Understanding Induction

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Induction is a powerful method of proving mathematical statements. It is an important tool for computer scientists because it allows us to make rigorous arguments that our programs are correct. Understanding induction is also intimately connected with understanding recursive programs. You will be learning about both program correctness and recursive programming in CSC148/A58.

Most or all of you have come across induction in high school. However, we find that many students are not completely comfortable with it and have trouble writing inductive proofs that are correct. This section of the Handbook is intended to help you understand induction — not just the familiar structure ("base case", "induction step", etc.), but *why* it works, so that you can recognize flaws in inductive proofs, and write flawless proofs yourself. This will also help you later when you learn about recursive programs.

If you already feel you understand induction, you may skim over this document to become familiar with our format for writing inductive proofs. Focus mainly on sections 4 and 5.

1 Introduction to induction

Induction can be used for proving statements that say that something is true for all positive integers, or more generally, for all integers greater than or equal to some base value, which is often 1. For instance, we can use induction to prove the following statements:

1. For all integers $n \geq 1$, the sum of the first n integers is $\dfrac{n(n+1)}{2}$.
2. For all integers $n \geq 1$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
   This is, of course, the same as statement (1), except that it is written using more mathematical notation and less English.
3. For all integers $r \geq 2$, the product of any r consecutive integers is divisible by $r!$.
4. For every iteration i of the loop in procedure `findMax`, the value of variable `x` is greater than the value of variable `y` both at the beginning and the ending of that iteration.

Notice that these statements can express equalities, as in (1) and (2), but they need not, as in (3) and (4). They need only express things that are either true or false. And although you are probably most familiar with proving statements about numbers, induction can be used to prove statements about other things — even programs, as in (4). Also notice that we can write down a statement to be proven (and even the proof) using ordinary English, or we can choose to use some mathematical notation. Either form of expression is fine, *as long as the meaning is clear and unambiguous*. So if you find mathematical notation a little intimidating, you can start out using plain English. As you get used to more and more notation, you will find that it allows you to write statements and proofs that are far more concise.

Now, imagine that you have never heard of induction and you want to prove a statement such as (1) above. How would you go about it? Statement (1) is rather strong. It says that no matter what positive integer n you pick, out of the infinite number of choices, you will find that the sum of all the positive integers up to and including n will equal $n(n+1)/2$. So statement (1) is equivalent to an infinite set of statements:

- the sum of the first 1 integers = 1(1+1)/2
- the sum of the first 2 integers = 2(2+1)/2
- the sum of the first 3 integers = 3(3+1)/2
- the sum of the first 4 integers = 4(4+1)/2

$\vdots$

You would probably want to verify the first few of these statements by checking that the sums do in fact equal the amounts stated. (This would be a fine way to prove the first few of these statements.) But in order to prove (1) above, you need to prove *all* of these statements, which would require an infinite amount of time if you were to prove them one at a time.

This is where induction comes in. It is a way of proving statements like (1) without having to prove an infinite number of individual things.

2 Why induction works

You have probably seen an inductive proof with a structure that looks something like this:

Prove: *Statement to be proven.*
Proof:
BASE CASE:
INDUCTION HYPOTHESIS/ASSUMPTION:
INDUCTION STEP:
INDUCTION CONCLUSION:

Some students use this structure as a template to be filled in, without really understanding it. As a result, they often write "proofs" with errors that make them completely invalid.

Let's look at an example of a very simple inductive proof (see figure 1) and use it to discuss exactly how induction works. Read over the proof in figure 1. Why is this proof valid? That is, why do the base case, induction hypothesis, and induction step allow us to conclude the induction conclusion? To help see what's going on, let's look at just the essential structure of the proof. We can get rid of some distracting details by using the following definition:

Definition: Let $S(n)$, for any integer $n \geq 1$, represent the statement "$\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$".

For example, $S(18)$ represents the statement "$\sum_{i=1}^{18} i = \frac{18(18+1)}{2}$". We can restate the whole proof in terms of $S(n)$, as shown in figure 2. Look at this structure to see how $S(n)$ is used, and to confirm that it corresponds to the structure of the original proof in figure 1.

Now what do the base case, induction hypothesis, and induction step tell us? The base case is simple; it just tells us that $S(1)$ is true. But the induction hypothesis and induction step (the "induction part") are quite subtle.

Prove: For all integers $n \geq 1$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof:

Base Case:

Prove that the statement is true for $n = 1$; that is, $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$.

$\sum_{i=1}^{1} i$ is simply 1.

$\frac{1(1+1)}{2} = \frac{1 \times 2}{2} = 1$.

Thus $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$, so the statement is true for $n = 1$.

Let $k \geq 1$ be an arbitrary integer.

Induction Hypothesis:

Assume that the statement is true for $n = k$; that is, $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Induction Step:

Prove that the statement must also be true for $n = k+1$; that is, $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$.

$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$.

But $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ by the Induction Hypothesis.

$$\begin{aligned}
\text{So } \sum_{i=1}^{k+1} i &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)(k+1+1)}{2}
\end{aligned}$$

Thus $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$.

Induction Conclusion:

For all integers $n \geq 1$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Figure 1: Example of a very simple inductive proof. For clarity, the proof includes even small steps that involve only simple algebra. You may choose to leave such steps out of your own proofs.

Prove: For all $n \geq 1$, $S(n)$ is true.
Proof:

BASE CASE:

Prove that $S(1)$ is true.

⋮

Thus $S(1)$ is true.

Let $k \geq 1$ be an arbitrary integer.

INDUCTION HYPOTHESIS:

Assume that $S(k)$ is true.

INDUCTION STEP:

Prove that $S(k+1)$ must also be true.

⋮

Thus $S(k+1)$ is true.

INDUCTION CONCLUSION:

For all $n \geq 1$, $S(n)$ is true.

Figure 2: Simplified structure of an inductive proof. $S(n)$ is used to replace the details of the statement to be proven.

The meaning of the induction part of the proof

First of all, what is the purpose of the statement "Let $k \geq 1$ be an arbitrary integer"? It is actually *essential* to the proof — without it, we cannot make the induction conclusion. But it is hard to see its purpose because inductive proofs often omit a key intermediate conclusion that it makes possible. In figure 3, we have rewritten the structure of an inductive proof to include this key conclusion (in italics).

The italicized conclusion is made possible by a proof technique called "universal generalization". It is called this because it allows us to generalize a statement by using a universal quantifier ("for all"). Universal generalization works like this:
If you

(a) introduce a new variable, possibly with some limits on its range
(In our case, we introduced a new variable k, which must be at least 1.)

(b) and then prove some statement that mentions the variable, without making any assumptions about its value other than the restrictions stated when it was introduced
(In our case, we proved that if $S(k)$ is true, then $S(k+1)$ must also be true, in the part of the proof between the lines.)

then you can conclude that

(c) the statement must be true for *any* value of the variable within the limits stated, since you never assumed anything else about its value.
(In our case, we conclude that: "For $\boxed{\text{all}}$ $k \geq 1$, if $S(k)$ is true, then it follows that $S(k+1)$ must also be true".)

Prove: For all $n \geq 1$, $S(n)$ is true.
Proof:

BASE CASE:

Prove that $S(1)$ is true.

$\vdots$

Thus $S(1)$ is true.

Let $k \geq 1$ be an arbitrary integer.

INDUCTION HYPOTHESIS:

Assume that $S(k)$ is true.

INDUCTION STEP:

Prove that $S(k+1)$ must also be true.

$\vdots$

Thus $S(k+1)$ is true.

Thus, for all $k \geq 1$, if $S(k)$ is true, then it follows that $S(k+1)$ must also be true.

INDUCTION CONCLUSION:

For all $n \geq 1$, $S(n)$ is true.

Figure 3: Simplified structure of an inductive proof, with a key conclusion explicitly stated (in italics).

Universal generalization is something that you have used throughout high school mathematics, probably without even realizing it. For instance, if you were to write:

$$\begin{aligned} x(x+1)(2x+1) &= (x^2+x)(2x+1) \\ &= 2x^3+x^2+2x^2+x \\ &= 2x^3+3x^2+x \end{aligned}$$

you would mean that this is true for *any* value of x. If you wanted to be very explicit about this, you could write the following instead:

> Let x be any real number.
> $$\begin{aligned} x(x+1)(2x+1) &= (x^2+x)(2x+1) \\ &= 2x^3+x^2+2x^2+x \\ &= 2x^3+3x^2+x \end{aligned}$$
> Therefore, for $\boxed{\text{all}}$ real numbers x, $x(x+1)(2x+1) = 2x^3+3x^2+x$.

You can be sure that the final conclusion is correct because you made no assumptions about the value of x in the indented part. If you were to make assumptions about the value of x (for instance by dividing by x, which is only valid if $x \neq 0$), you could incorrectly "prove" things that aren't true.

All of this explains why the induction part of the proof in figure 3 allows us to conclude that

> for all $k \geq 1$, if $S(k)$ is true, then it follows that $S(k+1)$ must also be true.

We will see shortly how this is the key to making the induction conclusion valid. But first, notice that the part of our proof between the lines also has a structure of its own. It is a small "conditional proof": it proves a fact of the form "if P is true, then Q must also be true", which can also be written as "P implies Q", or $P \Rightarrow Q$. A conditional proof works like this:
If you

(a) assume that P is true
(In our case, we assume that $S(k)$ is true.)

(b) and while using this assumption, show that Q must be true
(In our case, we show that $S(k+1)$ must also be true.)

then you can conclude that

(c) $P \Rightarrow Q$.
(In our case, we can conclude that $S(k) \Rightarrow S(k+1)$.)

It can be helpful to write a conditional proof in a way that makes its structure obvious:

> Let us assume that P is true.
> > *Indent while we are under the assumption.*
> > *Steps leading to the conclusion Q go here.*
> > ⋮
> > Q is true.
>
> Therefore P implies Q.

This format makes it clear exactly when we are under the assumption and when we are not. This becomes very helpful when a large proof involves several small conditional proofs.

So why does induction work?

Now that we have figured out what's going on in the induction part of the proof, we are ready to go back to the big question: why is the induction conclusion valid?

Let's re-cap what in total we have proven in an inductive proof:

(I) $S(1)$ is true.

(II) For all $k \geq 1$, if $S(k)$ is true then it follows that $S(k+1)$ must also be true.

It is (II) that is the power of induction. It says that we can pick *any* k (as long as it's greater than or equal to 1), and we'll know that as long as $S(k)$ is true, then $S(k+1)$ must also be true. So we could, for instance, pick $k = 1$. Plugging 1 in for k, we know that as long as $S(1)$ is true, then $S(1+1)$ must be true, that is $S(2)$ must be true. In other words

$$S(1) \text{ implies } S(2).$$

But we already know that $S(1)$ is true, from the base case. $S(1)$, together with the implication above, implies that $S(2)$ must be true. So we now know that

$$S(1) \text{ and } S(2) \text{ are both true.}$$

But we can use (II) again, this time picking the value 2 for k. Plugging that in, we get

$$S(2) \text{ implies } S(3).$$

But we just proved that $S(2)$ is true. This means that $S(3)$ must also be true. So now we know that

$$S(1),\ S(2),\ \text{and } S(3) \text{ are all true.}$$

By now, the pattern is obvious. We can continue substituting larger and larger values for k, each time adding a new statement to the ones we know are true. Induction is like an engine for proving things. Once you prove facts (I) and (II) above, you can "turn the crank" over and over to get more and more facts out that are proven to be true. In principle, we could do this any number of times to prove that $S(n)$ is true for *any* n we might pick, as long as it's at least 1. In other words, once you have proven facts (I) and (II), you can be sure that

$$\text{For all } n \geq 1,\ S(n) \text{ must be true.}$$

This is why induction works!

Figure 4 summarizes the process visually.

Since induction is well known, you don't have to make this argument every time you write an inductive proof; you need only prove facts in the form of (I) and (II), and everyone familiar with induction will know that your induction conclusion is valid. However, you must *understand* the argument above in order to be sure that you have put your inductive proof together properly. If you go on to take CSC238, you will learn about how one can formally prove that induction works.

3 An analogy

You may have heard induction described using an analogy to dominoes. If you line up a bunch of dominoes, how do you know that they'll all fall down? Well you don't, unless you know that the first domino is going to fall over. That's analogous to knowing that the base case is true. But is that enough to be sure that all of the dominoes will fall? Of course not – we must also know that the dominoes are close enough together that if any one falls over, it will knock over the next one. That's analogous to the induction part of the proof. As long as it and the base case are both true, *i.e.*,

(I) The first domino will fall over.

(II) For every domino, if it falls over, then it follows that the next one must fall over

you can be sure that all the dominoes will fall.

4 Variations on this sort of induction

Base cases other than 1

The base case is the smallest value for which you are proving the statement to be true. This needn't be 1 of course. Often, it is zero, but it could be 12, for instance. In this case, your induction conclusion would be that

$$\text{For all } n \geq 12, S(n) \text{ is true.}$$

Figure 5 shows an inductive proof with a base case of 3.

Increments other than 1

Sometimes it is difficult to prove that $S(k)$ implies $S(k+1)$ in the induction step because the increment of 1 causes problems. When evenness and oddness are important to the statement being proven, it can be helpful to use an increment of 2. In other words, it can be convenient to prove that

$$\text{(II}'\text{)} \quad \text{For all } k \geq 1, S(k) \text{ implies } S(k+2).$$

However, if you were to prove the above, plus the base case $S(1)$, your induction conclusion could only be

$$\text{For all odd } n \geq 1, S(n) \text{ is true.}$$

Since the base value was 1 and the induction part goes up in increments of 2, nothing was ever proven about even values of n.

To prove that the statement $S(n)$ is true also for *even* values of $n \geq 1$, you need only prove the even base case $S(2)$; starting with $S(2)$ and going up by twos, you can reach any even integer ≥ 1. Figure 6 shows the structure of a proof that has increments of 2.

What if you wanted to prove the statement true for all integers greater than or equal to 0, rather than 1? Why not just prove the even base case $S(0)$ instead of proving $S(2)$? This would not work. The problem is that you couldn't use $S(0)$ and (II$'$) to deduce that $S(2)$ must be true (and from there, all the other even cases). Why not? Because (II$'$) talks only about $k \geq 1$. To make the whole proof work, you would simply have to re-do the induction step to prove:

$$\text{For all } k \geq 0, S(k) \text{ implies } S(k+1).$$

Strong induction

In the induction step, you use the assumption that $S(k)$ is true, the induction hypothesis, to help you prove that $S(k+1)$ is true (assuming an increment of 1). With "strong induction", you assume that the statement is true not only for k, but for all the values from the base value up to k. So if the base value were 1, you would assume that

$$\text{for all } 1 \leq j \leq k,\ S(j) \text{ is true.}$$

That is, you assume that $S(1)$ is true, and $S(2)$ is true, and $S(3)$ is true, all the way up to $S(k)$. This is a much stronger assumption to lean on, and can make it a lot easier to prove that $S(k+1)$ is true.

But is it okay to use this strong assumption? That is, will the induction conclusion still be valid? Look again at figure 4. As we "turn the crank" repeatedly, we do in fact know that not only is the statement true for one more value of n, it's also true for all the previous ones down to the base value.

Figure 7 summarizes how the process works for strong induction.

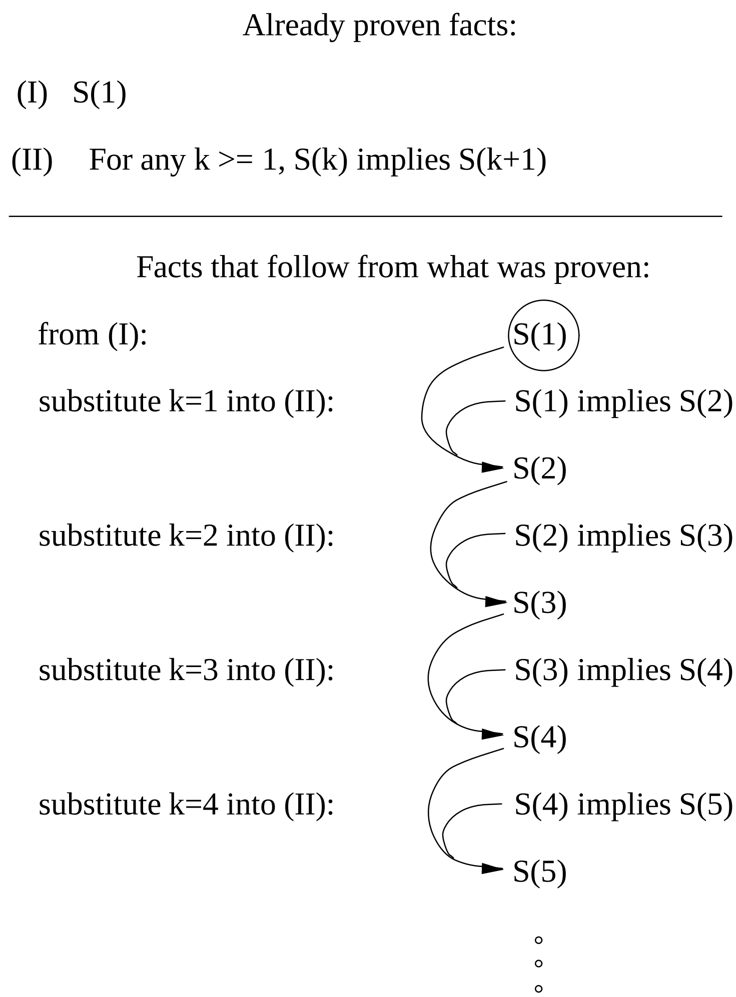


Figure 4: Summary of the "process" of induction.

Let $S(n)$, n an integer, represent the statement "$n! \geq 2n$".
Prove: For all $n \geq 3$, $S(n)$ is true.
Proof:

Base Case:

Prove that $S(3)$ is true.

$3! = 3 \times 2 \times 1 = 6$.
$2 \times 3 = 6$.
Thus $3! \geq 2 \times 3$.
Thus $S(3)$ is true.

Let $k \geq 3$ be an arbitrary integer.

Induction Hypothesis:

Assume that $S(k)$ is true.

Induction Step:

Prove that $S(k+1)$ must also be true; that is, $(k+1)! \geq 2(k+1)$.

$(k+1)! = k! \times (k+1)$, by definition of factorial.
But $k! \geq 2k$, by the Induction Hypothesis.
Thus $(k+1)! \geq 2k(k+1)$.
And $2k(k+1) \geq 2(k+1)$, since $k \geq 3$.
So $(k+1)! \geq 2(k+1)$.
Thus $S(k+1)$ is true.

Thus, for all $k \geq 3$, if $S(k)$ is true, then it follows that $S(k+1)$ must also be true.

Induction Conclusion:

For all $n \geq 3$, $S(n)$ is true.

Figure 5: An inductive proof with a base case of 3. Note that $S(1)$ and $S(2)$ are not true.

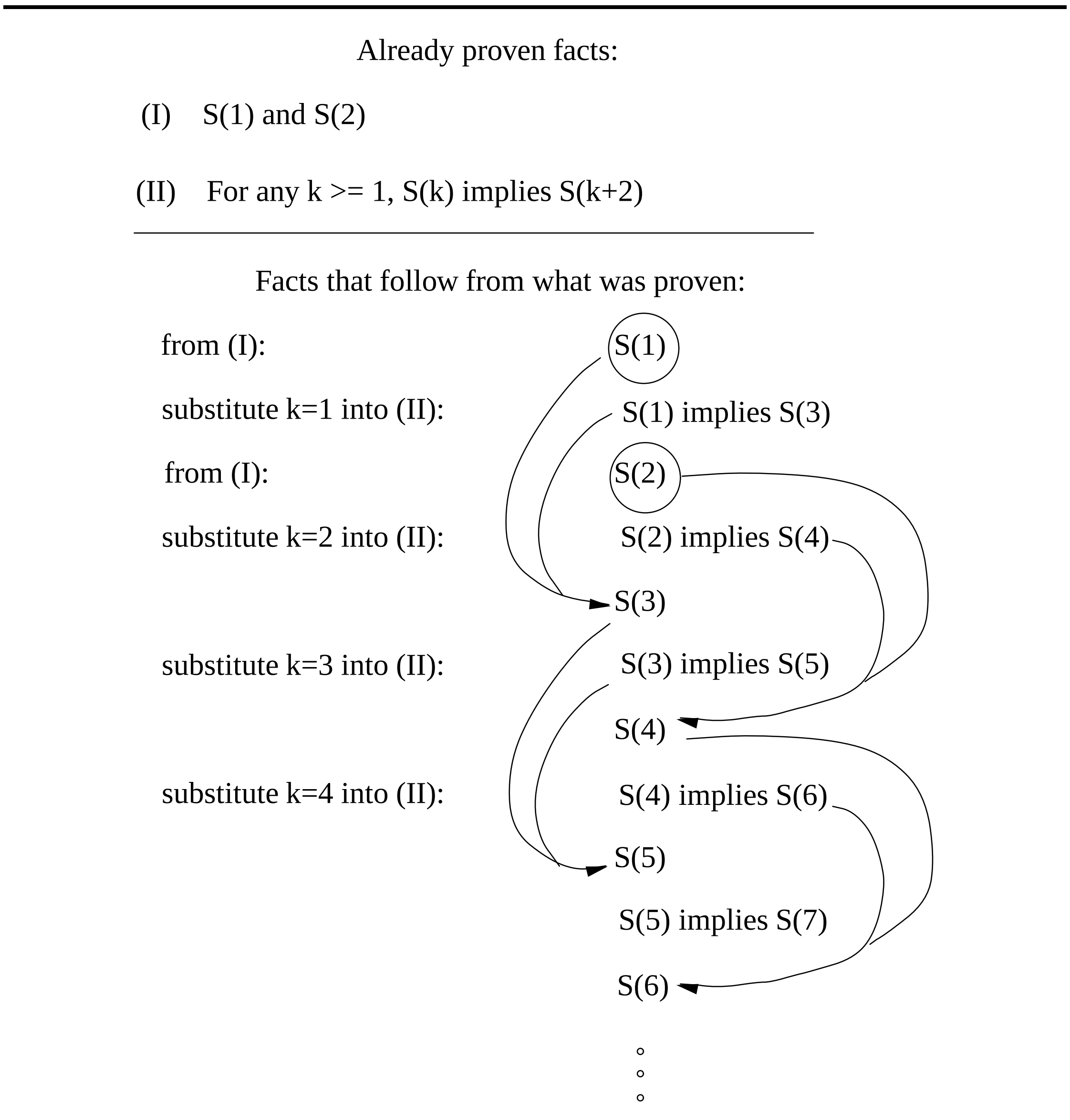


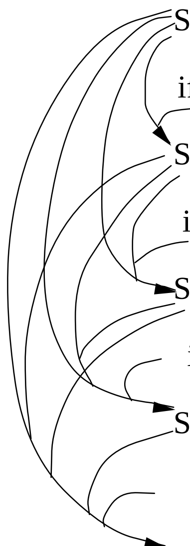
Figure 6: Summary of the process of induction, with an increment of 2 (and assuming a base value of 1). Notice that now we must prove two different base cases, in order to anchor the two chains of reasoning.

Already proven facts:

(I) S(1)

(II) For any k >= 1,
if S(j) is true for all 1 <= j <= k, then S(k+1) must be true

Facts that follow from what was proven:



from (I): S(1)

substitute k=1 into (II): if S(1) then S(2)

S(2)

substitute k=2 into (II): if S(1) and S(2), then S(3)

S(3)

substitute k=3 into (II): if S(1), S(2) and S(3), then S(4)

S(4)

substitute k=4 into (II): if S(1), S(2), S(3) and S(4), then S(5)

S(5)

∘
∘
∘

Figure 7: Summary of the “process” of strong induction.

5 Common flaws in inductive proofs

This section presents some common flaws in inductive proofs, all of which are taken from students' answers to a question on a recent final exam. The question required them to prove that:

$$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}, \text{ for all } m \geq 1.$$

Many of the flaws involve careless writing. If you are very careful in how you say things, you will be far less likely to make reasoning errors.

Figure 8 shows a correct version of the proof.

Induction hypothesis = original statement

Instead of writing the correct induction hypothesis:

Assume that $S(k)$ is true

many students wrote:

Assume that $S(k)$ is true, for all $k \geq 1$.

This is the original statement we are trying to prove! These students have based the whole proof on the assumption that the statement we are trying to prove is true. This is clearly invalid.

Base case doesn't connect to the induction hypothesis

What if the proof were to say the following?

> Let $k \geq 2$ be an arbitrary integer.
> INDUCTION HYPOTHESIS:
> Assume that $S(k)$ is true.

The base case would tell you

$$\text{(I)} \quad S(1)$$

as before, but the induction part would now tell you

$$\text{(II}''\text{)} \quad \text{for any } k \geq 2,\ S(k) \text{ implies } S(k+1).$$

This means that the smallest value you can substitute for k in (II$''$) would be 2, yielding

$$S(2) \text{ implies } S(3).$$

But this plus the base case does not tell you that $S(3)$, or anything else new, must be true. You are stuck — you can't turn the induction crank even once — because $S(2)$ was never proven to be true.

This may seem like a dumb mistake to make, but it is extremely common. It's especially easy to make this mistake when the base case value is zero. In such cases, the following would not be adequate:

> Let $k \geq 1$ be an arbitrary integer.
> INDUCTION HYPOTHESIS:
> Assume that $S(k)$ is true.

You would need to use this instead:

> Let $k \geq 0$ be an arbitrary integer.
> INDUCTION HYPOTHESIS:
> Assume that $S(k)$ is true.

Putting the quantifier "for all" in twice

Consider this:

Let $S(m)$ represent the statement "$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$ for all $m \geq 1$".

Prove: For all $m \geq 1$, $S(m)$ is true.

This doesn't make sense. It says that we are to prove that

$$\text{For all } m \geq 1, \sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6} \text{ for all } m \geq 1.$$

The correct way to do this only "quantifies" the variable m once:

Let $S(m)$ represent the statement "$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$".

Prove: For all $m \geq 1$, $S(m)$ is true.

Inconsistent use of $S(m)$

Because the summation comes up frequently in the proof, it is convenient to define a short-form for it. For instance, one might say:

$$\text{Let } S(m) = \sum_{i=1}^{m} i^2.$$

This is perfectly fine on its own, but cannot be used in the context of the proof we have written. $S(m)$ is already defined to be the statement

$$\text{“}\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}\text{”}.$$

To correctly use a short-form for the summation, one need only use different names for the summation and the statement. For example:

Let $\mathrm{Sum}(m) = \sum_{i=1}^{m} i^2$.

Let $S(m)$ represent the statement "$\mathrm{Sum}(m) = \frac{m(m+1)(2m+1)}{6}$".

Now everything will be fine, as long as S and Sum must be used appropriately. For example, the following statements are *not* appropriate:

Assume that $\mathrm{Sum}(m)$ is true.

$S(k+1) = S(k) + (k+1)^2$.

Think about why.

Sloppy definition of $S(m)$

Many students were sloppy in defining $S(m)$ in the first place. They took the line of the exam that gave the statement to prove, and tacked onto the front of it "Let $S(m) =$", so that their definition read:

$$\text{Let } S(m) = \sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}, \text{ for all } m \geq 1.$$

They have defined $S(m)$ to equal the whole statement

$$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}, \text{ for all } m \geq 1.$$

However, this was not what any student meant, and it often led to the "putting the quantifier in twice" problem. In most cases, the student actually meant that $S(m)$ should equal just

$$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$$

(with no "for all" quantifier). In other cases, the student went on to use $S(m)$ as if it meant

$$\sum_{i=1}^{m} i^2.$$

Neither of these is what they defined $S(m)$ to be.

Mixing up dependent and independent variables in a summation

Can you see what is wrong with the following?

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (i+1)^2.$$

It's easy to see the problem if you write the statement out another way:

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = 1^2 + 2^2 + 3^2 + \cdots + k^2 + (i+1)^2.$$

The value of the sum depends on k. It has nothing to do with i, which is like a local variable of the summation. So the value of the sum cannot be re-expressed in terms of i.

Below is another example of this sort of error:

$$\text{INDUCTION HYPOTHESIS: Assume that } \sum_{k=1}^{m} k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Here, the left side of the equality can be rewritten

$$1^2 + 2^2 + 3^2 + \cdots + m^2$$

which depends on m, not k.

Sloppy chaining of equalities

Consider this segment of a proof:

INDUCTION STEP:

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{2k^3 + 9k^2 + 13k + 6}{6} \end{aligned}$$

$$
\begin{aligned}
&= \cdots \textit{ algebra and application of induction hypothesis} \\
&= \sum_{i=1}^{k+1} i^2
\end{aligned}
$$

This is extremely confusing. It is simply wrong to begin by asserting that

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

This hasn't been shown to be true — in fact, the very goal of the induction step is to prove it. Forgetting this problem, what has been shown?

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k+1} i^2.$$

Not terribly interesting! A correct way to write this proof segment would be:

INDUCTION STEP:

We must prove that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$.

$$
\begin{aligned}
\frac{(k+1)(k+2)(2k+3)}{6} &= \frac{2k^3+9k^2+13k+6}{6} \\
&= \cdots \textit{ algebra and application of induction hypothesis} \\
&= \sum_{i=1}^{k+1} i^2
\end{aligned}
$$

Thus, $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Sloppy use of "LHS/RHS"

The sort of sloppiness shown above is often accompanied by an attempt to do the induction step by proving that some left-hand side ("LHS") equals some right hand side ("RHS"). There is nothing wrong with this, but it completes the inductive proof only if "LHS = RHS" is another way of saying "$S(k+1)$ is true", which is, after all, the thing you must prove in the induction step. In the case of our proof, we could use

$$\sum_{i=1}^{k+1} i^2$$

as our LHS and

$$\frac{(k+1)(k+2)(2k+3)}{6}$$

as our RHS, since our goal in the induction step is to prove that they are equal.

Sloppiness about relationships between statements

Sometimes something as seemingly insignificant as a period can make a big difference in a proof. Consider this segment of a student's proof:

INDUCTION STEP:

Show also for $k+1$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

At first glance, it looks like the second line is stating what is to be shown for $k+1$ (especially since it begins with a sum to $k+1$). That is, it looks as though the student meant:

INDUCTION STEP:

Show that $S(k+1)$ is also true; that is show that

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

But this was not what the student meant, and in fact, didn't make sense given how they had defined $S(n)$. The student actually meant:

INDUCTION STEP:

Show that $S(k+1)$ is also true.

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

The second line is the beginning of the proof of that $S(k+1)$ is also true. A period at the end of the first line would have made this clear.

Vague language

Notice also, in the previous example, that the statement "Show also for $k+1$" was very vague. Show *what* for $k+1$? Here is another example of vagueness:

INDUCTION HYPOTHESIS: Assume true for $m=k$.
INDUCTION STEP: Prove true for $m=k+1$.

Assume what? Prove what?

It is a good idea to avoid this sort of vague language. Notice how we were careful in our proof (see figure 8) to be precise in making these statements. If you make yourself write precisely, you will have to think precisely, which is necessary anyway if you plan to write correct proofs and correct programs. This is a case where ordinary English allows you to be vague in potentially dangerous ways. Sometimes, a little notation can help you to be more precise. In our proof, $S(m)$ helped us to be both precise and concise.

Let $S(m)$, for all integers $m \geq 1$, represent the statement "$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$".

Prove: For all integers $m \geq 1$, $S(m)$ is true.

Proof:

Base Case:

Prove that $S(1)$ is true.

$$\sum_{i=1}^{1} i^2 = 1^2 = 1.$$

$$\frac{1(1+1)(2 \times 1 + 1)}{6} = \frac{1(2)(2+1)}{6} = 1.$$

Thus $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2 \times 1 + 1)}{6}$.

Thus $S(1)$ is true.

Let $k \geq 1$ be an arbitrary integer.

Induction Hypothesis:

Assume that $S(k)$ is true, that is, $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

Induction Step:

Prove that $S(k+1)$ must also be true.

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

But $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ by the Induction Hypothesis.

$$\text{Thus } \sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{2k^3 + 9k^2 + 13k + 6}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Thus, $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Thus $S(k+1)$ is true.

Thus, for all $k \geq 1$, if $S(k)$ is true, then $S(k+1)$ must also be true.

Induction Conclusion:

For all integers $m \geq 1$, $S(m)$ is true.

Figure 8: Another example of a correct inductive proof. In this proof, some simple algebra has been omitted from the induction step. Note that here we are proving a statement about m rather than n. It doesn't matter what we name this integer, as long as it's not a name already in use for something else.