Wrap up from last time:

- briefly: Sandholm and Conitzer’s work on automated mechanism design; Blumrosem, Nisan, Segal: limited communication auctions
- note: review material on auction design from last week’s slides (we won’t go over in class due to time limitations)

Intro to Social Choice

Announcements

- Make up class next week: Tues, Dec.9, 1-3PM, PT266
- Assignment 2: marker not quite done (sorry!)
- Assignment 3 (short): posted today, due Dec.15
- Projects due on Dec.17
Social Choice

Social choice

- more general version of the mechanism design problem
- assume agents (society, club, ...) have preferences over outcomes
- we have a social choice function that specifies the “right” outcome given the preferences of the population

Focus is different than mechanism design

- preferences are usually orderings (qualitative, not quantitative)
- no monetary transfers considered (“mechanism design w/o money”)
- often focus on design and analysis of aggregation schemes (or “voting rules”) that satisfy specific axioms, usually assuming sincere reporting of preferences
- computational focus: winner determination, approximation, communication complexity, manipulability, …
Social Choice: Basic Setup

- Set of $m$ possible alternatives (outcomes) $A$
- $n$ players
  - each with preference ordering $\succ_k$ (or ranking/vote $v_k$) over $A$
  - assume $\succ_k$ is a linear order (no indifference): not a critical assumption
  - let $v = (\succ_1, \ldots, \succ_n)$ denote preference profile
  - let $L$ denote the set of linear orderings over $A$

Two settings considered
- A social choice function (SCF) $C: L^n \rightarrow A$ (i.e., consensus winner)
- A social welfare function (SWF) $C: L^n \rightarrow L$ (i.e., consensus ranking)
Why Should We Care?

- Computational models/tradeoffs inherently interesting
  - Winner determination, manipulation, approximations, computational/communication complexity
- Decision making/resource allocation in multi-agent systems
- Preference and rank learning in machine learning
  - *Ready availability of preference data* from millions of individuals
  - Web search data, ratings data in recommender systems, …
  - Often implicit; but explicit preferences available at low cost
Voting Rules

- Often SCFs are specified using voting rules
  - each player specifies a vote (her ranking or some part of it)
  - given vote profile, rule \( r: V^n \rightarrow A \) specifies consensus choice
    - distinguish resolute, irresolute rules; assume sincere voting
- Three simple rules (with different forms of votes)
  - **plurality vote**: each voter specifies their preferred alternative; winner is candidate with largest number of votes (with some tie-breaking rule)
  - **Borda rule**: each voter specifies ranking; each alternative receives \( m-1 \) points for every 1\(^{\text{st}}\)-place rank, \( m-2 \) points for every 2\(^{\text{nd}}\)-place, etc.; alternative with highest total score wins
  - **approval vote**: each voter specifies a subset of alternatives they “approve of;” a point given for each approval; alternative with highest total score wins (variant: \( k\)-approval, list exactly \( k \) candidates)

Notice: each of these can be defined by assigning a score to each rank position

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How do they differ?

- **Example preference profile (3 alternatives, bold=approval):**
  - \( A > B > C \): 5 voters (approve of only top alternative)
  - \( C > B > A \): 4 voters (approve of only top alternative)
  - \( B > C > A \): 2 voters (approve of top two alternatives)

- **Winners:**
  - plurality: A wins (5 votes)
  - Borda: B wins (scores B: 13; A: 10; C: 10)
  - approval: C wins (scores C: 6; A: 5; B: 2)

- **Which is voting rule is better?**
  - hard to say: depends on social objective one is trying to meet
  - common approach: identify axioms/desirable properties and try to show certain voting rules satisfy them
    - we will see it is not possible in general!
Some Voting Systems/Rules

- **Plurality, Borda, k-approval, k-veto**
  - all implementable with *scoring rules*: assign score $\alpha$ to each rank position; winner $a$ with max total: $\sum_i \alpha(v_i(a))$
  - for two candidates, plurality sometimes called *majority* voting

- **Approval**
  - can’t predict how sincere voters will vote based on ranking alone

- **Single-transferable vote (STV) or Hare system**
  - Round 1: vote for favorite candidate; eliminate candidate with lowest plurality score;
  - Round $t$: if your favorite eliminated at round $t-1$, recast vote for favorite remaining candidate; eliminate candidate with lowest plurality score
  - Round $m-1$: winner is last remaining candidate
    - terminate at any round if plurality score of top candidate > $m/2$
  - Needn’t be online: voters can submit rankings once
  - used in Australia, New Zealand, Ireland, …
Small Sampling of Voting Systems/Rules

- **Egalitarian** *(maxmin fairness)*
  - Winner maximizes min rank: \( \text{argmax}_a \text{min}_j (m-v_j(a)) \)

- **Copeland**
  - Let \( W(a,b,v) = 1 \) if more voters rank \( a \succ b \); 0 if more \( b \succ a \); \( \frac{1}{2} \) if tied
  - Score \( s_c(a,v) = \sum_{b \neq a} W(a,b,v); \) winner is \( a \) with max score
    - i.e., *winner is candidate that wins most pairwise elections*

- **Nanson’s rule**
  - Just like STV, but use Borda score to eliminate candidates

- **Tournament/Cup**
  - Arrange a (balanced) tournament tree of pairwise contests
  - Winner is last surviving candidate

- Lots of others!!!
Condorcet Principle

- **Condorcet winner (CW):** an alternative that beats any other in a pairwise majority vote
  - if it exists, must be unique
  - a rule is *Condorcet-consistent* if it selects the Condorcet winner whenever one exists

- **Condorcet paradox:** CW may not exist
  - and pairwise majority preferences may induce cycles in “societal ranking”
    - A > B > C: \( m/3 \) voters
    - C > A > B: \( m/3 \) voters
    - B > C > A: \( m/3 \) voters
Violations of Condorcet Principle

- **Plurality violates Condorcet**
  - 499 votes: \( A > B > C \)
  - 3 votes: \( B > C > A \)
  - 498 votes: \( C > B > A \)
  - plurality chooses A; but B is a CW (\( B>A\, 501:499; \, B>C\, 502:498 \))

- **Borda violates Condorcet**
  - 3 votes: \( A > B > C \)
  - 2 votes: \( B > C > A \)
  - 1 vote: \( B > A > C \)
  - 1 vote: \( C > A > B \)
  - Borda choose B (9 pts); but A is a CW (\( A>B\, 4:3; \, A>C\, 4:3 \))
  - notice *any* scoring rule (not just Borda) will choose B if scores strictly decrease with rank

- **Nanson, Copeland, Kemeny** rules are Condorcet consistent
Consensus Rankings

- May wish to determine a *societal preference order*
  - notice: *any rule that scores candidates* can determine a societal ranking
- Another important rule: *Kemeny rule*
  - Distance measure between rankings—Kendall’s $\tau$
    $$\tau(r, v) = \sum_{\{c, c'\}} I[r(c) > r(c') \text{ and } v(c') > v(c)]$$
  - *Kemeny ranking* $\kappa(V)$: minimizes sum of distances
    $$\kappa(V) = \min_r \kappa(r, V); \quad \kappa(r, V) = \sum_{\ell=1}^{n} \tau(r, v_\ell)$$
- Can determine winner too: top of Kemeny ranking
  - Condorcet consistent
  - Example of a voting rule that is hard to compute: NP-hard
  - Other difficult rules include Dodgson’s rule, Slater’s rule

Also co-inventor of BASIC

Also co-inventor of BASIC
Other Principles

- **Weak monotonicity**: Let profile $V'$ be identical to $V$ except that some candidate $a$ is ranked higher in some votes. Then:
  - Rule: If $a \in r(V)$ then $a \in r(V')$;
  - Ranking: If $a \succ b$ in $r(V)$ then $a \succ b$ in $r(V')$;
  - STV violates weak monotonicity
    - 22 votes: $A \succ B \succ C$
    - 21 votes: $B \succ C \succ A$
    - 20 votes: $C \succ A \succ B$
    - $A$ wins (C, then B eliminated)…
    - … but if 2-9 voters in BCA group “promote” $A$ to top of ranking, $C$ wins (B, then A eliminated)
  - Lot of rules satisfy it (plurality, Borda, …)
Strong monotonicity: Let \( a = r(V) \). Let \( V' \) be s.t. for every \( k \), every \( b \neq a \), if \( a \succ b \) in \( v_k \), then \( a \succ b \) in \( v_k \). Then \( a = r(V') \).

- i.e., if no voter “demotes” \( a \) relative to any other candidate, \( a \) still wins
- unlike WeakMon, can reorder non-winning candidates w.r.t. each other
- Plurality (and many others) violate SM
  - 22 votes: \( A \succ B \succ C \)
  - 21 votes: \( B \succ C \succ A \)
  - 20 votes: \( C \succ A \succ B \)
  - \( A \) wins; but if 3 or more BCA voters “promote” \( C \), \( C \) wins (even though relative standing of \( A \) to \( B \), \( C \) unchanged by any voter)
Other Principles

- Independence of Irrelevant Alternatives (IIA): \( V' \) different from \( V \), but relative ordering of \( a, b \), same in each vote
  - Rule: If \( a \in r(V), b \notin r(V) \), then \( b \notin r(V') \);
    - i.e., if \( b \) wasn’t strong enough to beat \( a \) given \( V \), it shouldn’t be given \( V' \)
  - Rank: if \( a > b \) in \( r(V) \) then \( a > b \) in \( r(V') \);
  - Most rules violate IIA: easy to construct examples
Other Principles (Relatively Uncontroversial)

- In what follows, assume all preference/vote profiles are possible

- **Unanimity**: if all \( v \in V \) rank \( a \) first, \( r(V) = a \); if all rank \( a \succ b \), then \( a \succ b \) in \( r(V) \)
  - relatively uncontroversial (sometimes called weak Pareto)

- **Weak Pareto**: if all \( v \in V \) rank \( a \succ b \), then \( b \notin r(V) \)
  - relatively uncontroversial

- **Non-dictatorial**: there is no voter \( k \) s.t. \( r(V) = a \) whenever \( k \) ranks \( a \) first
  - for rankings, no \( k \) s.t. \( a \succ b \) in \( r(V) \) whenever \( k \) ranks \( a \succ b \)

- **Anonymity**: permuting votes within a profile doesn’t change outcome
  - e.g., if all votes identical, but provided by “different” voters
  - implies non-dictatorship

- **Neutrality**: permuting alternatives in a profile doesn’t change outcome
  - i.e., result depends on relative position in votes, not identity
  - implies non-imposition (any candidate can win, i.e., for some profile)
Arrow’s Theorem

- **Arrow’s Theorem (1951):** Assume at least three alternatives. No voting rule can satisfy IIA, unanimity (weak Pareto), and non-dictatorship. Equivalently, there is no SWF that satisfies these properties.
  - *(Recall SWF produces “societal ranking,” not just a winner; c.f. SCF)*
  - Most celebrated theorem in social choice
  - Broadly (perhaps too broadly) interpreted as stating there is no good way to aggregate preferences

- There are a wide variety of alternative proofs around
  - see text for one
  - we’ll consider a simple proof
Brief Proof Sketch

- Fix SWF $F$; let $\succ_F$ denote social preference order given input profile.
- A coalition $S \subseteq N$ is decisive for $a$ over $b$ if, whenever $a \succ_k b$, $\forall k \in S$, and $a \succ_j b$, $\forall j \notin S$, we have $a \succ_F b$.
- **Lemma 1**: if $S$ is decisive for $a$ over $b$ then, for any $c$, $S$ is decisive for $a$ over $c$ and $c$ over $b$.
- **Sketch**: Let $S$ be decisive for $a$ over $b$.
  - Suppose $a \succ_k b \succ_k c$, $\forall k \in S$ and $b \succ_j c \succ_j a$, $\forall j \notin S$.
  - Clearly, $a \succ_F b$ by decisiveness.
  - Since $b \succ_j c$ for all $j$, $b \succ_F c$ (by unanimity), so $a \succ_F c$.
  - If $b$ placed anywhere in ordering of any agent, by IIA, we must still have $a \succ_F c$.
  - Hence $S$ is decisive for $a$ over $c$.
  - Similar argument applies to show $S$ is decisive for $c$ over $b$.
- **Lemma 2**: If $S$ is decisive for $a$ over $b$, then it’s decisive for every pair of alternatives $(c,d) \in A^2$.
- **Sketch**: By Lemma 1, $S$ decides $c$ over $b$. Reapplying Lemma 1, $S$ decides $c$ over $d$.
Brief Proof Sketch

- So now we know a coalition $S$ is either *decisive* for all pairs or for no pairs.
- Notice that *entire group $N$ is decisive for any pair of outcomes* (by unanimity)
- **Lemma 3**: For any $S \subseteq N$, and any partition $(T,U)$ of $S$. If $S$ is decisive then either $T$ is decisive or $U$ is decisive.
- **Sketch**: Let $a >_k b >_k c$ for $k \in T$; $b >_j c >_j a$ for $j \in U$; $c >_q a >_q b$ for $q \in N \setminus S$;
  - Social ranking has $b >_F c$ since $S$ is decisive.
  - Suppose social ranking has $a >_F b$, which implies $a >_F c$ (by transitivity).
    - Notice only agents in $T$ rank $a > c$, and those in $U, N \setminus S$ rank $c > a$.
    - But if we reorder prefs for any other alternatives (keeping $a > c$ in $T$, $c > a$ in $U$ and $N \setminus S$), by IIA, we must still have $a >_F c$ in this new profile.
    - Hence $T$ is decisive for $a$ over $c$ (hence decisive for all pairs).
  - Suppose social ranking has $b >_F a$
    - Since only agents in $U$ rank $b > a$, similar argument shows $U$ is decisive.
    - So either $T$ is decisive or $U$ is decisive.
- **Proof of Theorem**: Entire group $N$ is decisive. Repeatedly partition, choosing the decisive subgroup at each stage. Eventually we reach a singleton set that is decisive for all pairs… the dictator!
**Muller-Satterthwaite Theorem**

- Arrow’s theorem tells us: impossible to produce a *societal ranking* satisfying our desired conditions (in a fully general way)
  - Maybe producing a full ranking is too much to ask
  - What if we only want a unique winner?
  - Also not possible…

**Muller-Satterthwaite Theorem (1977):** Assume at least three alternatives. No resolute voting rule satisfies strong monotonicity, non-imposition, and non-dictatorship. Equivalently, there is no SCF that satisfies these properties.
May’s Theorem

- Should Arrow’s Thm cause complete despair? Not really…
  - dismiss some of the desiderata as too stringent
  - live with “general” impossibility, but use rules that tend to (in practice) give desirable results (behavioral social choice)
  - look at restrictions on the assumptions (number of alternatives, all possible preference/vote profiles, …)

- Here’s a positive result (and characterization)…

- May’s Theorem (1952): Assume \textit{two} alternatives. Plurality (which is \textit{majority} in case of two alternatives) is the only voting rule that satisfies anonymity, neutrality, and positive responsiveness (a slight variant of weak monotonicity).

- Social choice has a variety of interesting (and not so interesting) characterizations of this type (we’ll see some more)
Manipulability

- As with mechanism design, most voting rules provide positive incentive to misreport preferences to get a more desirable outcome
  - political phenomena such as vote splitting are just one example

- Plurality:
  - 100 votes: Bush $\succ$ Gore $\succ$ Nader
  - 12 votes: Nader $\succ$ Gore $\succ$ Bush
  - 95 votes: Gore $\succ$ Nader $\succ$ Bush
  - Bush wins sincere plurality vote; in the interest of Nader supporters to vote for Gore. *Notice that Borda, STV would give election to Gore*

- Borda: same example with different numbers
  - 100 votes: Bush $\succ$ Gore $\succ$ Nader
  - 17 votes: Nader $\succ$ Gore $\succ$ Bush
  - 90 votes: Gore $\succ$ Nader $\succ$ Bush
  - Bush wins sincere Borda vote (B:200 pts; G:197pts); in the interest of Nader supporters to rank Gore higher than Nader
Manipulability

- **Strategyproofness** defined for voting procedures just as it is for mechanisms
  - no profiles where insincere report by \( k \) leads to preferred outcome for \( k \)
    - *strategyproof*: dominant strategy truthful
    - *incentive compatible*: truthful in (voting) equilibrium (e.g., Bayes-Nash)

- Alternatively, we can define SCFs themselves as being strategyproof
  - there is no profile, agent \( k \) s.t. \( C(\succ^1, \ldots, \succ_k, \ldots, \succ^m) >_k C(\succ^1, \ldots, \succ_k, \ldots, \succ^m) \)

- Manipulability unavoidable in general (for general SCFs)
  - already seen our old friend GS in the context of mechanism design

- **Thm (Gibbard73, Sattherwaite75)**: Let \( C \) (over \( N, O \)) be s.t.:
  - (i) \( |O| > 2 \);
  - (ii) \( C \) is onto (every outcome is selected for some profile \( v \));
  - (iii) \( C \) is non-dictatorial;
  - (iv) all preference profiles \( L^n \) are possible.

Then \( C \) cannot be strategy-proof.
Single-peaked Preferences

- Special class of preferences for which GS circumvented
- Let $\gg$ denote some “natural” ordering over $A$
  - e.g., order political candidates on left-right spectrum
  - e.g., locations of park, warehouse on real-line (position on highway)

- $k$’s preferences are single-peaked (with respect to the given ordering of $A$) if there is alternative $a^*[k]$ s.t.:
  - $a^*[k]$ is $k$’s ideal point, i.e., $a^*[k] >_k a$ for any $a \neq a^*[k]$
  - $b >_k c$ if (a) $c \gg b \gg a^*[k]$ or (b) $a^*[k] \gg b \gg c$
Median Voting

- Suppose all voter’s prefs are single-peaked (same domain order!)
- **Median voting scheme**: voter specifies only her peak; winner is median of reported peaks (Black 1948)
  - result is a Condorcet winner (if \( n \) odd)
  - result is Pareto efficient
  - voting scheme is *strategyproof* (easy to see)
Generalized Median Voting

- Suppose we add \( n-1 \) “phantom voters” with arbitrary peaks
  - announced in advance, chosen for “some purpose”
- Winner is median of the \( 2n-1 \) total votes (\( n \) real, \( n-1 \) phantom)
  - e.g., in example, the phantom votes implement selection of 33rd percentile (or 1/3 quantile) among true peaks
- **Generalized Median**: if preferences are single-peaked, any anonymous, Pareto efficient, strategyproof rule must be a generalized median mechanism (Moulin 1980)
  - some mild generalizations (e.g., multiple dimensions) possible
  - **Recent work**: can you find an axis/axes that render profile \( V \) SP?
  - … are there natural approximations of SP? how does it impact incentives?

Median of genuine and phantom peaks
Complexity as Barrier to Manipulation

- Topic of considerable study in CS
  - started with seminal work of Bartholdi, Tovey, Trick (1989, 1991)
  - widely ignored for many years, now well-studied
- Given $n-1$ votes, desired candidate $a^*$: can $n^{th}$ voter ensure $a^*$ wins?
  - constructive manipulation; also destructive variant (prevent winner)
  - can also consider manipulating coalitions (and size needed)
- Decision problem is tractable for some rules
  - plurality: easy, if manipulable, it is accomplished by voting for $a^*$
  - Borda: easy (for single voter): place $a^*$ at top of ballot, greedily add candidates in next positions so they don’t “overtake” $a^*$ (if not possible, not manipulable)
- Intractable for others:
  - STV: determining (constructing) manipulating vote NP-hard (BTT91)
  - many voting rules subsequently analyzed this way
- Analysis more nuanced for coalitions, weighted voters, etc.
Complexity as Barrier to Manipulation

- These results should be taken with a grain of salt
  - worst-case manipulation: *some* vote profiles are hard to handle; but doesn’t mean *typical* case is (and that’s *crucial* for “resistance” claims)
    - increasing work on empirical analysis and avg. case behavior
  - assumptions are beneficial to manipulators: know votes cast by others!
    - hence a conclusion of manipulability under this model may not be very meaningful (too pessimistic, unrealistic)
    - further analysis needed with realistic knowledge constraints (min entropy, sample complexity, etc.)

- Other forms of manipulation
  - control: adding, deleting candidates; setting agenda (tournament); setting up electoral “boundaries” or groups (gerrymandering); …
  - bribery: pay someone to change their vote
Example: Control of Tournament (Cup Rule)

- Set a *balanced* binary tree of pairwise contests
- Person setting the agenda can sometimes choose whichever winner they want (if they know the votes)
  - 35 votes: \( A > C > B \)
  - 33 votes: \( B > A > C \)
  - 32 votes: \( C > B > A \)
    - If \((a,b)\) paired first, \(c\) wins; If \((b,c)\) first, \(a\) wins; If \((a,c)\) first, \(b\) wins
- Complexity of determining if a (dynamic) schedule can make \(a\) win:
  - known votes: still unknown if polynomial!
  - probabilistic votes: NP-hard (even for \(v \in \{0, \frac{1}{2}, 1\}\))
- Other interesting questions in this space (esp. for sports, etc):
  - throwing matches, maximizing competitiveness/revenue, etc.
“Complexity” as a barrier to manipulation

The Doge of Venice:
- chief magistrate of the Most Serene Republic of Venice c.700-1797
- elected for life by the city-state's aristocracy
- concern about the influence of powerful families!

Voting Protocol in 15\textsuperscript{th} Century \textit{(courtesy Wikipedia via Mike Trick ADT-09)}
- 30 members of the Great Council are chosen by lot
- The 30 are reduced by lot to 9
- The 9 choose 40 representatives
- The 40 are reduced by lot to 12
- The 12 choose 20 representatives
- The 20 twenty are reduced by lot to 9
- The nine elect 45 representatives
- The 45 are reduced by lot to 11
- The 11 choose 41 representatives
- These 41 actually elect the doge
Objective Rankings

- A different perspective: rankings as beliefs (not preferences)
  - suppose there is a true underlying objective ranking $r^*$
    - e.g., quality of sports teams, ability to lead a nation, impact of policy $P$ on economy, relevance of document/web page to a query, ...
  - agents have opinions on the matter: correlated (noisily) with obj. $r^*$
- Rank aggregation aimed at ascertaining true $r^*$, not some SCF
- Condorcet addressed this in 1785:
  - Suppose $n$ voters (e.g., jury) vote on two alternatives (e.g., guilt/innocence). If each votes independently and is correct with $p > \frac{1}{2}$, then plurality rule gives maximum likelihood estimate of correct alternative, and converges to correct decision as $n \to \infty$.
  - Young (1995) generalized: if each voter noisily ranks arbitrary pairs $(a,b)$ correctly with probability $p > \frac{1}{2}$, the Kemeny consensus is a maximum likelihood estimate of the true underlying ranking.
  - See Conitzer, Sandholm (2005) for treatment of several other rules (e.g., Borda) using specific noise models tuned to that rule.
Other Issues

- Multi-winner elections
  - proportional assemblies, committees, multiple projects, etc.
  - diversity a key consideration: “first $k$ past the post” usually a bad idea

- Behavioral social choice
  - designing, analyzing rules based on empirical preferences
  - modeling preference distributions (econometrics, psychometrics)

- Combinatorial preference aggregation
  - preferences over complex domains (multi-issue)
  - appropriate preference rep’ns, aggregation methods, algorithms

- Communication complexity, privacy concerns (à la mech. design)

- Preference Elicitation
  - ballot complexity a barrier to wider-spread use of rank-based voting

- Approximation of Social Choice Functions
  - does ability to approximate winner ease burden:
    - communication? computation? privacy?