ii) *If two edges of E share a single vertex, then this vertex must be vertex 1.*

(iii) *If two edges share two vertices, they are $\{1, 2\}$.*

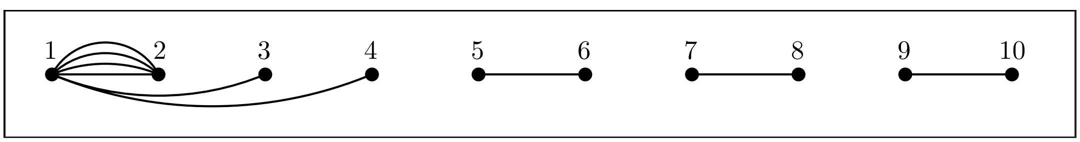
Proof. Property (i) follows by comparing the three possible ways of matching i, j, k, l by two disjoint edges. In (ii) and (iii), any single edge incident to a vertex $i \neq 1$ with degree $d_i \geq 2$ can be rerouted to 1 or 2 instead of i, unless the edge is $\{1, 2\}$. □

From this lemma we can deduce the structure of the optimal solution: The only multiple edge can be $\{1, 2\}$. When we disregard the multiplicity of this edge and look at the resulting simple graph, all vertices must have degree one except for vertex 1. Thus the graph consists of a star with center 1 and additional edges which form a matching. By property (i), these matching edges must come after all vertices adjacent to 1, and each of them connects two neighbors in the sequence $1, \ldots, N$. Let us summarize this:

Theorem 2. *An optimal graph subject to the constraints* (2–3) *consist of the following edges, for some k, $0 \leq k \leq N/2 - 1$.*

- *k "matching edges" $\{N, N-1\}, \{N-2, N-3\}, \ldots, \{N-2k+2, N-2k+1\}$,*
- *$k+1$ copies of the edge $\{1, 2\}$,*
- *and $N - 2k - 2$ edges $\{1, 3\}, \{1, 4\}, \ldots, \{1, N-2k\}$.* □

A typical solution with $k = 3$ and $N = 10$ is shown in the following figure.



Lemma 3. *The graphs described in Theorem 2 can be realized by a feasible schedule.*

Proof. We proceed by induction on N. The base cases $N = 2$ and $N = 3$ can be checked directly. For $N \geq 4$, we distinguish two cases.

Case I. $k \geq 1$, and the edge $\{N, N-1\}$ is present. We start the schedule with

$$+\{1, 2\} - 1 + \{N, N-1\} - 2.$$

This reduces the graph to a solution for $N - 2$ persons with $k - 1$ matching edges.

Case II. $k = 0$, and the edges $\{N, 1\}$ and $\{N-1, 1\}$ are present. We start with

$$+\{1, N\} - 1 + \{1, N-1\} - 1.$$

The graph is again reduced to a graph for $N-2$ persons (with $k = 0$ matching edges). □

One easily checks that the cost of the solution in Theorem 2 according to (4) is given by C_k in (1). This concludes the proof of Theorem 1.

Cases I and II both reduce the problem from N persons to $N-2$ persons by bringing persons N and $N-1$ to the other shore. This suggests an easy greedy-like algorithm for constructing the optimal solution:

> For $N \geq 4$, select the better solution of Case I and Case II for starting (i. e., compare $t_1 + 2t_2 + t_N$ with $2t_1 + t_{N-1} + t_N$), and then solve the problem for the remaining $N-2$ persons recursively.
>
> For $N = 2$ and $N = 3$, the solutions are $+\{1,2\}$ and $+\{1,3\} - 1 + \{1,2\}$, respectively.

Sillke [7] has proposed this as a conjectured optimal solution, but he does not claim it exclusively for himself, as he has seen it (without proof) in various newsgroups, and "almost anybody who thinks about the n-person generalization will arrive at this result."[3]

References

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[7] Torsten Sillke. *Crossing the bridge in an hour*. `http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/crossing-bridge`, last updated September 2001.

[8] Moshe Sniedovich. *The Bridge and Torch Problem*, `http://www.tutor.ms.unimelb.edu.au/bridge/`, version of June 2002.

[3] However, even after deriving Theorem 2 and Lemma 3, I was not aware of this form of presenting the solution until I saw it.

I think the essential step towards the optimality proof is the abstraction from the *sequence* of crossings to the *set* of crossings which is achieved in the graph model. See [2] for a similar case.