Mathematics Preparedness Course

Prepared by: Z. Shahbazi and D. M. Dang

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University of Toronto Scarborough

1 Functions

In the majority of all physical phenomenon, we observe that one quantity depends on another.

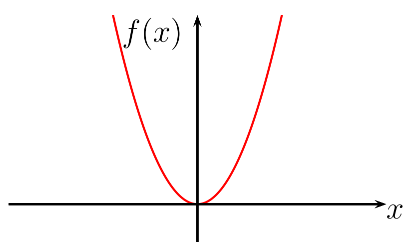
- Example: Height is a function of age
- Example: Temperature is a function of the date.
- Example: Cost of mailing a package is a function of weight.

Any others?

Functions are basically rules, the dependence of one quantity on a different quantity. In order to describe it we need to give it a name. We can use letters such as $f, g, h, \ldots$ to represent functions. There are three different ways to represent a function: algebraically, visually and numerically.

Example:

- Algebraic: $f(x) = x^2$.
- Visual:

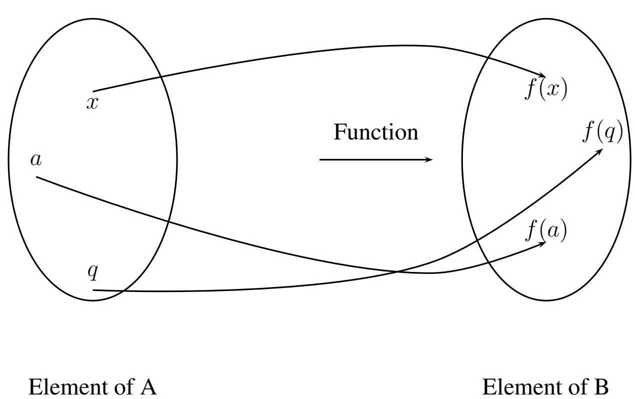


- Numeric:

x	1	2	3	4
$y = f(x)$	1	4	9	16

1.1 Definition of function

A function is a rule that assigns to each element in a set A exactly one element, called $f(x)$, in a set B. When reading $f(x)$ we say —— "f of x". Think of a function as the following:



The set of all input numbers to which the rule of the function applies is called the **domain** of the function and the set of all output numbers is called the **range** of the function

Example: Evaluate the function $f(x) = 2x^2 + 3x$ at (a) $f(3)$ (b) $f(-2)$

1.2 Piecewise defined functions

A **piecewise defined function** is defined by different formulas on different parts of its domain (x values).

Examples:

1. A cell phone plan costs $\$39$ a month. The plan includes 400 free minutes and charges 20 cents for each additional minute of usage. The monthly charges are a function of the number of minutes used by

$$C(x) = \begin{cases} 39 & \text{if} \quad 0 \le x \le 400 \\ 39 + 0.2(x - 400) & \text{if} \quad x \ge 400 \end{cases}$$

   Find $C(100)$, $C(400)$ and $C(480)$.

2. Evaluate the following piecewise function

$$g(x) = \begin{cases} 3 - x & \text{if} \quad x \le 3 \\ x^2 & \text{if} \quad x > 3 \end{cases}$$

   at $g(-2), g(3)$ and $g(10)$.

3. If $f(x) = 2x^2 + 3x - 1$, evaluate the following
   (a) $f(a)$ (b) $f(-a)$
   (c) $f(a + h)$ (d) $\frac{f(a+h)-f(a)}{h}$, $h \neq 0$.

4. If $f(x) = 2x^2$, evaluating the following
   (a) $f(x^2)$ (b) $(f(x^2))^2$

1.3 Polynomial functions

A polynomial in is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0,$$

where $n \geq 0$. If $a_n \neq 0$, then the polynomial is of degree n and the number a_n is called the leading coefficient. A polynomial that is written with descending powers of x is said to be in **standard form**. A polynomial with only one term is called a **monomial**, with two terms is called a **binomial** and with three terms is called a **trinomial**.

Example: Find the leading coefficient and degree of each polynomial function.

Polynomial Function	Leading Coefficient	Degree
$f(x) = 5x^4 + 3x^3 - x^2 + x + 1$	5	4

The graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0,$$

has several characteristics. First, its end behavior (behavior for large values of x, either positive or negative) resembles the graph of the power function $f(x)$. Second, the graph is smooth, which means the graph contains no sharp corners or cusps. Third, the graph is continuous, which means the graph has no gaps or holes and can be drawn without lifting pencil from paper.

1.3.1 Zeros of a polynomial function and their multiplicity

If f is a polynomial function and r is a real number for which $f(r) = 0$, that is, if r is a (real) zero of f then

a. r is an x-intercept of the graph of f, and

b. $(x - r)$ is a factor of f.

If $(x - r)^m$ is a factor of a polynomial and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity** m **of** f.

If r is a zero of even multiplicity then the graph of f **touches** the x-axis at r and the sign of f does not change from one side to the other side of r.

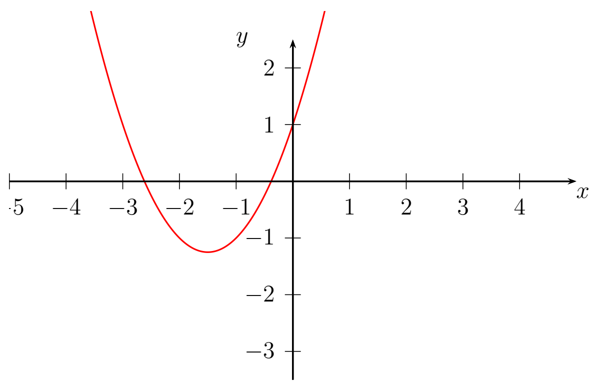
If r is a zero of odd multiplicity then the graph of f **crosses** the x-axis at r and the sign of f changes from one side to the other side of r.

1.3.2 Turning points

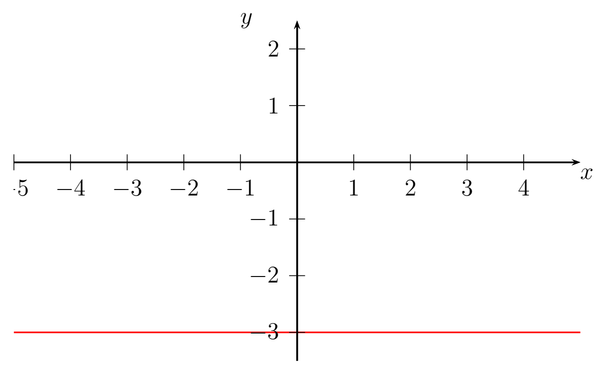
The points at which a graph changes direction are called turning points. These points are called **local maxima** or **local minima** in calculus and their location is important for many real world applications. How to find them algebraically will be discussed in calculus. For our class it is sufficient just to know how many, at most, that there are. The following theorem tells us the maximum possible turning points a polynomial function can have.

Theorem 1 *If f is a polynomial function of degree n, then f has at most $n - 1$ turning points. If the graph of a polynomial function f has $n - 1$ turning points, the degree of the f is at least n.*

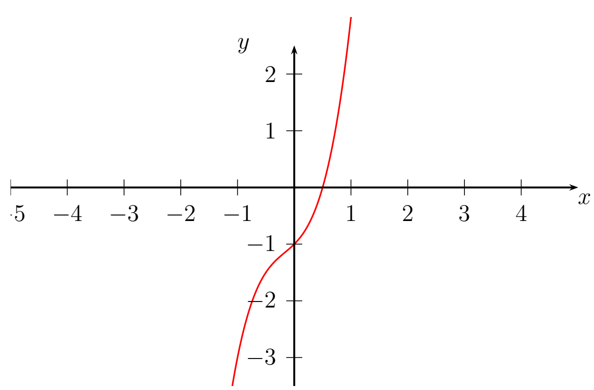
1.3.3 Graphs of polynomial functions



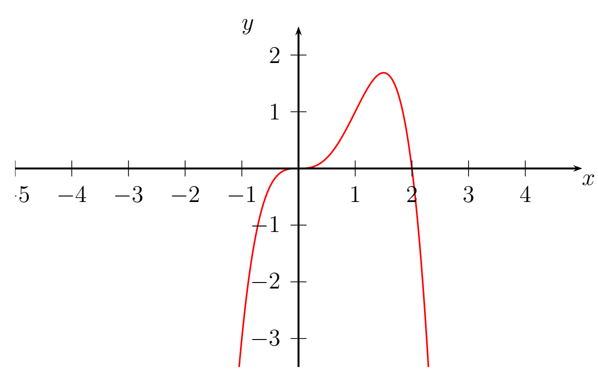
(a) $f(x) = x^2 + 3x + 1$



(b) $f(x) = -3$

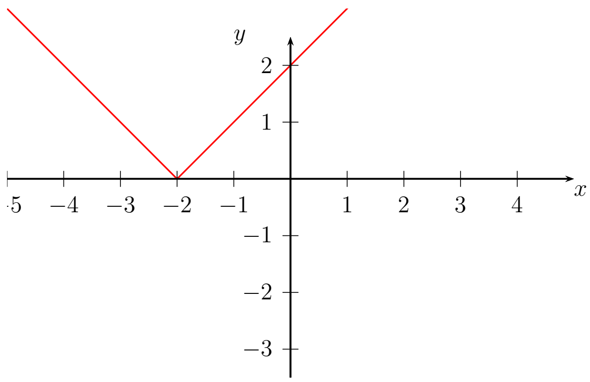


(c) $f(x) = 2x^3 + x^2 + x - 1$

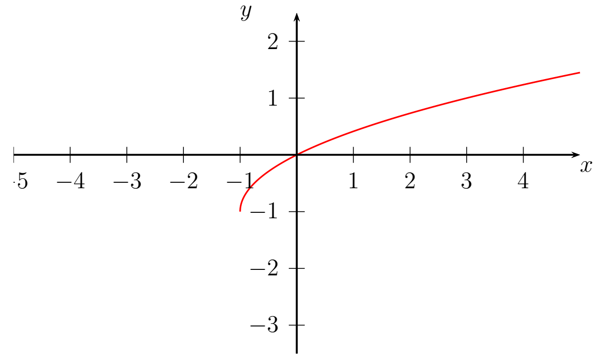


(d) $f(x) = -x^4 + 2x^3$

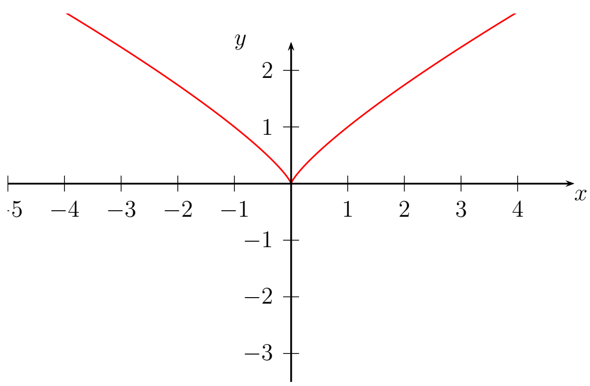
1.3.4 Graphs of non-polynomial functions



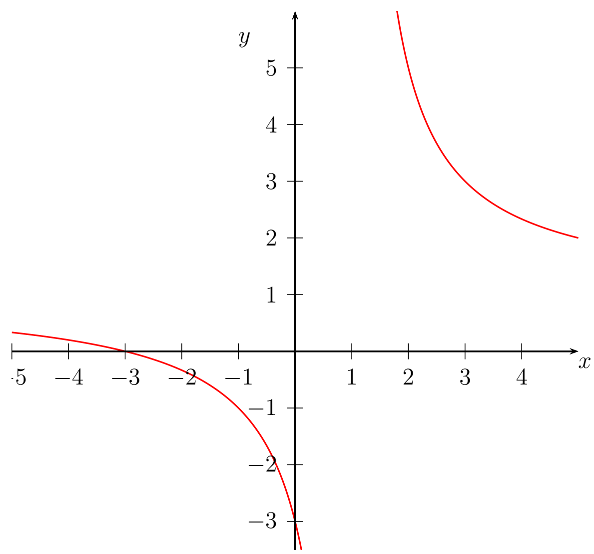
(e) $f(x) = |x+2|$



(f) $f(x) = \sqrt{x+1} - 1$



(g) $f(x) = x^{4/5}$



(h) $f(x) = \frac{x+3}{x-1}$

1.4 Rational functions

A function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, is called a $rational$ function. If the degree of $q(x)$ is greater than that of $p(x)$, the function is called a **proper rational function**. Otherwise, it is called an **improper rational function**. Any improper rational function can be expressed as the sum of a polynomial function and a proper rational function.

Example: Express $f(x) = \dfrac{x^4+2}{x^2+x+1}$ as the sum of a polynomial and a proper rational function.

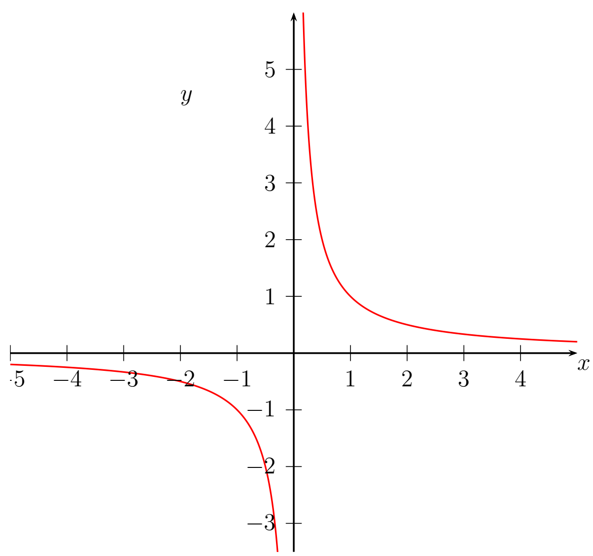
$$
\begin{array}{rlr}
 & & x^2 - x \\ \cline{2-3}
x^2+x+1 \big) & x^4 & +2 \\
 & -x^4 - x^3 - x^2 & \\ \cline{2-2}
 & -x^3 - x^2 & \\
 & x^3 + x^2 + x & \\ \cline{2-3}
 & & x+2
\end{array}
$$

At this stage the division terminates with remainder $x+2$ so that

$$f(x) = x^2 - x + \frac{x+2}{x^2+x+1}.$$

1.4.1 Asymptotes

Rational functions give rise to the concept of **asymptotes**. An asymptote can be thought of as a tangent to a curve at infinity. As a simple example consider the function $f(x) = \dfrac{1}{x}$. As can be seen on the graph below the curve approaches the y-axis as the value of x approaches zero and approaches the x-axis as x becomes large. In this case the y-axis is a vertical asymptote and the x-axis is a horizontal asymptote.



(i) $f(x) = \frac{1}{x}$

Example: Let $f(x) = \dfrac{x^2-2x+2}{(x-1)^2}$. To draw its graph we proceed by the following steps:

i. $x^2 - 2x + 2$ has no real roots so that $y \neq 0$ at any point on the graph.

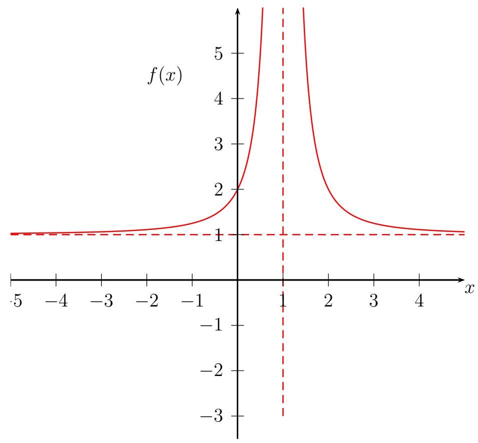
ii. When x has values close to 1, y becomes large.

iii. Notice that it does not matter whether x is slightly less than 1 or slightly more than 1, y is positive in both cases

iv. By noticing that we can rewrite the function as follows

$$y = \frac{x^2 - 2x + 2}{(x-1)^2} = 1 + \frac{1}{(x-1)^2}$$

v. When $x = 0$ then $y = 2$. Thus $(0, 2)$ is the y-intercept.

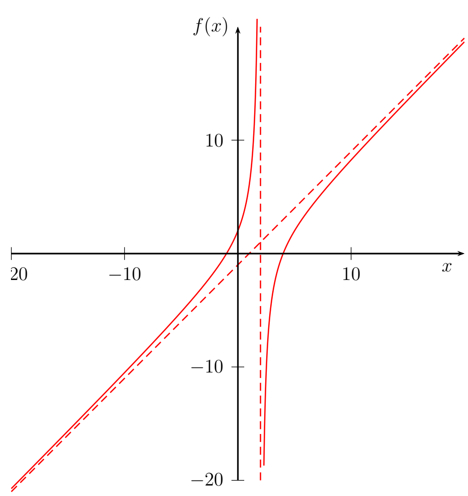


(j) $f(x) = \frac{x^2-2x+2}{(x-1)^2}$

The dotted lines are the lines $x = 1$ and $y = 1$.

- $x = 1$ is a **vertical asymptote**
- $y = 1$ is a **horizontal asymptote**

Generally, rational functions may have vertical, horizontal and slant asymptotes. An example of a rational function with slant asymptote is below.



(k) $f(x) = \frac{x^2-3x-4}{x-2}$

1.5 Composition of functions

Definition 1 $(f \circ g)(x)$ *means* $f(g(x))$.

1.6 Even and odd functions

In calculus if you can recognize that a function is even or odd you can save yourself a lot of work.

Definition 2 *A function is* ***even*** *if* $f(-x) = f(x)$. *This means that the function's graph will have* y***-axis symmetry***.

Definition 3 *A function is* ***odd*** *if* $f(-x) = -f(x)$. *This means that the function's graph will have* ***origin symmetry***.

Remember, a function cannot ever have x-axis symmetry. Why not?

1.7 Inverse functions

There are several things about inverse functions that are important for you to know.

1. What is an inverse function?
   An inverse of a function f is a function that **reverses** what the function f does.

2. How do you determine whether a function HAS an inverse?
   The easiest way is to graph a function and see if the function passes the horizontal line test

3. How do you find the inverse of a function if it exists?
   Given an equation for a function, you need to switch x and y and then solve for y.

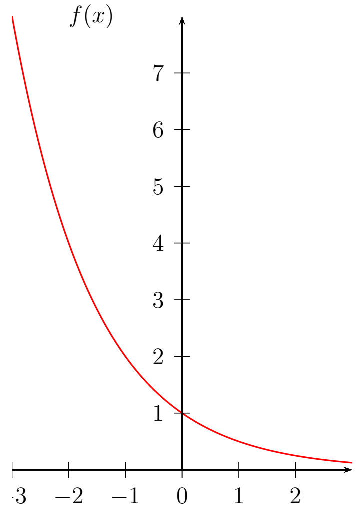
4. How do you determine whether two functions are inverses of each other?
   Graphically, you can determine this if the two graphs reflect across the line $y = x$. Analytically, f and g would be inverses of each other if $f(g(x)) = g(f(x)) = x$.

1.8 Exponential functions

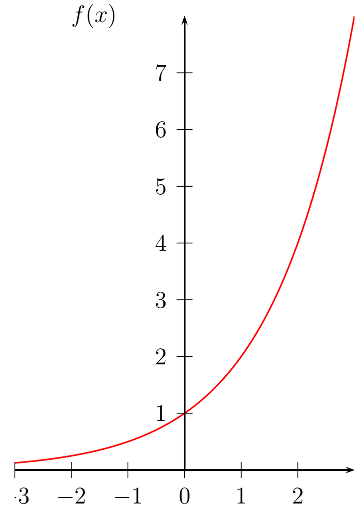
The exponential function f with base a is denoted by $f(x) = a^x$, where $a \neq 1$, and x is any real number. The function value will be positive because a positive base raised to any power is positive. This means that the graph of the exponential function $f(x) = a^x$ will be located in quadrants I and II. For example, if the base is 2 and $x = 4$, the function value $f(4)$ will equal 16. A corresponding point on the graph of $f(x) = 2^x$ would be $(4, 16)$.

1.9 Logarithmic functions

For $x > 0$, $a > 0$, and $a \neq 1$, we have $f(x) = \log_a(x)$ if and only if $a^{f(x)} = x$. Since $x > 0$, the graph of the above function will be in quadrants I and IV.



(l) $f(x) = (\frac{1}{2})^x$



(m) $f(x) = (2)^x$

1.9.1 Comments on logarithmic functions

- The exponential equation $4^3 = 64$ could be written in terms of a logarithmic equation as $\log_4(64) = 3$.
- The exponential equation $5^{-2} = \frac{1}{25}$ can be written as the logarithmic equation $\log_5 \frac{1}{25} = -2$.
- Since logarithms are nothing more than exponents, you can use the rules of exponents with logarithms.
- Logarithmic functions are the inverse of exponential functions. For example if $(4, 16)$ is a point on the graph of an exponential function, then $(16, 4)$ would be the corresponding point on the graph of the inverse logarithmic function.

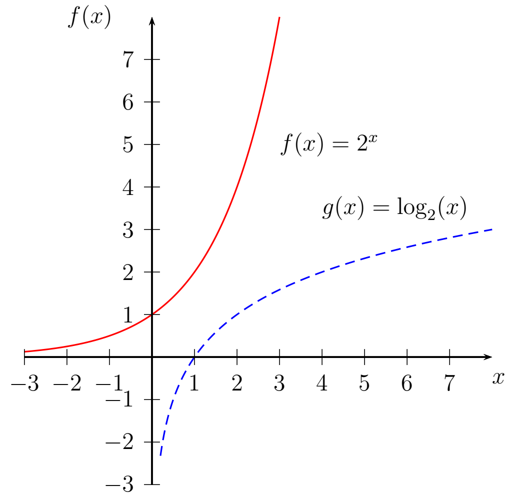
  The two most common logarithms are called common logarithms and natural logarithms. Common logarithms have a base of 10, and natural logarithms have a base of e. On your calculator, the base 10 logarithm is noted by $\log$, and the base e logarithm is noted by $\ln$.

1.9.2 Properties of logarithms

Property 1: $\log_a 1 = 0$ because $a^0 = 1$.

- Example 1: In the equation $14^0 = 1$, the base is 14 and the exponent is 0. Remember that a logarithm is an exponent, and the corresponding logarithmic equation is $\log_{14} 1 = 0$, where the 0 is the exponent.
- Example 2: In the equation $\frac{1}{2}^0 = 1$, the base is $\frac{1}{2}$ and the exponent is 0. Remember that a logarithm is an exponent, and the corresponding logarithmic equation is $\log_{\frac{1}{2}} 1 = 0$, where the 0 is the exponent.

Property 2: $\log_a a = 1$ because $a^1 = a$.
Property 3: $\log_a a^x = x$ because $a^x = a^x$.



(n) $f(x) = 2^x$ and $g(x) = \log_2(x)$

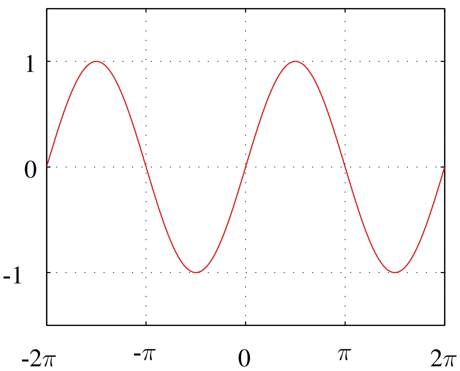
1.9.3 Rules of logarithms

Let a be a positive number such that $a \neq 1$, n be a real number, and u and v be positive real numbers.

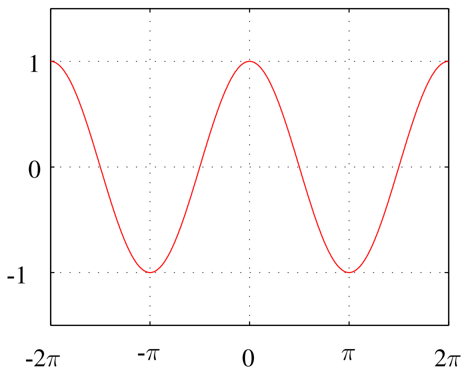
Logarithmic rule 1: $\log_a(uv) = \log_a(u) + \log_a(v)$
Logarithmic rule 2: $\log_a(\frac{u}{v}) = \log_a(u) - \log_a(v)$
Logarithmic rule 3: $\log_a(u)^n = n \log_a(u)$

Since logarithms are nothing more than exponents, these rules come from the rules of exponents. Let a be greater than 0 and not equal to 1, and let n and m be real numbers.

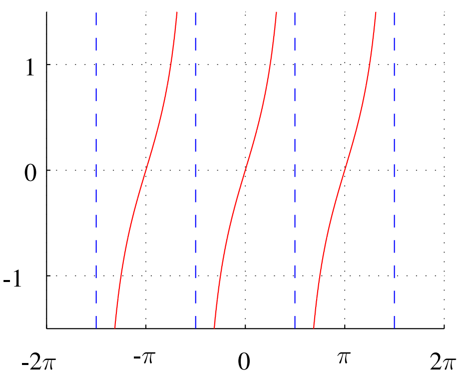
1.10 Trigonometric functions



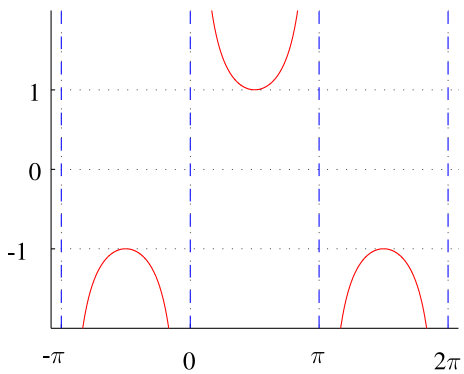
$f(x) = \sin(x)$
Domain:$(-\infty, \infty)$
Range: $[-1, 1]$
Period: 2π



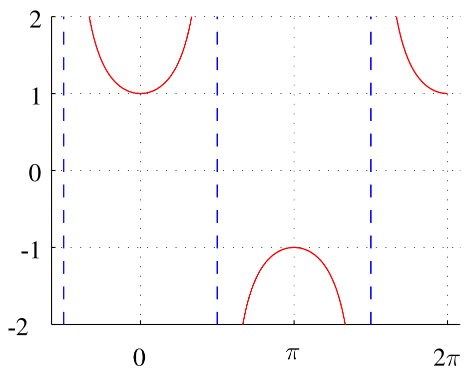
$f(x) = \cos(x)$
Domain:$(-\infty, \infty)$
Range: $[-1, 1]$
Period: 2π



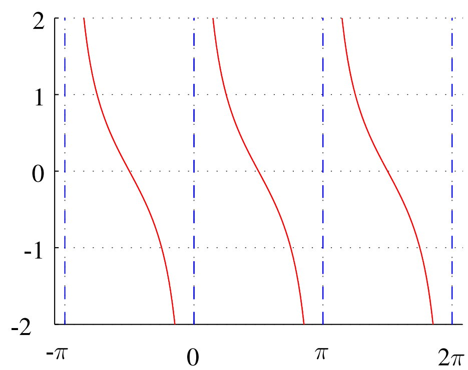
$f(x) = \tan(x)$
Domain: $((k - \frac{1}{2})\pi, (k + \frac{1}{2})\pi)$
Range: $(-\infty, \infty)$
Period: π



$f(x) = \csc(x) = \frac{1}{\sin(x)}$
Domain: $((k-1)\pi, k\pi)$
Range: $(-\infty, -1] \cup [1, \infty)$



$f(x) = \sec(x) = \frac{1}{\cos(x)}$
Domain: $((k - \frac{1}{2})\pi, (k + \frac{1}{2})\pi)$
Range: $(-\infty, -1] \cup [1, \infty)$



$f(x) = \cot(x) = \frac{1}{\tan(x)}$
Domain: $((k-1)\pi, k\pi)$
Range: $(-\infty, \infty)$

This abbreviated chart consists of the values you should know

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\dfrac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\dfrac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\dfrac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\dfrac{\pi}{2}$	1	0
π	0	-1
$\dfrac{3\pi}{2}$	-1	0
2π	0	1

1.10.1 Trigonometric identities

Even/odd identities:

$$\sin(-x) = -1 \times \sin(x) \qquad \cos(-x) = \cos(x) \qquad \tan(-x) = -1 \times \tan(x)$$

Pythagorean identities:

$$\sin^2(x) + \cos^2(x) = 1 \qquad \tan^2(x) + 1 = \sec^2(x) \qquad 1 + \cot^2(x) = \csc^2(x)$$

Sum and difference identities:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double angle identities:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Power reducing identities:

$$\sin^2(\frac{x}{2}) = \frac{1-\cos x}{2} \qquad \cos^2(\frac{x}{2}) = \frac{1+\cos x}{2}$$

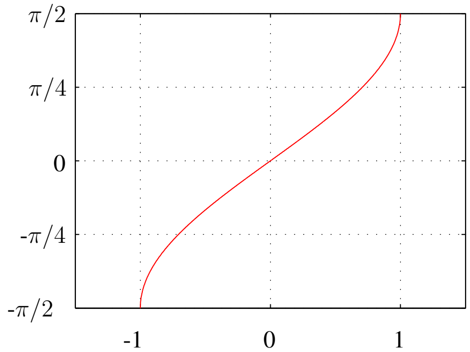
1.10.2 Inverse trigonometric identities

You will also be required to remember information about inverse trigonometric functions, or the "arc" functions. Trigonometric functions technically do not have inverses when we consider the entire domains of these functions. But if the following restrictions are placed on these functions, these functions will have inverses:

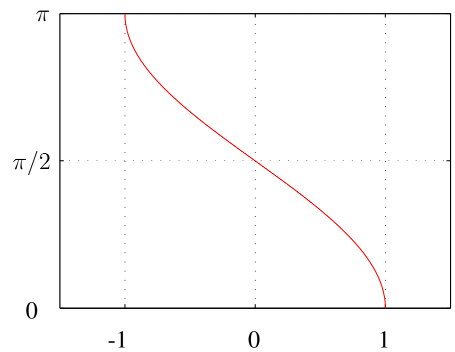
$$\begin{aligned} y &= \sin x \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ y &= \cos x \quad x \in [0, \pi], \\ y &= \tan x \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}). \end{aligned}$$

Remember that the domain of a function becomes the range of the corresponding inverse function. So, the appropriate ranges for these inverse functions are as follows:

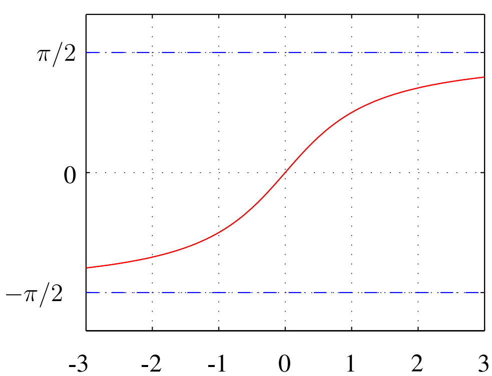
$$\begin{aligned} y &= \sin^{-1} x &&= \arcsin x, \quad y \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ y &= \cos^{-1} x &&= \arccos x, \quad y \in [0, \pi], \\ y &= \tan^{-1} x &&= \arctan x, \quad y \in (-\frac{\pi}{2}, \frac{\pi}{2}). \end{aligned}$$



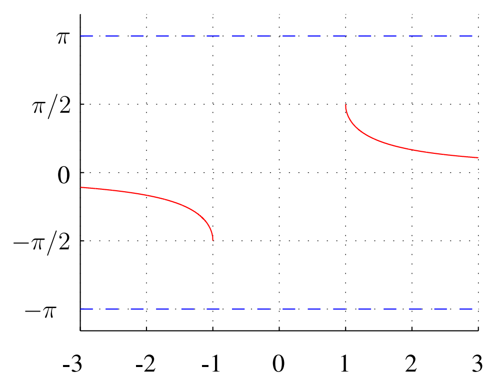
$f(x) = \sin^{-1}(x) \equiv \arcsin(x)$
Domain:$[-1, 1]$
Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$



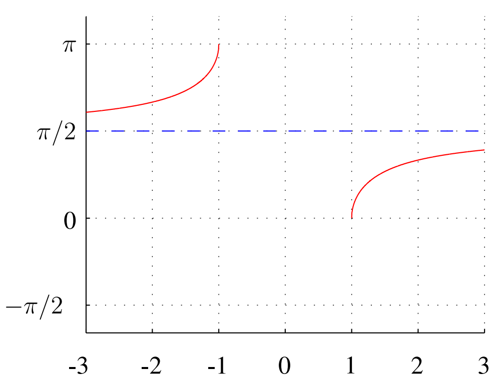
$f(x) = \cos^{-1}(x) \equiv \arccos(x)$
Domain:$[-1, 1]$
Range: $[0, \pi]$



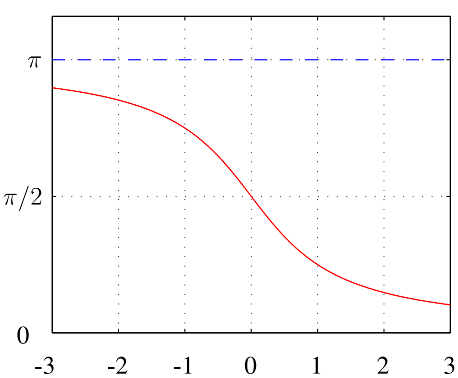
$f(x) = \tan^{-1}(x) \equiv \arctan(x)$
Domain: $(-\infty, \infty)$
Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$



$f(x) = \csc^{-1}(x) \equiv \text{arccsc}(x)$
Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\frac{\pi}{2}, \frac{\pi}{2}], \quad y \neq 0$



$f(x) = \sec^{-1}(x) \equiv \text{arcsec}(x)$
Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi], \quad y \neq \pi/2$



$f(x) = \cot^{-1}(x) \equiv \text{arccot}(x)$
Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Practice questions: Functions

1. Given sets $A = \{x \in \mathbb{R} | x \geq -2\}$, $B = \{x \in \mathbb{R} | x < 5\}$, $C = \{x \in \mathbb{R} | -1 \leq x \leq 7\}$. Then the new sets $A \cap (B \cup C)$ is

   a. $(-2, 7)$
   b. $[-2, 7]$
   c. $[-1, 5)$
   d. none of (a), (b), or (c).

2. The domain of the function $f(x) = \sqrt{4x+3} + \sqrt{x^2 - 1}$ is

   a. $(-1, -\frac{3}{4}] \cup [1, +\infty)$
   b. $[-\frac{3}{4}, 1]$
   c. $[-\frac{3}{4}, +\infty)$
   d. $[1, +\infty)$

3. The domain of the function $f(x) = \dfrac{\sqrt[3]{x}}{x^4 - 1}$ is

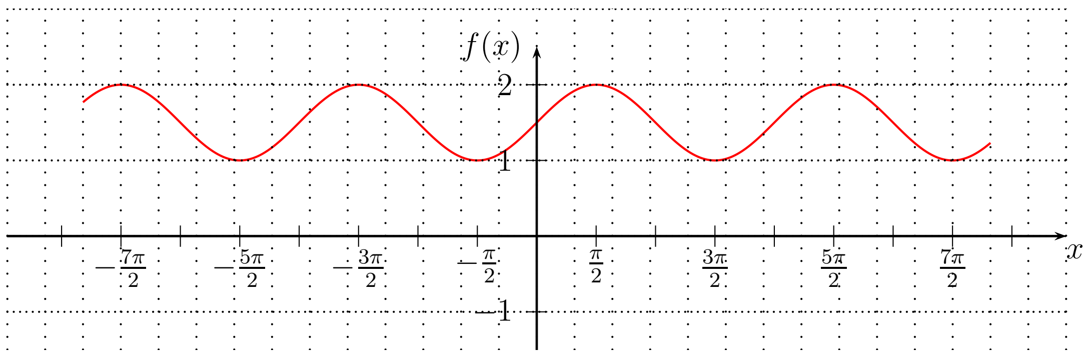
   a. $[0, \infty)$
   b. $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
   c. $[0, 1) \cup (1, \infty)$
   d. none of (a), (b), or (c).

4. The domain of the function
$$f(x) = \begin{cases} 4 & x = 3, \\ x^2 & 1 \leq |x| < 3, \end{cases}$$
   is

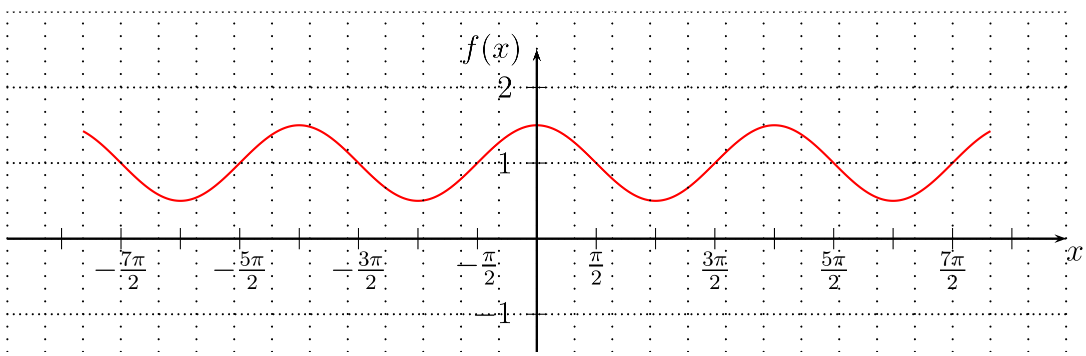
   a. $[1, 3]$
   b. $(-3, -1] \cup [1, 3]$
   c. $[-1, 3]$
   d. none of (a), (b), or (c)

5. Choose the graph of the function $f(x) = 1 + \dfrac{\cos(x)}{2}$

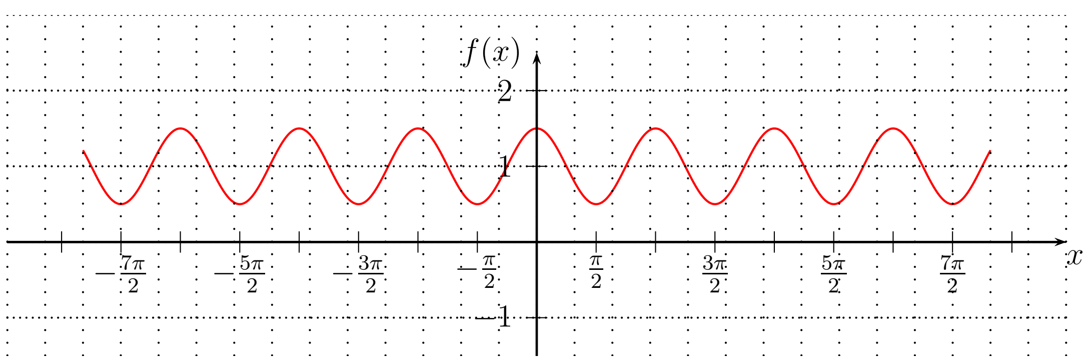
a .



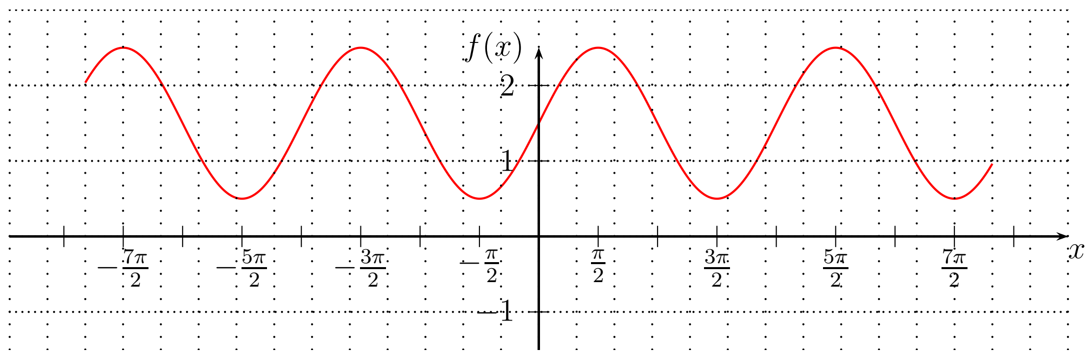
b .



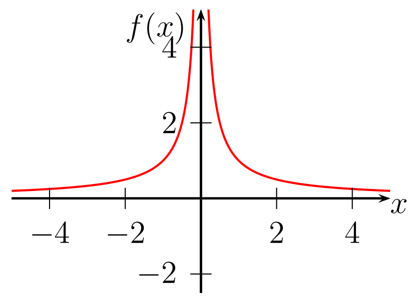
c .



d .



6. The domain and range of the function f whose graph appears below are



a. $\text{dom}(f(x))$ = all real numbers, $\text{range}(f(x))$ = all real numbers

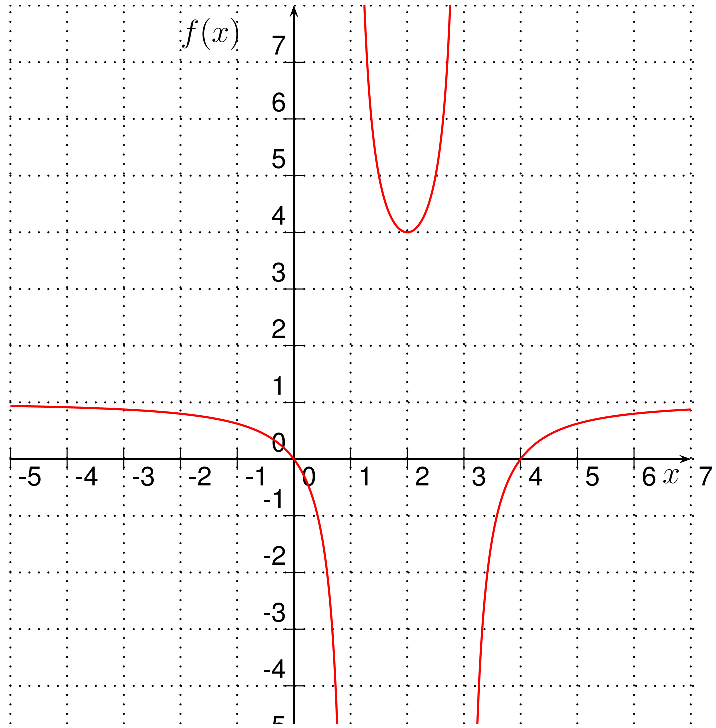
b. $\text{dom}(f(x))$ = all real numbers, $\text{range}(f(x))$ = all real numbers not equal to 0

c. $\text{dom}(f(x))$ = all real numbers not equal to 0, $\text{range}(f(x))$ = all real numbers

d. $\text{dom}(f(x))$ = all real numbers not equal to 0, $\text{range}(f(x))$ = all nonnegative real numbers

f. $\text{dom}(f(x))$ = all real numbers not equal to 0, $\text{range}(f(x))$ = all positive real numbers

7. The domain and range of the function f whose graph appears below are



a. dom($f(x)$) = all real numbers, range($f(x)$) = all real numbers

b. dom($f(x)$) = all real numbers, range($f(x)$) = all real numbers not equal 1 and 3

c. dom($f(x)$) = all real numbers not equal 1 and 3, range($f(x)$) = all real numbers

d. dom($f(x)$) = all real numbers not equal 1 and 3, range($f(x)$) = all real numbers less than 1 or greater than or equal to 4

f. dom($f(x)$) = all real numbers not equal 1 and 3, range($f(x)$) = all real numbers less than or equal 1 or greater than or equal to 4

8. The amount of a saving account is given by $S(t) = 50,000e^{0.045t}$ where t is number of years after 2009. The amount at the end of June 2011 is

   a. $51,137.75$

   b. $52,301.39$

   c. $53,491.51$

   d. $55,953.61$

9. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100,000}{100 + 900e^{-t}}$$

   where t is measured in years. The inverse of this function and its meaning are

   a. $t = \dfrac{100,000}{100 + 900e^{-P}}$; the time required for the population to reach the capacity

b. $t = -\ln\left(\dfrac{9P}{1000-P}\right)$; the time required for the population to reach a given number P

c. $t = -\ln\left(\dfrac{1000-P}{9P}\right)$; the time required for the population to reach a given number P

d. $t = -\ln\left(\dfrac{1000-P}{9P}\right)$; the time required for the population to reach the capacity

10. Starting from a principal of $\$P$, the amount of a saving account grows according to the formula $S(t) = P(1.021)^{2t}$ where t is number of years. The least number of full half-years it takes for the principal to increase by 40% is

    a. 16

    b. 17

    c. 18

    d. none of (a), (b), or (c)

11. Let $f(x) = \sqrt{3x - x^2}$. The equation of $g(x)$ whose graph is the graph of $f(x)$ shifted 2 units to the right and then stretched vertically by a factor of 2 is

    a. $g(x) = 2\sqrt{-x^2 - x + 2}$

    b. $g(x) = \sqrt{-2x^2 - 2x + 4}$

    c. $g(x) = 2\sqrt{-x^2 + 7x - 10}$

    d. none of (a), (b), or (c)

12. If $f(x) = 5x + 3$, then $\dfrac{f(3+h) - f(3)}{h}$ is

    a. 1

    b. 5

    c. $\dfrac{5h-12}{h}$

    d. $\dfrac{5h-3}{h}$

13. The function $g(x) = 1 - x^4$ is

    a. neither

    b. odd.

    c. even.

    d. both even and odd.

14. The graph of the function $f(x) = x^5 + x$ is

    a. symmetric with respect to the y-axis.

    b. symmetric with respect to the x-axis.

c. symmetric about the origin.

d. none of (a), (b), or (c)

15. A radioactive element decays such that after t days the number of milligrams present is given by $N = 100e^{-0.062t}$. The number of hours it takes for the 53.8 milligrams to remain is

    a. approximately 240 hours

    b. approximately 200 hours

    c. approximately 160 hours

    d. approximately 140 hours

16. If $f(x) = x^3$ and $g(x) = 2x - 1$ then $(f \circ g)(x)$ is

    a. $8x^3 - 12x^2 + 6x - 1$

    b. $2x^3 - 1$

    c. $2x^4 - x^3$

    d. none of (a), (b), or (c)

17. If $f(x) = x^2 + 2x - 1$, then $(f \circ f)(1)$ is

    a. 2

    b. 4

    c. 0

    d. 7

18. Given $f(x) = x^2 + 1$, $g(x) = x^2 - x$, and $h(x) = x^3$, then $h\big((f \circ g)(1) - (g \circ f)(1)\big)$ is

    a. 1

    b. 0

    c. -1

    d. none of (a), (b), or (c)

19. If $f(1) = 5$ and $f(5) = -1$ then the value of $f^{-1}(5)$ is

    a. $\frac{1}{5}$

    b. 1

    c. -1

    d. none of (a), (b), or (c)

20. If $f(-10) = 2, f(1) = 5, f(3) = 7, f(5) = 30, f(7) = 20, f(8) = -10$, then $f^{-1}(7) + f^{-1}(5) + f^{-1}(-10)$ is

    a. 12

b. $\dfrac{7}{12}$

c. $\dfrac{17}{70}$

d. none of (a), (b), or (c)

21. Consider $f(x) = (4x-5)^2$, for $x \geq \frac{5}{4}$. The inverse function $f^{-1}(x)$ is

 a. $\dfrac{5}{4} - \dfrac{\sqrt{x}}{4}$

 b. $\dfrac{\sqrt{x+5}}{4}$

 c. $\dfrac{\sqrt{x}}{4} + 5$

 d. $\dfrac{\sqrt{x}}{4} + \dfrac{5}{4}$

22. Exactly how many of the following statements are always true?

 i. Every polynomial function is also a rational function.

 ii. An even function can not be odd.

 iii. Given function $f(x)$ and $g(x) = f(x+c)$ with constant $c \in \mathbb{R}$, then the graph of $g(x)$ is that of $f(x)$ shifted left c units.

 iv. For any functions f and g, the function $(f \circ g)$ is always different from $(f+g)$.

 v. For any function f, $(f^{-1})^{-1} = f$.

 a. 5

 b. 4

 c. 3

 d. 2

 f. 1

 g. 0

23. The domain of the function $f(x) = \sqrt{3 - e^{2x}} + \cos(1-x^2)$ is

 a. $-\infty < x \leq \ln(\sqrt{3})$

 b. $-1 \leq x \leq \dfrac{\ln(3)}{2}$

 c. $-\infty < x < \infty$

 d. none of (a), (b), or (c) is correct

24. The domain of the function $f(x) = 4\csc(x)$ is

 a. $x \in \mathcal{R}$

b. $x \neq \frac{\pi}{2}$
c. $x \neq \pm\frac{n\pi}{2}$ where $n = 0, 1, 2, \ldots$
d. $|x| \leq 4$

25. Given $f(x) = \sqrt{1 - x^2}$ and $g(x) = \cos(2x)$, the domain of $f \circ g$ is

a. $x \neq \pm k\frac{\pi}{4}$ where $k = 0, 1, 2, \ldots$
b. $x \neq \pm k\pi$ where $k = 0, 1, 2, \ldots$
c. $-\infty < x < \infty$
d. none of (a), (b), or (c) is correct

26. The domain of the function $f(x) = \ln(\arctan(x))$ is

a. $0 < x < \infty$
b. $-\infty < x < \infty$
c. $x \neq \pm k\frac{\pi}{2}$ where $k = 0, 1, \ldots$
d. none of (a), (b), or (c) is correct

27. The function f such that $f \circ g = F$ given that $g(x) = 3x$ and $F(x) = 2\sin(3x)$ is

a. $f(x) = 2\sin(x)$
b. $f(x) = \sin(2x)$
c. $f(x) = 2\sin(3x)$
d. $f(x) = \sin(3x)$

28. Which of the following functions is (are) symmetric about the origin?

i. $f(x) = \sin(x)$
ii. $f(x) = \tan^2(x)$
iii. $f(x) = \cos(\dfrac{x}{2})$
iv. $f(x) = \tan(x)$

a. (i) only
b. (ii) and (iii)
c. (iv) only
d. (ii) and (iv)
e. (i) and (iv)

29. Exactly how many of the following four mathematical statements are always true:

i. The product of two even functions is an even function.
ii. The product of two odd functions is an even function.

iii. The product of an odd and an even function is an odd function.

iv. The composition of two even functions is an even function.

a. 0

b. 1

c. 2

d. 3

e. 4

30. The exact value of $\cos(\arcsin \frac{\sqrt{3}}{2})$ is

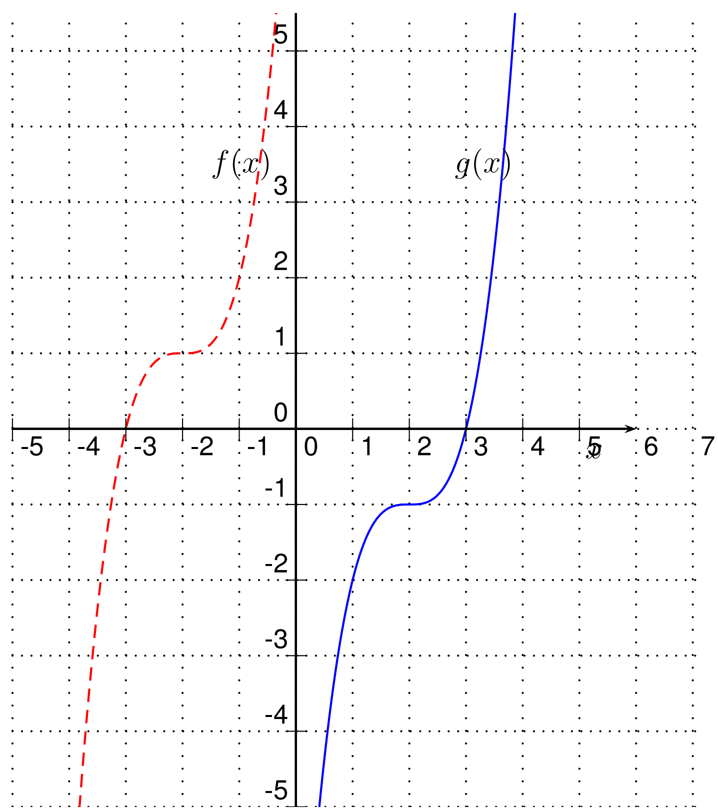
a. $\dfrac{\sqrt{2}}{2}$

b. $-\dfrac{1}{2}$

c. 0

d. $\dfrac{1}{2}$

31. Consider functions $f(x)$ and $g(x)$ whose graphs appear below



Exactly how many of the following four mathematical statements are true about these two functions

i. $f^{-1}(x) = g(x)$ and $g^{-1}(x) = f(x)$ for all real numbers x.

ii. $f(x) = g(x)$ for all real numbers x.

iii. dom$(f(x)) \neq$ dom$(g(x))$

iv. range$(f(x)) \neq$ range$(g(x))$

v. $f(x) = f(-x)$ and $g(x) = g(-x)$ for all real numbers x.

vi. $f(x) = -f(-x)$ and $g(x) = -g(-x)$ for all real numbers x.

vii. $f(x+1) - 2 = g(x-2) + 1$ for all real numbers x.

viii. $f(x+2) - 1 = g(x-2) + 1$ for all real numbers x.

ix. $f(x-2) - 1 = g(x+2) + 1$ for all real numbers x.

x. $f(x+1) - 2 = g(x-1) + 2$ for all real numbers x.

a. 0
b. 1
c. 2
d. 3
e. 4
f. 5
g. 6
h. 7
i. 8
j. 9
k. 10

2 What is a limit?

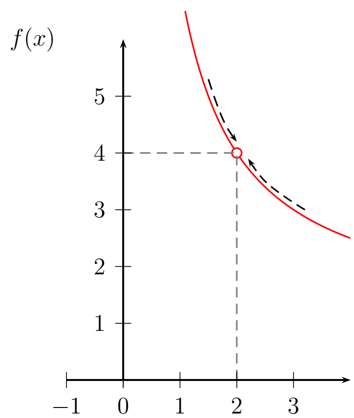
Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ gets closer and close to L as the value of x gets closer to a (not equal to a).

Example 1: Evaluate the following limit

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$



(o) $f(x) = \frac{x^2+4x-12}{x^2-2x}$

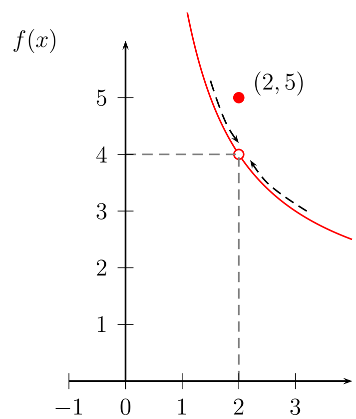
In other words, we are asking what the graph is doing around the point $x = 2$. In our case we can see that as x moves in towards 2 (from both sides), the function is approaching $y = 4$. Thus,

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4.$$

Example 2: Evaluate the following limit

$$\lim_{x\to 0} g(x) \quad \text{where} \quad g(x) = \begin{cases} \dfrac{x^2+4x-12}{x^2-2x} & \text{if} \quad x \neq 2, \\ 5 & \text{if} \quad x = 2. \end{cases}$$

The limit is **NOT** 5!!! Remember from the discussion after the first example that limits do not care what the function is actually doing at the point in question. Limits are only concerned with what is going on **around** the point. Since the only thing about the function that we actually changed was its behavior at $x = 2$ this will not change the limit.

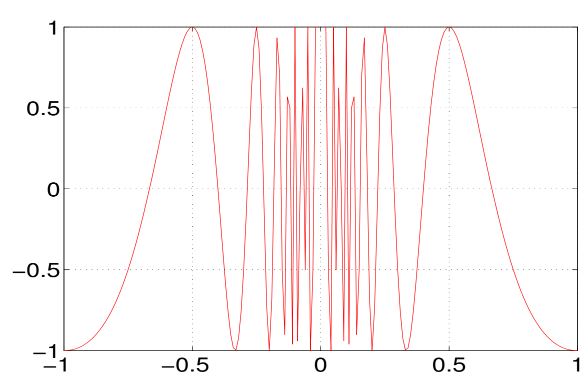


(p) $g(x)$

Example 3: Evaluate the following limit

$$\lim_{t\to 0} \cos\left(\frac{\pi}{t}\right).$$

From this graph we can see that as we move in towards $t = 0$ the function starts oscillating wildly and in fact the oscillations increases in speed the closer to $t = 0$ that we get. Recall from our definition of the limit that in order for a limit to exist the function must be settling down in towards a single value as we get closer to the point in question.



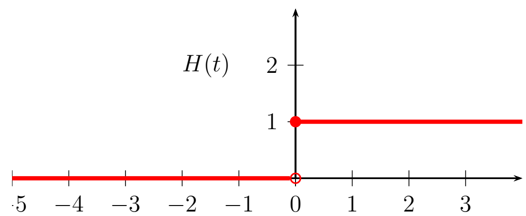
This function clearly does not settle in towards a single number and so this limit does not exist!

Example 4: Evaluate the following limit

$$\lim_{t\to 0} H(t) \quad \text{where} \quad H(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ 1 & \text{if} \quad t \geq 0. \end{cases}$$

This is called a step function, or a heaviside function.

We can see from the graph that if we approach $t = 0$ from the right side the function is moving in towards a y value of 1. Well actually it's just staying at 1, but in the terminology that we've been using in this section it's moving in towards 1. Also, if we move in towards $t = 0$ from the left the function is moving in towards a y value of 0.



(q) $f(x) = H(t)$

According to our definition of the limit the function needs to move in towards a single value as we move in towards $t = a$ (from both sides). This isn't happening in this case and so in this example we will also say that the limit doesn't exist.

What have we learned?

- In the first three examples we saw that limits do not care what the function is actually doing at the point in question. They only are concerned with what is happening around the point. In fact, we can have limits at $x = a$ even if the function does not take a value at that point. Likewise, even if a function exists at a point there is no reason (at this point) to think that the limit will have the same value as the function at the point.
- Sometimes the limit and the function will have the same value at a point and other times they won't have the same value.

Right-hand limits

$$\lim_{x\to a^+} f(x) = L$$

Left-hand limits

$$\lim_{x\to a^-} f(x) = L$$

Note: one-sided limits do not care about what's happening at the point any more than normal limits do. They are still only concerned with what is going on around the point. The only real difference between one-sided limits and normal limits is the range of x that we look at when determining the value of the limit.

Example 5: The function in Example 4

$$\lim_{t\to 0^+} H(x) = 1 \qquad \lim_{t\to 0^-} H(x) = 0.$$

Example 6: The function in example 3. We can see that both of the one-sided limits suffer the same problem that the normal limit did. The function does not settle down to a single number on either side of $t = 0$. Therefore, neither the left-handed nor the right-handed limit will exist.

Note: when sided limits exists and are equal then the limit exists. That is,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

then $\lim\limits_{x \to a} f(x)$ exists.

2.1 Properties of limits

- $\lim\limits_{x \to c} b = b$
- $\lim\limits_{x \to c} x = c$
- $\lim\limits_{x \to c} x^n = c^n$
- $\lim\limits_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$

Let $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = K$

- $\lim\limits_{x \to c} [bf(x)] = bL$
- $\lim\limits_{x \to c} [f(x) \pm g(x)] = L \pm K$
- $\lim\limits_{x \to c} [f(x)g(x)] = LK$
- $\lim\limits_{x \to c} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L}{K} \quad , K \neq 0$
- $\lim\limits_{x \to c} [f(x)]^n = L^n$

Example 7: Evaluate by using the properties of limits. Show each step and which property was used.

$$\lim_{x \to 2} (4x^2 + 3) =$$

Fact: If f and g are functions such that $\lim\limits_{x \to c} g(x) = L$ and $\lim\limits_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left[\lim_{x \to c} g(x)\right] = f(L).$$

Example 8:

$$\lim_{x \to 2} \sqrt[3]{2x^2 - 10} =$$

By now you should have already arrived at the conclusion that many algebraic functions can be evaluated by direct substitution.

Theorem 2 ***The Squeeze Theorem***
If

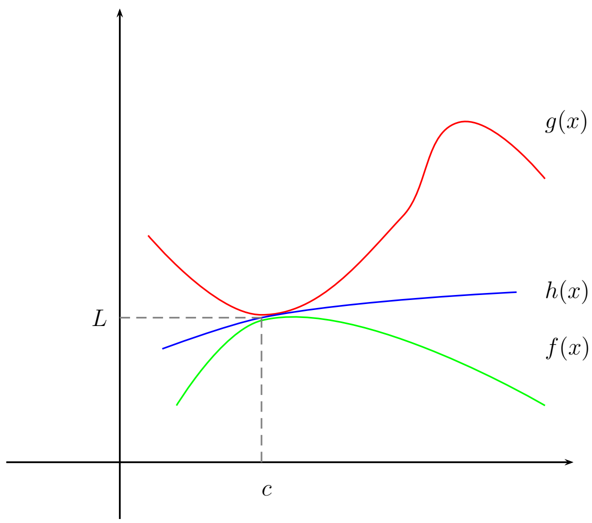
$$f(x) \le h(x) \le g(x)$$

for all x in $[a, b]$ *and* $a \le c \le b$

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = l,$$

then

$$\lim_{x \to c} h(x) = l.$$



(r) An example of the Squeeze theorem

Example 9: Find the following limit

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$$

Solution: We will use the squeeze theorem. Since $-1 \le \cos x \le 1$, we have

$$\begin{aligned} -1 \le \cos\left(\frac{1}{x}\right) &\le 1 \\ -x^2 \le x^2 \cos\left(\frac{1}{x}\right) &\le x^2 \end{aligned}$$

and since

$$\lim_{x \to 0} -x^2 = 0, \quad \lim_{x \to 0} x^2 = 0,$$

we have

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

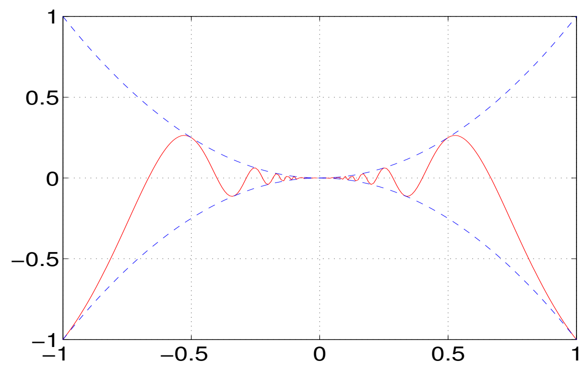


Figure 2.1: $f(x) = x^2 \cos\left(\frac{1}{x}\right)$

Practice questions: Limits

1. $\lim_{x \to -2} 3x^3 - 4x^2 + 2x - 3 =$

   a. -47.

   b. 13.

   c. 20.

   d. none of (a), (b), or (c).

2. If $h(x) = \dfrac{x^2 - 9x + 20}{x^2 - 3x - 4}$ then the value of $\lim_{x \to 4} h(x)$ is

   a. $4/5$

   b. $-4/5$

   c. $-1/5$

   d. none of (a), (b), or (c)

3. If $f(x) = x^3 - 4x^2$, then $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

   a. equals $3x^2 - 8x$.

   b. equals 0.

   c. equals h.

   d. does not exist.

4. For a particular predator-prey relationship, it was determined that the number y of prey consumed by an individual predator over a period of time was a function of prey density x (the number of prey per unit area). Suppose

$$y = f(x) = \frac{10x}{1 + 0.1x}.$$

   If the prey density were increase without bound, y would approach

a. 100
b. 10
c. 1
d. 1.1

5. $\lim_{x\to 2^+}(x\sqrt{x^2-4}) =$

a. -2
b. 0
c. 1
d. 2

6. $\lim_{x\to 3}\dfrac{1-\sqrt{x-2}}{x-3}$

a. equals $-\frac{1}{2}$.
b. equals $\frac{1}{2}$.
c. equals 0
d. does not exist

7. Given

$$f(x) = \begin{cases} x^2+1 & x \geq 1, \\ 3 & x < 1, \end{cases}$$

the value of $\lim_{x\to 1} f(x)$ is

a. 2
b. 3
c. 4
d. none of (a), (b), or (c)

8. If $h(t) = \dfrac{t^3+3t^2+2t}{t^2+t-2} - \dfrac{1}{3}\ln(5+2t)$, then the value of $\lim_{t\to -2} h(t)$ is

a. 0
b. -1
c. $-\frac{2}{3}$
d. undefined

9.

$$f(x) = \begin{cases} x^2-2 & x \leq 3, \\ x+c^2 & x > 3. \end{cases}$$

If $\lim_{x\to 3} f(x)$ exists, then

a. $c = 2$ only.
b. $c = -2$ only.
c. $c = \pm 2$ only.
d. none of (a), (b), or (c).

10.
$$f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & x < 2, \\ c^2 x^2 & x > 2. \end{cases}$$

If $\lim_{x \to 2} f(x)$ exists, then

a. $c = \pm\sqrt{3}$.
b. $c = \pm 3$.
c. $c = \pm\sqrt{2}$.
d. none of (a), (b), or (c).

11. For $f(x) = \left(\dfrac{1}{|x|} + \dfrac{1}{x}\right)$, $\lim_{x \to 0^-} f(x)$

a. equals 0
b. equals 1
c. equals 2
d. does not exist

12. $\lim_{x \to 1^+} \dfrac{x^2 - 1}{|x - 1|} - \lim_{x \to 1^-} \dfrac{x^2 - 1}{|x - 1|}$

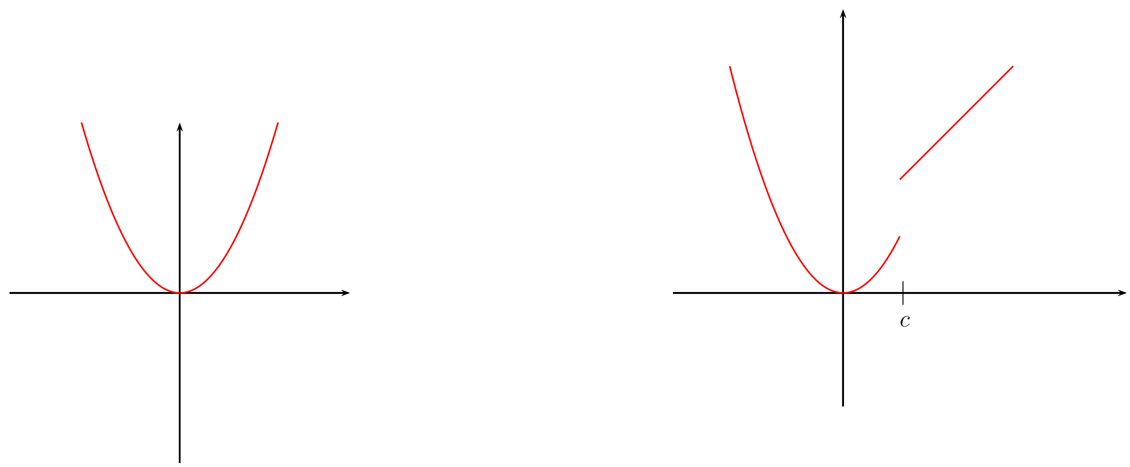
a. equals -4
b. equals -2
c. equals 2
d. equals 4

13. $\lim_{x \to \pi} \dfrac{\sin(x)}{2 + \cos(x)} =$

a. -1
b. 0
c. 1
d. 2

3 Continuity

Intuitively, a function is continuous at $x = c$ if you can draw it without lifting your pen from the paper. In the diagram below, the function on the left is continuous throughout, but the function on the right is not. It is "discontinuous" at $x = c$.

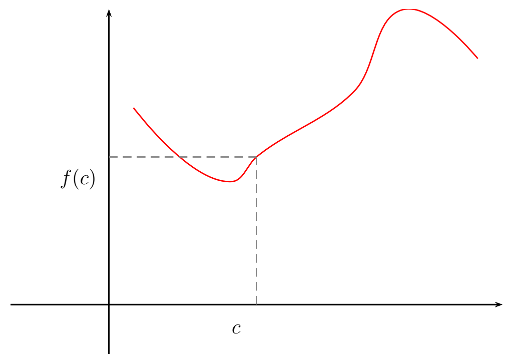


(a) continuity at c (b) discontinuity at c

Formally, a function is continuous at $x = c$ on the open interval (a, b) if the following three conditions are met:

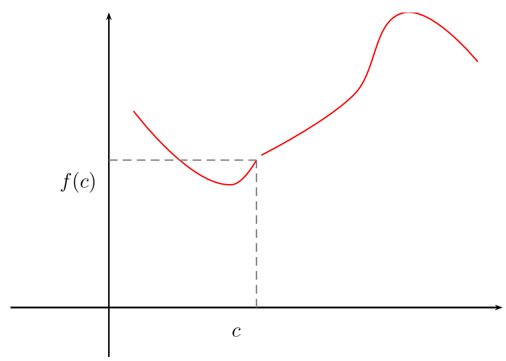
1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$.

Graphically the following function is continuous at c:



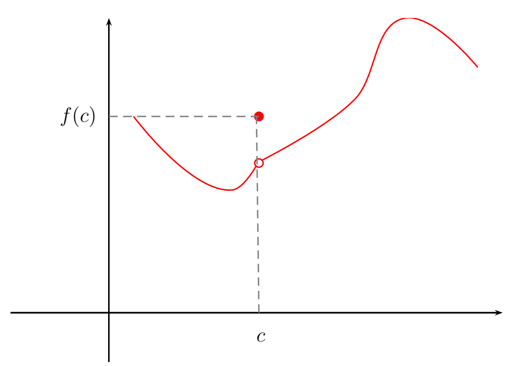
(c) continuity at c

But the following function is discontinuous at c:



(d) discontinuity at c

Here $\lim_{x\to c} f(x)$ does not exist. In the next example although $\lim_{x\to c} f(x)$ exists but $\lim_{x\to c} f(x) \neq f(c)$. Thus, the function is discontinues at c.

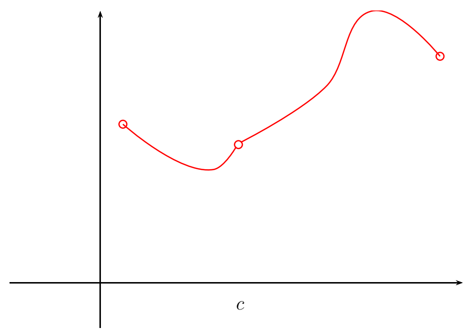


(e)

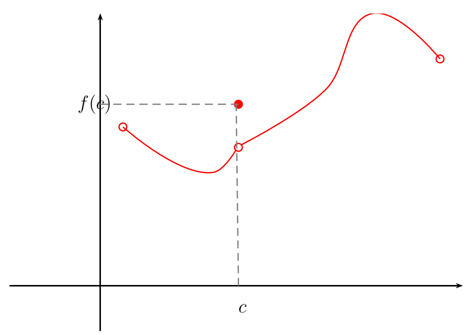
4 Discontinuity

There are two categories of discontinuities

- **Removable discontinuity**
  A discontinuity at $x = c$ is called *removable* if f can be made continuous by appropriately defining (or redefining) $f(c)$.

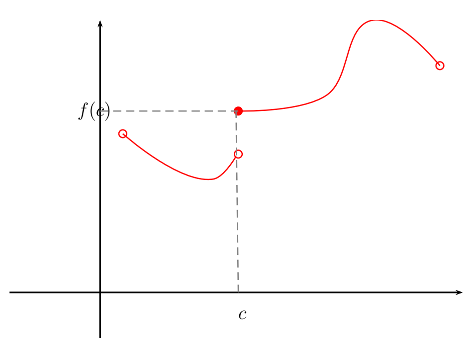


(f) removable discontinuity



(g) removable discontinuity

- **Non-removable discontinuity**
  A discontinuity is called *non-removable* if the one-sided limits each are tending towards different values. These discontinuities are often called jump discontinuities because the graph of the function seems to "jump" to another point. Jump discontinuities will occur where the graph has a break.



(h) non-removable

Practice questions: Continuity

1. The set of all points of continuity of the function $f(x) = \dfrac{x^2 + 3x - 4}{x^2 - 4}$ is

   a. $x = \pm 2$
   b. $(-\infty, -2) \cup (2, \infty)$
   c. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
   d. $(-2, 2)$

2. The set on which the function $g(x) = \sqrt{x+1} + \dfrac{x+1}{x-1} + \dfrac{x+2}{x^2-2}$ is continuous is

   a. $[-1, 1) \cup (1, \sqrt{2}) \cup (\sqrt{2}, +\infty)$
   b. $[-\sqrt{2}, 1) \cup (1, \sqrt{2}) \cup (\sqrt{2}, +\infty)$
   c. $(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, 1) \cup (\sqrt{2}, +\infty)$
   d. $(-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, +\infty)$

3. The function $f(x) = \dfrac{x^{-1} + c^{-1}}{x^{-1} - c^{-1}}$ where $c > 0$ is discontinuous at

   a. 0 only
   b. $\dfrac{1}{c}$ only
   c. 0 and $\dfrac{1}{c}$
   d. 0 and c

4. For what values of the constants a, b is the function f continuous on $(-\infty, +\infty)$ where

$$f(x) = \begin{cases} x + a & x > 2, \\ x^2 & x < 2, \\ b & x = 2. \end{cases}$$

   a. $a = 2, b = 4$
   b. $a = -2, b = 4$
   c. $a = 2, b = 2$
   d. $a = -2, b = -2$

5. Suppose that the long-distance rate for a phone call from Toronto to Ottawa is $\$0.1$ for the first minute and $\$0.06$ for each additional minutes of fraction thereof. Let $y = f(t)$ be a function that indicates the total charge for a call of t minutes with $0 < t \leq 4\frac{1}{2}$. The graph of $f(t)$ for $0 < t \leq 4\frac{1}{2}$ is

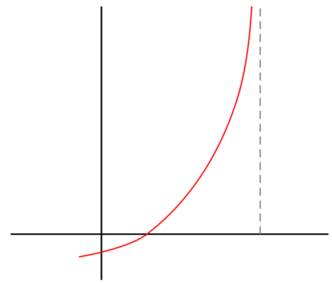
   a. continuous

b. discontinuous at exactly one value of t
c. discontinuous at exactly two values of t
d. discontinuous at exactly three values of t
e. discontinuous at exactly four values of t
f. discontinuous at exactly five values of t
g. nowhere continuous

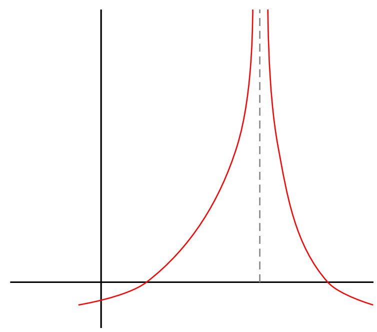
5 Infinite limits

There are two types of infinite limits.

- **The limit equals infinity**

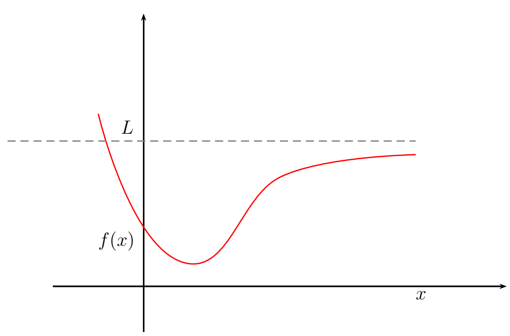


(i) $\lim_{x \to a^-} f(x) = +\infty$



(j) $\lim_{x \to a} f(x) = +\infty$

- **The limit is approaching infinity**
  We let x get very large in either the positive or negative sense.

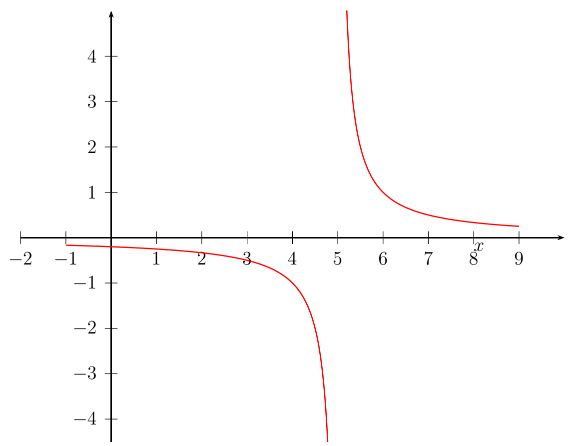


(k) As x gets larger and larger $f(x)$ moves closer and and closer to L

Example: $f(x) = \frac{1}{x-5}$.

- $\lim_{x \to 5^-} f(x) = -\infty$

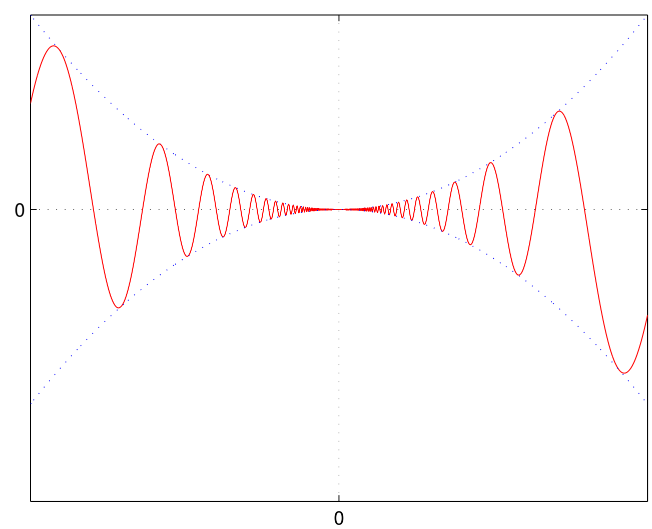
- $\lim_{x\to 5^+} f(x) = \infty$
- $\lim_{x\to 5} f(x) =$ d.n.e



(l) $f(x) = \frac{1}{x-5}$

Practice questions: Infinite limits

1. For function $f(x)$ whose graph appears below,



   a. $f(x)$ is periodic
   b. $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$
   c. $\lim_{x\to 0} = 0$
   d. both (a) and (c)

2. $\lim\limits_{x\to 2} \dfrac{3x^2 - x - 10}{x^2 + 5x - 14} =$

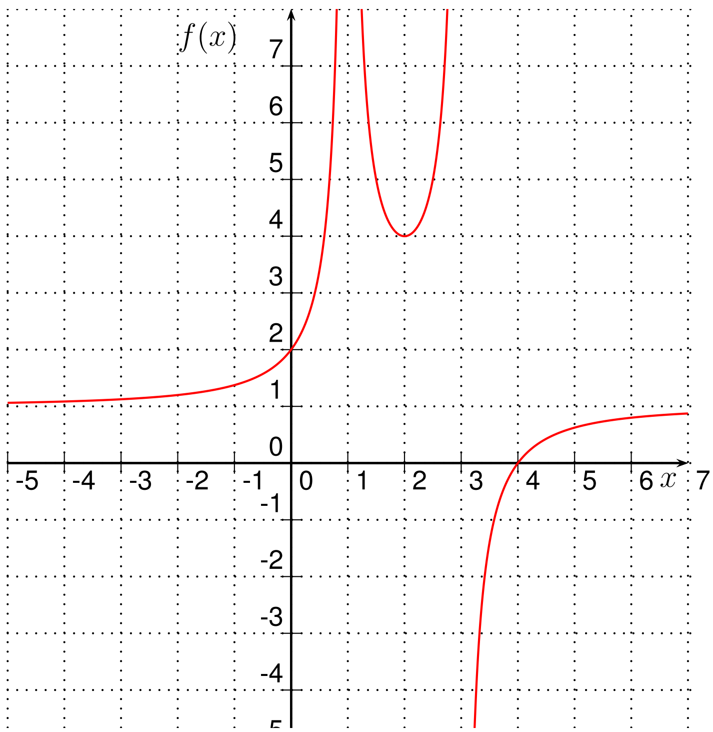
   a. $-\infty$

   b. 0

   c. $\dfrac{11}{9}$

   d. $+\infty$

3. Given function $f(x)$ whose graph appears below. Then



   a. $\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to \infty} f(x)$

   b. $\lim\limits_{x\to 3}$ exists

   c. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$ hence $\lim\limits_{x\to 1} f(x)$ exists

   d. none of (a), (b), or (c)

4. $\lim\limits_{x\to -2^+} \dfrac{x}{\sqrt{16 - x^4}} =$

   a. $+\infty$

   b. 2

   c. 0

   d. $-\infty$

5. $\lim\limits_{x\to -\infty} \left(\dfrac{5x - 4x^3 + 2}{x^3 + 8x^2} + \dfrac{x+1}{1-x} \right) =$

a. $-\infty$
b. -5
c. -4
d. $+\infty$

6. $\lim_{x\to -3^+} \left((1-x)^2 + \frac{x}{(x+3)^2}\right) =$

a. $-\infty$
b. 0
c. 16
d. $+\infty$

7. $\lim_{x\to +\infty} \frac{4^{x+1}+3^x}{1-4^x} =$

a. 4
b. -4
c. $-\infty$
d. $+\infty$

8. $\lim_{x\to\infty} \sqrt{x^2+x} - x$

a. equals $-\infty$
b. equals 0
c. equals $\frac{1}{2}$
d. equals $+\infty$

9. $\lim_{x\to\infty} \sin(4x)$

a. equals -1
b. equals 0
c. equals 1
d. does not exist

10. $\lim_{x\to 2\pi^-} x\csc(x)$ =

a. $-\infty$.
b. ∞.
c. 0.
d. none of (a), (b), or (c) is correct

References

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[5] S. Stewart, L. Redlin, and S. Watson. *Precalculus*. Brooks Cole, Fifth edition, 2006.