University of Toronto Department of Computer Science

Induction Exercises

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1. Prove that $0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, for all $n \geq 0$.
   This is solved in the "Understanding Induction" document in your course handbook.

2. Prove that $n! \geq 2^{n-1}$, for all $n > 0$.

3. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all $n > 0$.

4. Prove that $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$, for all $n > 0$.

5. Prove that $n^3 + 2n$ is divisible by 3, for all $n > 0$.
   After proving this by induction, also try proving it *without* induction.
   Hint: consider the cases $n \bmod 3 = 0$; $n \bmod 3 = 1$; and $n \bmod 3 = 2$.

6. Given stamps only in 3-cent and 5-cent denominations,
   prove that postage of exactly n-cents is possible, for all $n > 7$.

7. Find the error in the following proof:
   Prove that $\frac{n}{n+1} > 1$, for all $n > 0$.
   Proof:

   Induction Hypothesis: Assume that $\frac{k}{k+1} > 1$.

   Induction Step: Prove that $\frac{k+1}{k+2} > 1$.

   If $\frac{k+1}{k+2} > \frac{k}{k+1}$, then by the induction hypothesis (i.e., $\frac{k}{k+1} > 1$) we would be done.

   To see if $\frac{k+1}{k+2} > \frac{k}{k+1}$, multiply both sides by $(k + 1)(k + 2)$.

   So, is it true that $(k + 1)^2 > k(k + 2)$?

   Is $k^2 + 2k + 1 > k^2 + 2k$? Sure.

   Therefore, $\frac{k+1}{k+2} > 1$.

   In fact, not only is the proof incorrect, the statement itself is wrong.
   Instead, $\frac{n}{n+1} < 1$, for all $n > 0$.

8. Prove $1 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$, for all $n > 0, r > 1$.

9. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n > 0$.

10. Prove that $1^2 + 2^2 + \cdots + (n - 1)^2 < \frac{n^3}{3} < 1^2 + 2^2 + \cdots + n^2$, for all $n \geq 2$.

11. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{(n^2)(n+1)^2}{4}$, for all $n > 0$.

12. Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + 3 + ... + n)^2$, for all $n > 0$.