A PDE pricing framework for cross-currency interest rate derivatives

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1. Power Reverse Dual Currency (PRDC) swaps

2. The model and the associated PDE

3. Numerical methods

4. Numerical results

5. Summary and future work
Outline

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PRDC swaps: dynamics

- **Long-dated** cross-currency swaps (≥ 30 years);
- Two currencies (domestic and foreign) and their foreign exchange (FX) rate
- FX-linked PRDC coupon amounts in exchange for LIBOR payments,

\[
\nu_1 L_d(T_0, T_1) N_d \quad \nu_2 L_d(T_1, T_2) N_d \quad \nu_{\beta-1} L_d(T_{\beta-2}, T_{\beta-1}) N_d
\]

\[
T_0 \quad \text{--} \quad T_1 \quad \text{--} \quad T_2 \quad \text{--} \quad T_{\beta-1} \quad \text{--} \quad T_{\beta}
\]

\[
\nu_1 C_1 N_d \quad \nu_2 C_2 N_d \quad \nu_{\beta-1} C_{\beta-1} N_d
\]

- \(C_\alpha = \min \left( \max \left( c_f \frac{s(T_\alpha)}{F(0, T_\alpha)} - c_d, b_f \right), b_c \right)\)
  - \(s(T_\alpha)\): the spot FX-rate at time \(T_\alpha\)
  - \(F(0, T_\alpha) = \frac{P_f(0, T_\alpha)}{P_d(0, T_\alpha)} s(0)\), the forward FX rate
  - \(c_d, c_f\): domestic and foreign coupon rates; \(b_f, b_c\) : a cap and a floor
- When \(b_f = 0, b_c = \infty\), \(C_\alpha\) is a **call option on the spot FX rate**

\[
C_\alpha = h_\alpha \max(s(T_\alpha) - k_\alpha, 0), \quad h_\alpha = \frac{c_f}{f_\alpha}, \quad k_\alpha = \frac{f_\alpha c_d}{c_f}
\]
PRDC swaps: issues in modeling and pricing

- Essentially, a PRDC swap are long dated portfolio of FX options
  - effects of FX skew (log-normal vs. local vol/stochastic vol.)
  - interest rate risk (Vega ($\approx \sqrt{T}$) vs. Rho ($\approx T$))
  
  $\Rightarrow$ high dimensional model, calibration difficulties

- Moreover, the swap usually contains some optionality:
  - knockout
  - FX-Target Redemption (FX-TARN)
  - Bermudan cancelable

This talk is about

- Pricing framework for cross-currency interest rate derivatives via a PDE approach using a three-factor model
- Bermudan cancelable feature
- Local volatility function
- Analysis of pricing results and effects of FX volatility skew
Bermudan cancelable PRDC swaps

The issuer has the right to cancel the swap at any of the times \( \{ T_\alpha \}_{\alpha=1}^{\beta-1} \) after the occurrence of any exchange of fund flows scheduled on that date.

- **Observations**: terminating a swap at \( T_\alpha \) is the same as
  
  i. continuing the underlying swap, and
  
  ii. entering into the offsetting swap at \( T_\alpha \Rightarrow \) the issuer has a long position in an associated offsetting Bermudan swaption

- **Pricing framework**:
  
  - Over each period: dividing the pricing of a Bermudan cancelable PRDC swap into
    
    i. the pricing of the underlying PRDC swap (a “vanilla” PRDC swap), and
    
    ii. the pricing of the associated offsetting Bermudan swaption
  
  - Across each date: apply jump conditions and exchange information
  
  - Computation: 2 model-dependent PDE to solve over each period, one for the PRDC coupon, one for the “option” in the swaption
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The pricing model

Consider the following model under **domestic** risk neutral measure

\[
\frac{ds(t)}{s(t)} = (r_d(t) - r_f(t))dt + \gamma(t,s(t))dWs(t),
\]

\[
drd(t) = (\theta_d(t) - \kappa_d(t)r_d(t))dt + \sigma_d(t)dW_d(t),
\]

\[
rf(t) = (\theta_f(t) - \kappa_f(t)r_f(t) - \rho_{fs}(t)\sigma_f(t)\gamma(t,s(t)))dt + \sigma_f(t)dW_f(t),
\]

- \( r_i(t), i = d, f \): domestic and foreign interest rates with mean reversion rate and volatility functions \( \kappa_i(t) \) and \( \sigma_i(t) \)
- \( s(t) \): the spot FX rate (units domestic currency per one unit foreign currency)
- \( W_d(t), W_f(t), \) and \( W_s(t) \) are correlated Brownian motions with
  \[
  dW_d(t)dW_s(t) = \rho_{ds} dt, \quad dW_f(t)dW_s(t) = \rho_{fs} dt, \quad dW_d(t)dW_f(t) = \rho_{df} dt
  \]
- Local volatility function \( \gamma(t,s(t)) = \xi(t)\left(\frac{s(t)}{L(t)}\right)^{\varsigma(t)-1} \)
  - \( \xi(t) \): relative volatility function
  - \( \varsigma(t) \): constant elasticity of variance (CEV) parameter
  - \( L(t) \): scaling constant (e.g. the forward FX rate \( F(0, t) \))
The 3-D pricing PDE

Over each period of the tenor structure, we need to solve two PDEs of the form

\[
\frac{\partial u}{\partial t} + \mathcal{L}u \equiv \frac{\partial u}{\partial t} + (r_d - r_f) s \frac{\partial u}{\partial s} + \left( \theta_d(t) - \kappa_d(t) r_d \right) \frac{\partial u}{\partial r_d} + \left( \theta_f(t) - \kappa_f(t) r_f - \rho s \sigma_f(t) \gamma(t, s(t)) \right) \frac{\partial u}{\partial r_f} \\
+ \frac{1}{2} \gamma^2(t, s(t)) s^2 \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \sigma_d^2(t) \frac{\partial^2 u}{\partial r_d^2} + \frac{1}{2} \sigma_f^2(t) \frac{\partial^2 u}{\partial r_f^2} \\
+ \rho d \sigma_d(t) \gamma(t, s(t)) s \frac{\partial^2 u}{\partial r_d \partial s} + \rho_f \sigma_f(t) \gamma(t, s(t)) s \frac{\partial^2 u}{\partial r_f \partial s} + \rho_d \sigma_d(t) \sigma_f(t) \frac{\partial^2 u}{\partial r_d \partial r_f} - r_d u = 0
\]

- Derivation: multi-dimensional Itô’s formula
- Boundary conditions: Dirichlet-type “stopped process” boundary conditions
- Backward PDE: solved from \( T_\alpha \) to \( T_{\alpha-1} \) via change of variable \( \tau = T_\alpha - t \)
- Difficulties: high-dimensionality, cross-derivative terms
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Discretization

- **Space:** Second-order central finite differences on uniform mesh
- **Time:**
  - **Crank-Nicolson:** solving a system of the form $\tilde{A}^m u^m = b^{m-1}$ by preconditioned GMRES, where $\tilde{A}^m$ is block-tridiagonal
  - **Alternating Direction Implicit (ADI):** solving several tri-diagonal systems for each space dimension
GMRES with a preconditioner solved by FFT techniques

- Applicable to $\bar{A}^m u^m = b^{m-1}$ with nonsymmetric $\bar{A}^m$
- Starting from an initial guess update the approximation at the $i$-th iteration by linear combination of orthonormal basis of the $i$-th Krylov’s subspace
- **Problem**: slow converge (greatly depends on the spectrum of $\bar{A}^m$)
- **Solution**: preconditioning - find a matrix $P$ such that
  i. GMRES method applied to $P^{-1} \bar{A}^m u^m = P^{-1} b^{m-1}$ converges faster
  ii. $P$ can be solved fast
- **Our choice**:
  - $P = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial r^2_d} + \frac{\partial^2 u}{\partial r^2_f} + u$
  - $P$ is solved by Fast Sine Transforms (FST)
  - Complexity: $O(npq \log(npq))$ flops
**ADI**

Timestepping scheme from time $t_{m-1}$ to time $t_m$:

**Phase 1:**

$$v_0 = u^{m-1} + \Delta \tau (A^{m-1}u^{m-1} + g^{m-1}),$$

$$(I - \frac{1}{2} \Delta \tau A_i^m) v_i = v_{i-1} - \frac{1}{2} \Delta \tau A_i^{m-1} u^{m-1} + \frac{1}{2} \Delta \tau (g_i^m - g_i^{m-1}), \quad i = 1, 2, 3,$$

**Phase 2:**

$$\tilde{v}_0 = v_0 + \frac{1}{2} \Delta \tau (A^m v_3 - A^{m-1} u^{m-1}) + \frac{1}{2} \Delta \tau (g^m - g^{m-1}),$$

$$(I - \frac{1}{2} \Delta \tau A_i^m) \tilde{v}_i = \tilde{v}_{i-1} - \frac{1}{2} \Delta \tau A_i^{m} v_3, \quad i = 1, 2, 3,$$

$$u^m = \tilde{v}_3.$$

- $u^m$: the vector of approximate values
- $A_0^m$: matrix of all mixed derivatives terms; $A_i^m, i = 1, \ldots, 3$: matrices of the second-order spatial derivative in the $s$-, $r_d$-, and $r_s$- directions, respectively
- $g_i^m, i = 0, \ldots, 3$ : vectors obtained from the boundary conditions
- $A^m = \sum_{i=0}^{3} A_i^m; g^m = \sum_{i=0}^{3} g_i^m$
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Market Data

- Two economies: Japan (domestic) and US (foreign)
- \(s(0) = 105\), \(r_d(0) = 0.02\) and \(r_f(0) = 0.05\)
- Interest rate curves, volatility parameters, correlations:

\[
\begin{align*}
P_d(0, T) &= \exp(-0.02 \times T) & \sigma_d(t) &= 0.7\% & \kappa_d(t) &= 0.0\% \\
P_f(0, T) &= \exp(-0.05 \times T) & \sigma_f(t) &= 1.2\% & \kappa_f(t) &= 5.0\%
\end{align*}
\]

- Local volatility function:

<table>
<thead>
<tr>
<th>period (years)</th>
<th>((\xi(t)))</th>
<th>((\varsigma(t)))</th>
<th>period (years)</th>
<th>((\xi(t)))</th>
<th>((\varsigma(t)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 0.5]</td>
<td>9.03%</td>
<td>-200%</td>
<td>(7 10)</td>
<td>13.30%</td>
<td>-24%</td>
</tr>
<tr>
<td>(0.5 1]</td>
<td>8.87%</td>
<td>-172%</td>
<td>(10 15)</td>
<td>18.18%</td>
<td>10%</td>
</tr>
<tr>
<td>(1 3]</td>
<td>8.42%</td>
<td>-115%</td>
<td>(15 20)</td>
<td>16.73%</td>
<td>38%</td>
</tr>
<tr>
<td>(3 5]</td>
<td>8.99%</td>
<td>-65%</td>
<td>(20 25)</td>
<td>13.51%</td>
<td>38%</td>
</tr>
<tr>
<td>(5 7]</td>
<td>10.18%</td>
<td>-50%</td>
<td>(25 30)</td>
<td>13.51%</td>
<td>38%</td>
</tr>
</tbody>
</table>

- Truncated computational domain:

\[
\{(s, r_d, r_f) \in [0, S] \times [0, R_d] \times [0, R_f]\} \equiv \{[0, 305] \times [0, 0.06] \times [0, 0.15]\}
\]
Specification

Bermudan cancelable PRDC swaps

- Principal: $N_d$ (JPY); Settlement/Maturity dates: 1 Jun. 2010/1 Jun. 2040
- Details: paying annual PRDC coupon, receiving JPY LIBOR

\[
\text{Year} & \quad \text{coupon (FX options)} & \quad \text{funding leg} \\
1 & \max(c_f \frac{s(1)}{F(0, 1)} - c_d, 0)N_d & L_d(0, 1)N_d \\
\vdots & \quad \vdots & \quad \vdots \\
29 & \max(c_f \frac{s(29)}{F(0, 29)} - c_d, 0)N_d & L_d(28, 29)N_d
\]

- Leverage level

<table>
<thead>
<tr>
<th>level</th>
<th>low</th>
<th>medium</th>
<th>high</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_f$</td>
<td>4.5%</td>
<td>6.25%</td>
<td>9.00%</td>
</tr>
<tr>
<td>$c_d$</td>
<td>2.25%</td>
<td>4.36%</td>
<td>8.10%</td>
</tr>
</tbody>
</table>

- The payer has the right to cancel the swap on each of $\{ T_\alpha \}_{\alpha=1}^{\beta-1}$, $\beta = 30$ (years)
## Prices and convergence

<table>
<thead>
<tr>
<th>lev.</th>
<th>m</th>
<th>n</th>
<th>p</th>
<th>q</th>
<th>underlying swap</th>
<th>cancelable swap</th>
<th>performance</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>ADI – GMRES</td>
<td>ADI – GMRES</td>
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</tr>
<tr>
<td></td>
<td>value (%)</td>
<td>change</td>
<td>ratio</td>
<td>value (%)</td>
<td>change</td>
<td>ratio</td>
<td>time (s)</td>
</tr>
<tr>
<td>low</td>
<td>4 12</td>
<td>6 6</td>
<td>-11.41</td>
<td>11.39</td>
<td>0.78</td>
<td>1.19 (5)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8 24</td>
<td>12 12</td>
<td>-11.16</td>
<td>11.30</td>
<td>2.5e-3</td>
<td>11.30</td>
<td>8.6e-4</td>
</tr>
<tr>
<td></td>
<td>16 48</td>
<td>24 24</td>
<td>-11.11</td>
<td>11.28</td>
<td>5.0e-4</td>
<td>11.28</td>
<td>1.7e-4</td>
</tr>
<tr>
<td></td>
<td>32 96</td>
<td>48 48</td>
<td>-11.10</td>
<td>11.28</td>
<td>1.0e-4</td>
<td>11.28</td>
<td>4.1e-5</td>
</tr>
<tr>
<td>med.</td>
<td>4 12</td>
<td>6 6</td>
<td>-13.87</td>
<td>13.42</td>
<td>5.0</td>
<td>13.42</td>
<td>3.3e-3</td>
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<tr>
<td></td>
<td>8 24</td>
<td>12 12</td>
<td>-12.94</td>
<td>13.76</td>
<td>9.3e-3</td>
<td>13.76</td>
<td>9.5e-4</td>
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<td>24 24</td>
<td>-12.75</td>
<td>13.85</td>
<td>1.9e-3</td>
<td>13.85</td>
<td>2.6e-4</td>
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<tr>
<td></td>
<td>32 96</td>
<td>48 48</td>
<td>-12.70</td>
<td>13.88</td>
<td>5.0e-4</td>
<td>13.88</td>
<td>3.6</td>
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<tr>
<td>high</td>
<td>4 12</td>
<td>6 6</td>
<td>-13.39</td>
<td>18.50</td>
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<td>3.2</td>
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<td>19.31</td>
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<td>19.31</td>
<td>8.1e-3</td>
</tr>
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<td>16 48</td>
<td>24 24</td>
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<td>19.56</td>
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<td>19.56</td>
<td>2.5e-3</td>
</tr>
<tr>
<td></td>
<td>32 96</td>
<td>48 48</td>
<td>-11.12</td>
<td>19.62</td>
<td>8.0e-4</td>
<td>19.62</td>
<td>4.6</td>
</tr>
</tbody>
</table>

Computed prices and convergence results for the underlying swap and cancelable swap with the FX skew model
Effects of the FX volatility skew - underlying swap

<table>
<thead>
<tr>
<th>leverage ($\frac{Cd}{Cf}$)</th>
<th>low (50%)</th>
<th>medium (70%)</th>
<th>high (90%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>underlying swap</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>model</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>skew</td>
<td>-11.10</td>
<td>-12.70</td>
<td>-11.11</td>
</tr>
<tr>
<td>diff (skew - log-normal)</td>
<td>-2.09</td>
<td>-3.03</td>
<td>-1.26</td>
</tr>
</tbody>
</table>

- The bank takes a **short** position in **low strike** FX call options.
- Skewness $\uparrow$ the implied volatility of low-strike options $\Rightarrow \downarrow$ value of the PRDC swaps.

Why total effect is the most pronounced for medium-leverage PRDC swaps?
- Total effect is a combination of: (i) **change in implied vol.** and (ii) **sensitivity** of the options (Vega) to those changes
- Low-leverage: the most change (lowest strikes) but smallest Vega
- High-leverage: reversed situation
- Medium-leverage: combined effect is the strongest
Effects of the FX volatility skew - cancelable swap

<table>
<thead>
<tr>
<th>leverage ( \left( \frac{c_d}{c_f} \right) )</th>
<th>low (50%)</th>
<th>medium (70%)</th>
<th>high (90%)</th>
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</thead>
<tbody>
<tr>
<td>cancelable swap model</td>
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<td></td>
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<tr>
<td>skew</td>
<td>11.28</td>
<td>13.88</td>
<td>19.62</td>
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<tr>
<td>log-normal</td>
<td>13.31</td>
<td>16.89</td>
<td>22.95</td>
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<tr>
<td>diff (skew - lognormal)</td>
<td>-2.03</td>
<td>-3.01</td>
<td>-3.33</td>
</tr>
</tbody>
</table>

![Graph showing the cancelable swap value vs. spot FX rate](image-url)
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Summary and future work

Summary

• PDE-based pricing framework for multi-currency interest rate derivatives with Bermudan cancelable features in a FX skew model

• Illustration of the importance of having a realistic FX skew model for pricing and risk managing PRDC swaps

Recent projects

• Parallelization on Graphics Processing Units (GPUs) - using two GPUs, each of which for a pricing subproblems which is solved in parallel

Future work

• Numerical methods: non-uniform/adaptive grids, higher-order ADI schemes

• Modeling: higher-dimensional/coupled PDEs for more sophisticated pricing models
Thank you!

   A PDE pricing framework for cross-currency interest rate derivatives  
   Available at http://ssrn.com/abstract=1502302

   Pricing of cross-currency interest rate derivatives on Graphics Processing Units  
   Available at http://ssrn.com/abstract=1498563

   GPU pricing of exotic cross-currency interest rate derivatives with a foreign exchange volatility skew model  
   Available at http://ssrn.com/abstract=1549661

   Parallel implementation on GPUs of ADI finite difference methods for parabolic PDEs with applications in finance  
   Available at http://ssrn.com/abstract=1580057