A PDE pricing framework for cross-currency interest rate derivatives

Duy Minh Dang
Department of Computer Science
University of Toronto, Toronto, Canada
dmdang@cs.toronto.edu

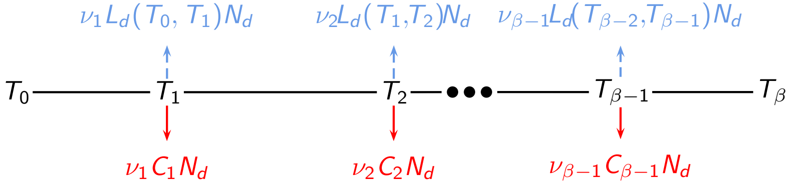
Joint work with Christina Christara, Ken Jackson and Asif Lakhany

Workshop on Computational Finance and Business Intelligence
International Conference on Computational Science 2010 (ICCS 2010)
Amsterdam, May 30–June 2, 2010

Outline

PRDC swaps: dynamics

- **Long-dated** cross-currency swaps ($\geq$ 30 years);
- Two currencies (domestic and foreign) and their foreign exchange (FX) rate
- FX-linked PRDC coupon amounts in exchange for LIBOR payments,



- $C_\alpha = \min\left(\max\left(c_f \dfrac{s(T_\alpha)}{F(0, T_\alpha)} - c_d, b_f\right), b_c\right)$
  - $s(T_\alpha)$: the spot FX-rate at time T_α
  - $F(0, T_\alpha) = \dfrac{P_f(0, T_\alpha)}{P_d(0, T_\alpha)} s(0)$, the forward FX rate
  - c_d, c_f: domestic and foreign coupon rates; b_f, b_c : a cap and a floor
- When $b_f = 0$, $b_c = \infty$, C_α is a **call option on the spot FX rate**

$$C_\alpha = h_\alpha \max(s(T_\alpha) - k_\alpha, 0), \quad h_\alpha = \frac{c_f}{f_\alpha}, k_\alpha = \frac{f_\alpha c_d}{c_f}$$

PRDC swaps: issues in modeling and pricing

- Essentially, a PRDC swap are long dated portfolio of FX options
  - effects of FX skew (log-normal vs. local vol/stochastic vol.)
  - interest rate risk (Vega ($\approx \sqrt{T}$) vs. Rho ($\approx T$))

  $\Rightarrow$ high dimensional model, calibration difficulties
- Moreover, the swap usually contains some optionality:
  - knockout
  - FX-Target Redemption (FX-TARN)
  - Bermudan cancelable

This talk is about

- Pricing framework for cross-currency interest rate derivatives via a PDE approach using a three-factor model
- Bermudan cancelable feature
- Local volatility function
- Analysis of pricing results and effects of FX volatility skew

Bermudan cancelable PRDC swaps

The issuer has the right to cancel the swap at **any** of the times $\{T_\alpha\}_{\alpha=1}^{\beta-1}$ *after* the occurrence of any exchange of fund flows scheduled on that date.

- **Observations**: terminating a swap at T_α is the same as
  1. continuing the underlying swap, and
  2. entering into the offsetting swap at $T_\alpha \Rightarrow$ the issuer has a long position in an associated offsetting Bermudan swaption
- **Pricing framework**:
  - Over each period: dividing the pricing of a Bermudan cancelable PRDC swap into
    1. the pricing of the underlying PRDC swap (a "vanilla" PRDC swap), and
    2. the pricing of the associated offsetting Bermudan swaption
  - Across each date: apply jump conditions and exchange information
  - Computation: 2 model-dependent PDE to solve over each period, one for the PRDC coupon, one for the "option" in the swaption

Outline

The pricing model

Consider the following model under domestic risk neutral measure

$$
\begin{aligned}
\frac{ds(t)}{s(t)} &= (r_d(t) - r_f(t))dt + \gamma(t,s(t))dW_s(t),\\
dr_d(t) &= (\theta_d(t) - \kappa_d(t) r_d(t))dt + \sigma_d(t) dW_d(t),\\
dr_f(t) &= (\theta_f(t) - \kappa_f(t) r_f(t) - \rho_{fs}(t)\sigma_f(t)\gamma(t,s(t)))dt + \sigma_f(t) dW_f(t),
\end{aligned}
$$

- $r_i(t), i = d, f$: domestic and foreign interest rates with mean reversion rate and volatility functions $\kappa_i(t)$ and $\sigma_i(t)$
- $s(t)$: the spot FX rate (units domestic currency per one unit foreign currency)
- $W_d(t)$, $W_f(t)$, and $W_s(t)$ are correlated Brownian motions with $dW_d(t)dW_s(t) = \rho_{ds}dt$, $dW_f(t)dW_s(t) = \rho_{fs}dt$, $dW_d(t)dW_f(t) = \rho_{df}dt$
- Local volatility function $\gamma(t, s(t)) = \xi(t)\left(\frac{s(t)}{L(t)}\right)^{\varsigma(t)-1}$
  - $\xi(t)$: relative volatility function
  - $\varsigma(t)$: constant elasticity of variance (CEV) parameter
  - $L(t)$: scaling constant (e.g. the forward FX rate $F(0, t)$)

The 3-D pricing PDE

Over each period of the tenor structure, we need to solve two PDEs of the form

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \mathcal{L}u \equiv \frac{\partial u}{\partial t} &+ (r_d - r_f)s\frac{\partial u}{\partial s} \\
&+ \Big(\theta_d(t) - \kappa_d(t)r_d\Big)\frac{\partial u}{\partial r_d} + \Big(\theta_f(t) - \kappa_f(t)r_f - \rho_{fS}\sigma_f(t)\gamma(t,s(t))\Big)\frac{\partial u}{\partial r_f} \\
&+ \frac{1}{2}\gamma^2(t,s(t))s^2\frac{\partial^2 u}{\partial s^2} + \frac{1}{2}\sigma_d^2(t)\frac{\partial^2 u}{\partial r_d^2} + \frac{1}{2}\sigma_f^2(t)\frac{\partial^2 u}{\partial r_f^2} \\
&+ \rho_{dS}\sigma_d(t)\gamma(t,s(t))s\frac{\partial^2 u}{\partial r_d \partial s} \\
&+ \rho_{fS}\sigma_f(t)\gamma(t,s(t))s\frac{\partial^2 u}{\partial r_f \partial s} + \rho_{df}\sigma_d(t)\sigma_f(t)\frac{\partial^2 u}{\partial r_d \partial r_f} - r_d u = 0
\end{aligned}
$$

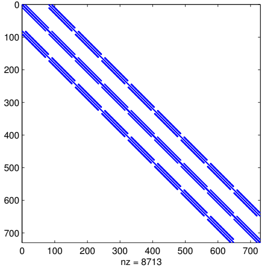
- Derivation: multi-dimensional Itô's formula
- Boundary conditions: Dirichlet-type "stopped process" boundary conditions
- Backward PDE: solved from T_α to $T_{\alpha-1}$ via change of variable $\tau = T_\alpha - t$
- Difficulties: high-dimensionality, cross-derivative terms

Outline

Discretization

- Space: Second-order central finite differences on uniform mesh
- Time:

- **Crank-Nicolson**:
  solving a system of the form $\bar{\mathbf{A}}^m \mathbf{u}^m = \mathbf{b}^{m-1}$ by preconditioned GMRES, where $\bar{\mathbf{A}}^m$ is block-tridiagonal



- **Alternating Direction Implicit (ADI)**:
  solving several tri-diagonal systems for each space dimension

GMRES with a preconditioner solved by FFT techniques

- Applicable to $\bar{\mathbf{A}}^m \mathbf{u}^m = \mathbf{b}^{m-1}$ with nonsymmetric $\bar{\mathbf{A}}^m$
- Starting from an initial guess update the approximation at the i-th iteration by by linear combination of orthonormal basis of the i-th Krylov's subspace
- **Problem**: slow converge (greatly depends on the spectrum of $\bar{\mathbf{A}}^m$)
- **Solution**: preconditioning - find a matrix $\mathbf{P}$ such that
  - i. GMRES method applied to $\mathbf{P}^{-1}\bar{\mathbf{A}}^m \mathbf{u}^m = \mathbf{P}^{-1}\mathbf{b}^{m-1}$ converges faster
  - ii. $\mathbf{P}$ can be solved fast
- **Our choice**:
  - $\mathbf{P} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial r_d^2} + \frac{\partial^2 u}{\partial r_f^2} + u$
  - $\mathbf{P}$ is solved by Fast Sine Transforms (FST)
  - Complexity: $\mathcal{O}(npq \log(npq))$ flops

ADI

Timestepping scheme from time t_{m-1} to time t_m:

Phase 1:

$$\mathbf{v}_0 = \mathbf{u}^{m-1} + \Delta\tau(\mathbf{A}^{m-1}\mathbf{u}^{m-1} + \mathbf{g}^{m-1}),$$

$$(\mathbf{I} - \frac{1}{2}\Delta\tau\mathbf{A}_i^m)\mathbf{v}_i = \mathbf{v}_{i-1} - \frac{1}{2}\Delta\tau\mathbf{A}_i^{m-1}\mathbf{u}^{m-1} + \frac{1}{2}\Delta\tau(\mathbf{g}_i^m - \mathbf{g}_i^{m-1}), \quad i = 1, 2, 3,$$

Phase 2:

$$\widetilde{\mathbf{v}}_0 = \mathbf{v}_0 + \frac{1}{2}\Delta\tau(\mathbf{A}^m\mathbf{v}_3 - \mathbf{A}^{m-1}\mathbf{u}^{m-1}) + \frac{1}{2}\Delta\tau(\mathbf{g}^m - \mathbf{g}^{m-1}),$$

$$(\mathbf{I} - \frac{1}{2}\Delta\tau\mathbf{A}_i^m)\widetilde{\mathbf{v}}_i = \widetilde{\mathbf{v}}_{i-1} - \frac{1}{2}\Delta\tau\mathbf{A}_i^m\mathbf{v}_3, \quad i = 1, 2, 3,$$

$$\mathbf{u}^m = \widetilde{\mathbf{v}}_3.$$

- $\mathbf{u}^m$: the vector of approximate values
- $\mathbf{A}_0^m$: matrix of all mixed derivatives terms; $\mathbf{A}_i^m, i = 1, \ldots, 3$: matrices of the second-order spatial derivative in the s-, r_d-, and r_s- directions, respectively
- $\mathbf{g}_i^m, i = 0, \ldots, 3$: vectors obtained from the boundary conditions
- $\mathbf{A}^m = \sum_{i=0}^3 \mathbf{A}_i^m$; $\mathbf{g}^m = \sum_{i=0}^3 \mathbf{g}_i^m$

Outline

Market Data

- Two economies: Japan (domestic) and US (foreign)
- $s(0) = 105$, $r_d(0) = 0.02$ and $r_f(0) = 0.05$
- Interest rate curves, volatility parameters, correlations:

$$P_d(0,T) = \exp(-0.02 \times T) \quad \sigma_d(t) = 0.7\% \quad \kappa_d(t) = 0.0\% \quad \rho_{df} = 25\%$$

$$P_f(0,T) = \exp(-0.05 \times T) \quad \sigma_f(t) = 1.2\% \quad \kappa_f(t) = 5.0\% \quad \rho_{dS} = -15\%$$

$$\rho_{fS} = -15\%$$

- Local volatility function:

period (years)	$(\xi(t))$	$(\varsigma(t))$	period (years)	$(\xi(t))$	$(\varsigma(t))$
(0 0.5]	9.03%	-200%	(7 10]	13.30%	-24%
(0.5 1]	8.87%	-172%	(10 15]	18.18%	10%
(1 3]	8.42%	-115%	(15 20]	16.73%	38%
(3 5]	8.99%	-65%	(20 25]	13.51%	38%
(5 7]	10.18%	-50%	(25 30]	13.51%	38%

- Truncated computational domain:

  $\{(s, r_d, r_f) \in [0, S] \times [0, R_d] \times [0, R_f]\} \equiv \{[0, 305] \times [0, 0.06] \times [0, 0.15]\}$

Specification

Bermudan cancelable PRDC swaps

- Principal: N_d (JPY); Settlement/Maturity dates: 1 Jun. 2010/1 Jun. 2040
- Details: paying annual PRDC coupon, receiving JPY LIBOR

Year	coupon (FX options)	funding leg
1	$\max(c_f \frac{s(1)}{F(0,1)} - c_d, 0) N_d$	$L_d(0,1) N_d$
...	...	...
29	$\max(c_f \frac{s(29)}{F(0,29)} - c_d, 0) N_d$	$L_d(28,29) N_d$

- Leverage level

level	low	medium	high
c_f	4.5%	6.25%	9.00%
c_d	2.25%	4.36%	8.10%

- The payer has the right to cancel the swap on each of $\{T_\alpha\}_{\alpha=1}^{\beta-1}$, $\beta = 30$ (years)

Prices and convergence

lev.	m	n	p	q	underlying swap value (%)	change	ratio	cancelable swap value (%)	change	ratio	performance ADI time (s) time (s)	GMRES time (s) (it.)
					ADI – GMRES						ADI	GMRES
low	4	12	6	6	-11.41			11.39			0.78	1.19 (5)
	8	24	12	12	-11.16	2.5e-3		11.30	8.6e-4		8.59	12.27 (6)
	16	48	24	24	-11.11	5.0e-4	5.0	11.28	1.7e-4	5.0	166.28	253.35 (6)
	32	96	48	48	-11.10	1.0e-4	5.0	11.28	4.1e-5	4.1	3174.20	4882.46 (6)
med.	4	12	6	6	-13.87			13.42				
	8	24	12	12	-12.94	9.3e-3		13.76	3.3e-3			
	16	48	24	24	-12.75	1.9e-3	4.7	13.85	9.5e-4	3.5		
	32	96	48	48	-12.70	5.0e-4	3.9	13.88	2.6e-4	3.6		
high	4	12	6	6	-13.39			18.50				
	8	24	12	12	-11.54	1.8e-2		19.31	8.1e-3			
	16	48	24	24	-11.19	3.5e-3	5.2	19.56	2.5e-3	3.2		
	32	96	48	48	-11.12	8.0e-4	4.3	19.62	5.4e-4	4.6		

Computed prices and convergence results for the underlying swap and cancelable swap with the FX skew model

Effects of the FX volatility skew - underlying swap

leverage $\left(\frac{c_d}{c_f}\right)$	low (50%)	medium (70%)	high (90%)
	underlying swap		
model			
skew	-11.10	-12.70	-11.11
log-normal	-9.01	-9.67	-9.85
diff (skew - lognormal)	-2.09	-3.03	-1.26

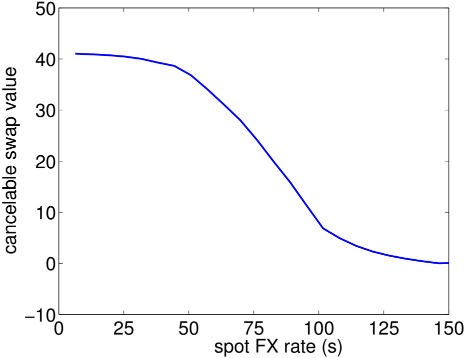
- The bank takes a <u>short</u> position in <u>low strike</u> <u>FX</u> call options.
- Skewness $\nearrow$ the implied volatility of low-strike options $\Rightarrow$ $\searrow$ value of the PRDC swaps.

Why total effect is the most pronounced for medium-leverage PRDC swaps?

- Total effect is a combination of: (i) **change in implied vol.** and (ii) **sensitivity** of the options (Vega) to those changes
- Low-leverage: the most change (lowest strikes) but smallest Vega
- High-leverage: reversed situation
- Medium-leverage: combined effect is the strongest

Effects of the FX volatility skew - cancelable swap

leverage $\left(\frac{c_d}{c_f}\right)$	low (50%)	medium (70%)	high (90%)
	cancelable swap		
model			
skew	11.28	13.88	19.62
log-normal	13.31	16.89	22.95
diff (skew - lognormal)	-2.03	-3.01	-3.33



Outline

Summary and future work

Summary

- PDE-based pricing framework for multi-currency interest rate derivatives with Bermudan cancelable features in a FX skew model
- Illustration of the importance of having a realistic FX skew model for pricing and risk managing PRDC swaps

Recent projects

- Parallelization on Graphics Processing Units (GPUs) - using two GPUs, each of which for a pricing subproblems which is solved in parallel

Future work

- Numerical methods: non-uniform/adaptive grids, higher-order ADI schemes
- Modeling: higher-dimensional/coupled PDEs for more sophisticated pricing models

Thank you!

1. D. M. Dang, C. C. Christara, K. R. Jackson and A. Lakhany (2009)
   A PDE pricing framework for cross-currency interest rate derivatives
   Available at http://ssrn.com/abstract=1502302

2. D. M. Dang (2009)
   Pricing of cross-currency interest rate derivatives on Graphics Processing Units
   Available at http://ssrn.com/abstract=1498563

3. D. M. Dang, C. C. Christara and K. R. Jackson (2010)
   GPU pricing of exotic cross-currency interest rate derivatives with a foreign exchange volatility skew model
   Available at http://ssrn.com/abstract=1549661

4. D. M. Dang, C. C. Christara and K. R. Jackson (2010)
   Parallel implementation on GPUs of ADI finite difference methods for parabolic PDEs with applications in finance
   Available at http://ssrn.com/abstract=1580057

More at http://ssrn.com/author=1173218