#1 $A = \{x \mid x \geq -3\}$, $B = \{x \mid x \leq 5\}$, $C = \{x \mid -2 \leq x \leq 6\}$

Write the following sets in the <u>interval notation</u> and graph on the real line: $B \cup C$, $B \cap C$, $A \cap B$, $A \cap C$.

<u>Solution.</u>

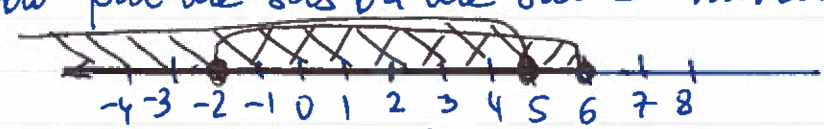
$B \cup C = \{x \mid x \in B \text{ or } x \in C\}$ – The <u>union</u> of sets B and C is the set consisting of the elements that belong either to B or to C or to both.

<u>Note:</u> that the word "OR" is used in the inclusive sense, allowing the possibility that x may belong to both sets.

B: 

C: 

Now put the sets on the same number line:



For union, we take all the elements involved.

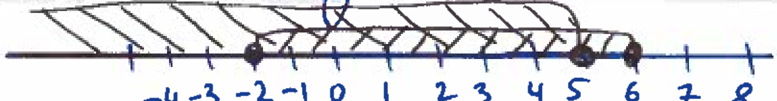
$B \cup C = \underbrace{\{x \mid x \leq 6\}}_{\text{this is set notetation}} = (-\infty, 6]$ this is interval notation

$\therefore$ $B \cup C = (-\infty, 6]$ 

$B \cap C$

$B \cap C = \{x \mid x \in B \text{ and } x \in C\}$

For the intersection of B and C, we take strickly the common elements.



Common elements are from -2 to 5 inclusive.

Set notation $B \cap C = \{x \mid -2 \leq x \leq 5\}$

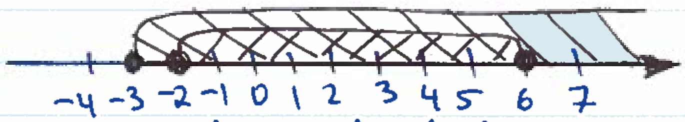
interval notation $B \cap C = [-2, 5]$

$\therefore$ $B \cap C = [-2, 5]$ 

A∩C $\quad A \cap C = \{x \mid x \in A \text{ and } x \in C\}$

A: 

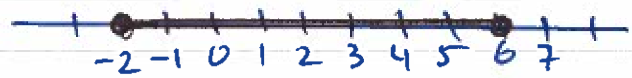
C: 



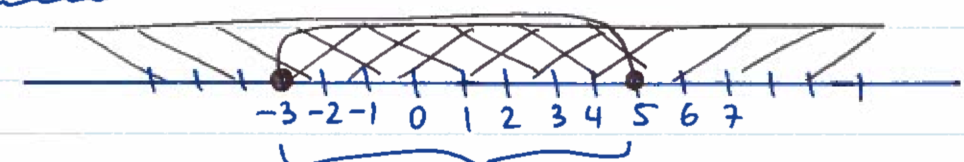
For $A \cap C$, take strictly common elements:

$A \cap C = \{x \mid -2 \le x \le 6\} \rightsquigarrow$ set notation

$= [-2, 6] \rightsquigarrow$ interval notation.

$\therefore A \cap C = [-2, 6]$ 

A∩B $\quad A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



intersection.

$A \cap B = \{x \mid -3 \le x \le 5\} \rightsquigarrow$ set notation

$= [-3, 5] \rightsquigarrow$ interval notation

$\therefore A \cap B = [-3, 5]$ 

#2 Simplify: a) $\dfrac{\frac{1}{a+h} - \frac{1}{a}}{h}$

Not to have a three-story fraction, we can rewrite it in the form

$$= \frac{1}{h}\left(\frac{1}{a+h} - \frac{1}{a}\right) = \frac{1}{h}\left(\frac{a-(a+h)}{(a+h)a}\right)$$

$$= \frac{1}{h} \times \frac{a-a-h}{a(a+h)} = \frac{1}{h}\,\frac{(-h)}{a(a+h)} = \frac{-1}{a(a+h)}$$

$$= \boxed{-\frac{1}{a(a+h)}}$$

b) $$\frac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} + \frac{1}{y}} = \frac{\frac{y^2-x^2}{x^2y^2}}{\frac{y+x}{xy}}$$

$$= \frac{y^2-x^2}{x^2y^2} \times \frac{xy}{y+x} = \frac{y^2-x^2}{(y+x)xy}$$

Use the formula for difference of squares:

$$a^2-b^2 = (a-b)(a+b)$$

$$= \frac{(y-x)(y+x)}{(y+x)xy} = \boxed{\frac{y-x}{xy}}$$

#3 Factor

a) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$

Let's first consider the following example:
$3y^5 + 2y^3 + 7y^2$.
Here, one would find the power with the smallest exponent, and then would factor it out. Here the power with the smallest exponent is y^2 so
$= y^2(3y^3 + 2y + 7)$

Now, back to our example:
The smallest exponent is $-1/2$
since $-1/2 < 3/2$ and $-1/2 < 1/2$
So one needs to factor the term $3x^{-1/2}$ out.
When we factor it out, we divide by it all the terms:

$$= 3x^{-1/2}\left(\frac{3x^{3/2}}{3x^{-1/2}} - \frac{9x^{1/2}}{3x^{-1/2}} + \frac{6x^{-1/2}}{3x^{-1/2}}\right)$$

$$= 3x^{-1/2}\left(x^{3/2-(-1/2)} - 3x^{1/2-(-\frac{1}{2})} + 2x^{-1/2-(-\frac{1}{2})}\right)$$

$$= 3x^{-1/2}\left(x^{3/2+\frac{1}{2}} - 3x^{1/2+\frac{1}{2}} + 2x^{-1/2+\frac{1}{2}}\right)$$

$$= 3x^{-1/2}(x^2 - 3x + 2) = \boxed{\frac{3}{\sqrt{x}}(x-1)(x-2)}$$

b) $8x^3 - 125y^6$

Here we can use the formula for difference of cubes:
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$= (2x)^3 - (5y^2)^3 = (2x - 5y^2)\left[(2x)^2 + (2x)(5y^2) + (5y^2)^2\right]$$

$$= \boxed{(2x - 5y^2)(4x^2 + 10xy^2 + 25y^4)}$$

#3 c) x^3+4x^2+x+4

When we have four terms, it's a good thing to try the method of grouping, but this method doesnot work always though.

$$= (x^3+4x^2)+(x+4) = x^2(x+4)+(x+4)$$

$$= \boxed{(x+4)(x^2+1)}$$

d) $(x^2+1)^2 - 7(x^2+1)+10$

Here, one can see a quadratic expression, but instead of usual variable x, we have an expression x^2+1.

So, let $y = x^2+1$.

Then

$$(x^2+1)^2-7(x^2+1)+10 = y^2-7y+10$$

$$= (y-5)(y-2)$$

Now back to the original expression:

$$= [(x^2+1)-5][(x^2+1)-2] = (x^2+1-5)(x^2+1-2)$$

$$= (x^2-4)(x^2-1)$$ Use the difference of squares formula: $a^2-b^2=(a-b)(a+b)$

$$= \boxed{(x-2)(x+2)(x-1)(x+1)}$$

#4 Rationalize the numerator or the denominator.

"Rationalize" means get rid of roots.

(a) $\dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}}$

There are roots in the denominator, so we need to rationalize the denominator.

To do so, we multiply the numerator and the denominator by the conjugate of the denominator.

denominator $= \sqrt{x} - \sqrt{y}$

conjugate $= \sqrt{x} + \sqrt{y}$

$$\frac{2(x-y)}{\sqrt{x}-\sqrt{y}} = \frac{2(x-y)}{\sqrt{x}-\sqrt{y}} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{2(x-y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}$$

use the formula of difference of squares in denominator:

$(a-b)(a+b) = a^2 - b^2$

Here $a = \sqrt{x}$, $b = \sqrt{y}$

<u>note</u>: $(\sqrt{x})^2 = x$

$(\sqrt{y})^2 = y$

$$= \frac{2(x-y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x})^2 - (\sqrt{y})^2}$$

$$= \frac{2(x-y)(\sqrt{x}-\sqrt{y})}{(x-y)} = \boxed{2(\sqrt{x}-\sqrt{y}),\ x \neq y}$$

Note! Here the restriction is $x \neq y$, so that the given fraction makes sense.

#4 b) $\sqrt{x+1} - \sqrt{x}$

Here there is no denominator, so we construct it

$$= \frac{\sqrt{x+1} - \sqrt{x}}{1}$$

Now we see that we have roots in the numerator, so we rationalize the numerator by multiplying by the conjugate of the numerator:

numerator $= \sqrt{x+1} - \sqrt{x}$; its conjugate $= \sqrt{x+1} \oplus \sqrt{x}$

$$\frac{\sqrt{x+1} - \sqrt{x}}{1} \times \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$$

$$= \frac{\left(\sqrt{x+1} - \sqrt{x}\right)\left(\sqrt{x+1} + \sqrt{x}\right)}{\sqrt{x+1} + \sqrt{x}}$$

Use difference of squares: $(a-b)(a+b) = a^2 - b^2$

Here $a = \sqrt{x+1}$, $b = \sqrt{x}$

$$= \frac{\left(\sqrt{x+1}\right)^2 - \left(\sqrt{x}\right)^2}{\sqrt{x+1} + \sqrt{x}}$$

$$= \frac{(x+1) - (x)}{\sqrt{x+1} + \sqrt{x}} = \boxed{\frac{1}{\sqrt{x+1} + \sqrt{x}}}$$

#5 Perform the operation and simplify
$(3x^2-4x+3) \div (3x+2)$.

Solution:

Perform the long division

$$
\begin{array}{r|l}
 & \quad x-2 \quad \leftarrow \text{quotient} \\
\text{divisor} \rightarrow 3x+2 & \overline{)\,3x^2-4x+3} \quad \leftarrow \text{divident} \\
 & -\,(3x^2+2x) \\
 & \quad -6x+3 \\
 & \quad -\,(-6x-4) \\
 & \qquad 7 \quad \leftarrow \text{Remainder}
\end{array}
$$

The division statement can be written in two ways:

1) $(3x^2-4x+3) \div (3x+2) = \underbrace{(x-2)}_{\text{quotient}} + \dfrac{\overset{\text{Remainder}}{7}}{\underbrace{3x+2}_{\text{divisor}}}$

OR

2) $3x^2-4x+3 = \underbrace{(x-2)}_{\text{quotient}}\underbrace{(3x+2)}_{\text{divisor}} + \underbrace{7}_{\text{Remainder}}$

#6 Solve for x

a) $a^2x + (a-1) = (a+1)x$.

$a^2x - (a+1)x = -(a-1)$

$[a^2-(a+1)]x = 1-a$

$(a^2-a-1)x = 1-a$

$x = \dfrac{1-a}{a^2-a-1}$

Since I cannot divide by zero, I should write a restriction on a so that

$a^2-a-1 \neq 0$

$a^2-a-1=0$

$a = \dfrac{1 \pm \sqrt{1^2-4(-1)(1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{5}}{2}$

$\therefore$ $\boxed{x = \dfrac{1-a}{a^2-a-1} \qquad a \neq \dfrac{1 \pm \sqrt{5}}{2}}$

#6 b) $x^2+2x-5=0$

Use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

with $a=1$, $b=2$, $c=-5$

$$x = \frac{-2 \pm \sqrt{(-2)^2-4(1)(-5)}}{2(1)} = \frac{-2 \pm \sqrt{4+20}}{2}$$

$$= \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 \pm 2\sqrt{6}}{2} = -1 \pm \sqrt{6}$$

$\therefore$ $\boxed{x_1 = -1+\sqrt{6}, \quad x_2 = -1-\sqrt{6}}$

c) $\dfrac{1}{x-1} + \dfrac{1}{x+2} = \dfrac{5}{4}$

$\dfrac{x+2+x-1}{(x-1)(x+2)} = \dfrac{5}{4}$ common denominator on the LHS

$\dfrac{2x+1}{(x-1)(x+2)} = \dfrac{5}{4}$ Cross multiply.

$4(2x+1) = 5(x-1)(x+2)$ Expand

$8x+4 = 5(x^2+x-2)$

$8x+4 = 5x^2+5x-10$

$0 = 5x^2-3x-14$ Factor

$(5x+7)(x-2)=0$

$x = -\frac{7}{5}$ or $x=2$.

Since we did cross-multiplication and got rid of denominators, we could get extra solutions. Therefore, check solutions:

$x=2$: LHS $= \dfrac{1}{2-1} + \dfrac{1}{2+2} = 1+\frac{1}{4} = \frac{5}{4} =$ RHS

$x=-\frac{7}{5}$: LHS $= \dfrac{1}{-\frac{7}{5}-1} + \dfrac{1}{-\frac{7}{5}+2} = \dfrac{1}{\frac{-7-5}{5}} + \dfrac{1}{\frac{-7+10}{5}}$

$= -\frac{5}{12} + \frac{5}{3} = \frac{20-5}{12} = \frac{15}{12} = \frac{5}{4} =$ RHS.

$\therefore$ $\boxed{x=2 \text{ or } x=-\frac{7}{5}}$

#6 d) $x^{4/3} - 5x^{2/3} + 6 = 0$

$(x^{2/3})^2 - 5x^{2/3} + 6 = 0$

Let $y = x^{2/3}$.

Then $y^2 - 5y + 6 = 0$

$(y-2)(y-3) = 0$

$y = 2$ or $y = 3$

$x^{2/3} = 2$	$x^{2/3} = 3$
$\sqrt[3]{x^2} = 2$	$\sqrt[3]{x^2} = 3$
$x^2 = 2^3$	$x^2 = 3^3$
$x^2 = 8$	$x^2 = 27$
$x = \pm\sqrt{8}$	$x = \pm\sqrt{27}$
$x = \pm 2\sqrt{2}$	$x = \pm 3\sqrt{3}$

$\therefore \boxed{x = \pm 2\sqrt{2} \quad \text{or} \quad x = \pm 3\sqrt{3}}$

#7

For #7, it's very important to understand the problem itself.
It's written that the commission for a sold machine is \$40 if the number of sold machines is less or equal to 600.

But if the number of sold machines exceeds 600, then the commission per <u>every</u> sold machine increases.

<u>Solution</u>

Let's first try if we can earn \$30,000 by selling 600 machines.

Check: the commission is \$40, because the number of sold machines ≤ 600.

$$40 \times 600 = 24{,}000 < 30{,}000$$

So, one should sell more than 600 machines.

Let n be the number of machines sold over 600.

The commission on each of $(600+n)$ sold machines is $\$40 + \$0.04n$

Equate total commission to \$30,000

$$\underbrace{(600+n)}_{\text{total number of sold machines}}\underbrace{(40+0.04n)}_{\text{commission per sold machine}} = 30{,}000$$

Expand

$$24000 + 24n + 40n + 0.04n^2 = 30{,}000$$

$$0.04n^2 + 64n - 6000 = 0$$

$$0.01n^2 + 16n - 1500 = 0$$

Use quadratic formula

$$n = \frac{-16 \pm \sqrt{16^2 - 4(0.01)(-1500)}}{2(0.01)}$$

$$= \frac{-16 \pm \sqrt{256 + 60}}{0.02} = \frac{-16 \pm \sqrt{316}}{0.02}$$

$$= \frac{-16 \pm 17.78}{0.02}$$

$\Rightarrow$ since n can be only positive,

$$n = \frac{-16 + 17.78}{0.02} = 88.81$$

$$n \doteq 89$$

$\therefore$ Total number of machines must be sold to get 30,000 commission is $600 + n = \boxed{689}$

#8

Recall: concentration $C = \frac{\text{amount}}{\text{Volume}}$

Then amount of pure juice $= \text{concentration} \times \text{volume}$
$= C \times V$

After Jill drinks 100 ml of fruit punch, the concentration is still 50%.
So, after she drinks, we have 650 ml of punch with 50% concentration in the bottle.

Then Jill adds another punch: 100 ml with unknown concentration and the final concentration is 48% for volume 750 ml.

We can write the equation for amount.

Let C_i be the initial concentration
Let V_i be the initial volume
} before adding new punch

Let C_{add} be the unknown concentration of added punch

Let V_{add} be the added volume

Let C_{fin} be the final concentration
V_{fin} be the final volume

Before adding new punch $C_i = 50\%$
$V_i = (750-100) = 650$ ml

Initial Amount of juice before adding $= (C_i)(V_i)$

Added punch: C_{add} = unknown
$V_{add} = 100$ ml

Added amount of juice $= (C_{add})(V_{add})$

Final punch: $C_{fin} = 48\%$
$V_{fin} = 750$ ml

Amount of pure juice $= C_{fin} V_{fin}$

Equation for amount of pure juice in the solution:
amount before adding + amound added = final amount

$$C_i V_i + C_{add} V_{add} = C_{fin} V_{fin}$$
$$(0.5)(650) + (C_{add})(100) = (0.48)(750)$$
$$325 + 100\, C_{add} = 360$$
$$100\, C_{add} = 360 - 325$$
$$100\, C_{add} = 35$$
$$C_{add} = \frac{35}{100} \cdot 100\% = 35\%$$

∴ The concentration of added punch was $\boxed{35\%}$

$\boxed{\text{Q9}}$ Prove that if $0 < a < b$, then
$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Solution. This inequality has three parts:
$a < \sqrt{ab}$; $\sqrt{ab} < \frac{a+b}{2}$; and $\frac{a+b}{2} < b$

Recall: Rules for inequalities:

$a > b \Rightarrow a + c > b + c$ (we can add or subtract a number)

$a > b,\ c > 0 \Rightarrow ac > bc$ multiply by positive number

$a > b,\ c < 0 \Rightarrow ac < bc$ multiply by negative and change inequality sign.

$a \geq 0,\ b \geq 0$

Then $a < b$, $a^2 < b^2$, $\sqrt{a} < \sqrt{b}$. For nonnegative a and b, we can square both sides or take a square root.

Prove $a < \sqrt{ab}$

Given $0 < a < b$, take $a < b$ and multiply by a.

$\Rightarrow a^2 < ab \Rightarrow$ take $\sqrt{\ }$ $\Rightarrow \boxed{a < \sqrt{ab}}$

Prove $\sqrt{ab} < \frac{a+b}{2}$: $(a-b)^2 > 0 \Rightarrow a^2 - 2ab + b^2 > 0$

$\Rightarrow a^2 + b^2 > 2ab \Rightarrow a^2 + 2ab + b^2 > 2ab + 2ab \Rightarrow (a+b)^2 > 4ab$

$\Rightarrow$ take a $\sqrt{\ }$ $\Rightarrow a + b > 2\sqrt{ab} \Rightarrow \sqrt{ab} < \frac{a+b}{2}$

Prove $\frac{a+b}{2} < b$. $a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b \Rightarrow \boxed{\frac{a+b}{2} < b}$