Assignment 2 Solutions

#1 a)

$$1 + \frac{2}{x+1} \leq \frac{2}{x}$$

$$1 + \frac{2}{x+1} - \frac{2}{x} \leq 0 \quad \text{(put everything on one side)}$$

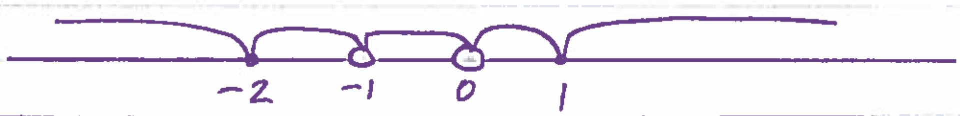
$$\frac{x(x+1) + 2(x) - 2(x+1)}{x(x+1)} \leq 0 \quad \text{(make a fraction)}$$

$$\frac{x^2 + x + \cancel{2x} - \cancel{2x} - 2}{x(x+1)} \leq 0$$

$$\frac{x^2 + x - 2}{x(x+1)} \leq 0$$

$$\frac{(x+2)(x-1)}{x(x+1)} \leq 0$$

There are several ways to solve this inequality. One way is to make a chart.
Zeroes of the LHS are $-2, -1, 0, 1$.
They divide a number line into 5 intervals:



	$x < -2$	$-2 < x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
test points	$x = -3$	$x = -1.5$	$x = -0.5$	$x = 0.5$	$x = 2$
$x+2$	$-$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$+$
x	$-$	$-$	$-$	$+$	$+$
$x+1$	$-$	$-$	$+$	$+$	$+$
$\frac{(x+2)(x-1)}{x(x+1)}$	$+$	$-$	$+$	$-$	$+$

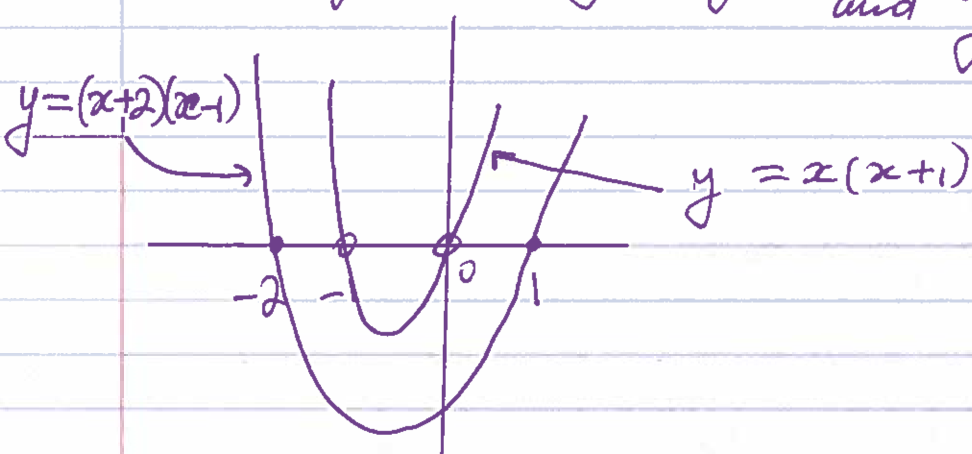
$\therefore x \in [-2, -1) \cup (0, 1]$

Another way to solve inequality $\frac{(x+2)(x-1)}{x(x+1)} \le 0$ is to consider inequalities for the numerator and denominator. In order the fraction to be non positive, we need the numerator and denominator to be of different signs

Case 1 Numerator ≥ 0, Denominator < 0 OR Case 2: Numerator $\le$, Denominator $>$

Case 1 $\begin{cases} (x+2)(x-1) \ge 0 \\ x(x+1) < 0 \end{cases}$

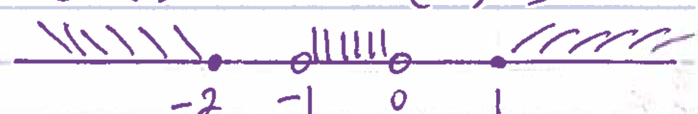
Solve graphically: graph $y=(x+2)(x-1)$ and $y=x(x+1)$ (just a sketch)



From graph: $(x+2)(x-1) \ge 0 \Leftrightarrow x \in (-\infty, 2] \cup [1, +\infty)$

$x(x+1) < 0 \Leftrightarrow x \in (-1, 0)$

Put on number line



intersection empty

Case 2

$\begin{cases} (x+2)(x-1) \le 0 \\ x(x+1) > 0 \end{cases}$

From graph above: $(x+2)(x-1) \le 0 \Leftrightarrow x \in [-2, 1]$

$x(x+1) > 0 \Leftrightarrow x \in (-\infty, -1) \cup (0, +\infty)$

intersection: $[-2, -1) \cup (0, 1]$

$\therefore x \in [-2, -1) \cup (0, 1]$

Instead of solving inequalities graphically, we can solve them analytically.
I will consider Case 2 from above.
Case 1 can be done analogously.

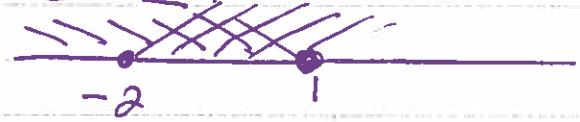
Case 2 $\begin{cases} (x+2)(x-1) \le 0 \\ x(x+1) > 0 \end{cases}$

$(x+2)(x-1) \le 0$

$\Leftrightarrow \begin{cases} x+2 \le 0 \\ x-1 \ge 0 \end{cases}$ OR $\begin{cases} x+2 \ge 0 \\ x-1 \le 0 \end{cases}$

$\Leftrightarrow \begin{cases} x \le -2 \\ x \ge 1 \end{cases}$ $\Leftrightarrow \begin{cases} x \ge -2 \\ x \le 1 \end{cases}$

empty



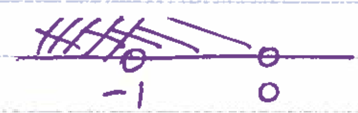
$[-2, 1]$

$x(x+1) > 0$

$\Leftrightarrow \begin{cases} x > 0 \\ x+1 > 0 \end{cases}$ OR $\begin{cases} x < 0 \\ x+1 < 0 \end{cases}$

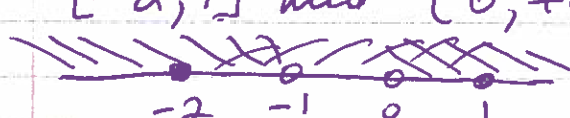
$\Leftrightarrow \begin{cases} x > 0 \\ x > -1 \end{cases}$ $\Leftrightarrow \begin{cases} x < 0 \\ x < -1 \end{cases}$

$(0, +\infty)$



$(-\infty, -1)$.

Combine (find intersection of) $[-2, 1]$ and $(0, +\infty)$ and $(-\infty, -1)$



: $[-2, -1) \cup (0, 1]$

$\therefore x \in [-2, -1) \cup (0, 1]$

note: for case 1 analogously it can be shown that all intersections are empty.

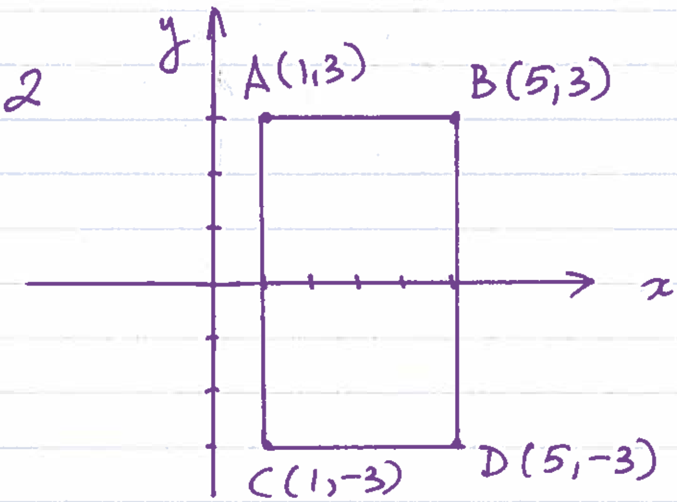
1 b)

$$\left|\frac{x-2}{3}\right| < 2 \iff -2 < \frac{x-2}{3} < 2$$

$$\iff -6 < x-2 < 6 \iff \boxed{-4 < x < 8}$$

$$\therefore \boxed{-4 < x < 8}$$

#2



There are several methods to show that ABCD is a rectangle. For example:

(i) By definition : A rectangle is a quadrileteral with four right angles.
We can do it by using coordinate geometry or by using vectors.

(ii) By the property that makes rectangles a special type of parallelogram :

The diagonals of a rectangle are congruent (equal in length)

We can use coordinate geometry or vectors here. Note: first show that the quadrilateral is a parallelogram.

Solution for #2.

① Show that all four angles of ABCD are right angles. Then by definition, ABCD is a rectangle.
Need to show $AB \perp AC$ which implies $\angle A = 90°$
$AB \perp BD \Rightarrow \angle B = 90°$; $AC \perp CD \Rightarrow \angle C = 90°$.
Then $\angle D = 360° - 3(90°) = 90°$.

(i) We can use coordinate geometry to show sides are perpendicular.

Fact: for nonvertical and nonhorizontal lines, the slopes of perpendicular lines are negative reciprocals of each other:
$m_1 = -\frac{1}{m_2}$.

We have horizontal & vertical lines.

Fact: Horizontal and vertical lines are always perpendicular; therefore, 2 lines, one of which has a zero slope and the other an undefined slope are perpendicular.

{Have to explain AB, CD – HL; AC, BD – VL.

Line AB is horizontal because the second coordinates of $A(1,3)$ and $B(5,3)$ are the same, and the first are different.
Analogously CD is horizontal.

Line AC is vertical, because $A(1,3)$ and $C(1,-3)$ have the same first coordinates, and different second coordinates.
Analogously, BD – vertical.

$m_{AB} = \frac{3-3}{5-1} = \frac{0}{4} = 0$

$m_{AC} = \frac{-3-3}{1-1} = -\frac{6}{0} = \text{undefined}$

Therefore $AB \perp AC \Rightarrow \angle A = 90°$

Analogously, we can show
$\angle B = \angle C = \angle D = 90°$.

$\therefore$ all four angles are 90°, by definition, ABCD is a rectangle.

(ii) we can use vectors to show sides are perpendicular:

$\overline{AB} = (5-1, 3-3) = (4, 0)$

$\overline{AC} = (1-1, -3-3) = (0, -6)$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = (4,0)\cdot(0,-6) = (4)(0)+(0)(-6) = 0$

$\therefore \overrightarrow{AB} \perp \overline{AC} \Rightarrow \angle A = 90°$

Analogously $\angle B = \angle C = \angle D = 90°$.

Area of ABCD:

$A_{ABCD} = |AB| \cdot |AC|$

$= \sqrt{(5-1)^2 + (3-3)^2} \cdot \sqrt{(1-1)^2 + (-3-3)^2}$

$= 4 \cdot 6 = \boxed{24}$

(4)

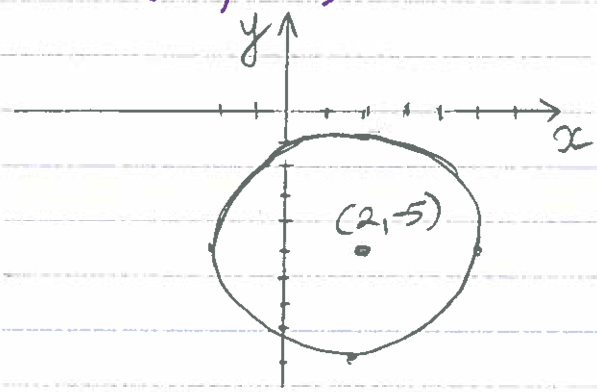
#3

a) Complete the squares for x and y:

$(x^2-4x)+(y^2+10y)+13=0$

$(x^2-4x+4)-4+(y^2+10y+25)-25+13=0$

$(x-2)^2+(y+5)^2=16 \rightarrow$ a circle centered at $(2,-5)$ with radius 4.



b) Complete the square for x and y:

$(2x^2-12x)+(y^2+2y)+19=0$

$2(x^2-6x)+(y^2+2y+1)-1+19=0$

$2(x^2-6x+9-9)+(y+1)^2+18=0$

$2(x^2-6x+9)-18+(y+1)^2+18=0$

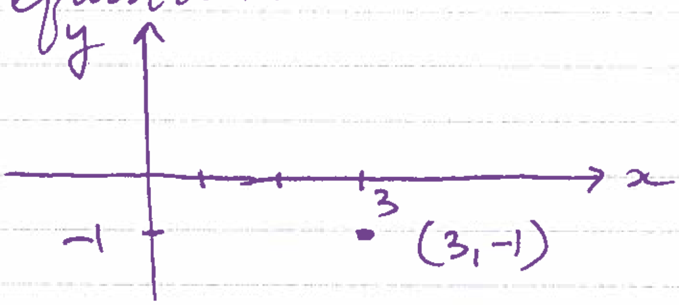
$2(x-3)^2+(y+1)^2=0$

Square are nonnegative, so the only way to get "0" is

$x-3=0$ and $y+1=0$

$x=3$ $y=-1$

$(3,-1)$ is the only solution to the equation.



The graph consists of just one point $(3,-1)$

#4

$$f(x) = \begin{cases} x^2+2x & \text{if } x \le -1 & ① \\ x & \text{if } -1 < x \le 1 & ② \\ -1 & \text{if } x \geq 1 & ③ \end{cases}$$

$-4 < -1 \Rightarrow$ use line ① : $f(-4) = (-4)^2 + 2(-4)$
$= \boxed{8}$

$-3/2 < -1 \Rightarrow$ line ① : $f(-\frac{3}{2}) = (-\frac{3}{2})^2 + 2(-\frac{3}{2})$
$= \frac{9}{4} + \frac{6}{2}$
$= \frac{9}{4} - \frac{12}{4} = \boxed{-\frac{3}{4}}$

$-1 \le -1$ $(-1 = -1) \Rightarrow$ line ① : $f(-1) = (-1)^2 + 2(-1)$
$= 1 - 2$
$= \boxed{-1}$

$-1 < 0 \le 1 \Rightarrow$ line ② : $f(0) = \boxed{0}$

$25 > 1 \Rightarrow$ line ③ : $f(25) = \boxed{-1}$

$f(-4) + f(-\frac{3}{2}) + f(-1) + f(0) + f(25)$

$= 8 - \frac{3}{4} - 1 + 0 - 1 = 6 - \frac{3}{4} = \frac{24}{4} - \frac{3}{4}$
$= \boxed{\frac{21}{4}}$

$\therefore \boxed{\frac{21}{4}}$

5

a) $f(x) = \begin{cases} 4 & \text{if } x = 3 \\ x^2 & \text{if } 1 \le x < 3 \end{cases}$

Domain is all x's for which the formula makes sense.
Since the function is defined for $1 \le x < 3$ and $x = 3$, the domain is $D = \{x \mid 1 \le x \le 3\}$ or in the interval notation $D = [1, 3]$.

b) Since it's a fraction, the domain is all x's such that the denominator $\neq 0$.

$x^2 + x - 6 = 0$
$(x+3)(x-2) = 0$
$x = -3,\ x = 2$

$\therefore$ The domain is $D = \{x \mid x \neq -3, x \neq 2\}$
OR in interval notation
$D = (-\infty, -3) \cup (-3, 2) \cup (2, +\infty)$.

c). Under $\sqrt{\ }$ nonnegative and denominator $\neq 0$

$x - 4 > 0$
$x > 4$

$\therefore\ D = \{x \mid x > 4\}$ OR $(4, +\infty)$

#6

Definition:

$f(x)$ is even if $f(-x) = f(x)$

$f(x)$ is odd if $f(-x) = -f(x)$

a) $f(-x) = (-x) + \frac{1}{(-x)} = -x - \frac{1}{x}$

$= -\left(x + \frac{1}{x}\right) = \boxed{-f(x)}$

$\therefore$ $f(x)$ is odd.

b) $f(-x) = 1 - \sqrt[3]{(-x)} = 1 - \sqrt[3]{-x}$

$= 1 + \sqrt[3]{x} \neq f(x)$

$\neq -f(x)$

$\therefore$ $f(x)$ is neither even, nor odd.

#7. $h(x) = \dfrac{x^2}{x^2+4}$

Find f, g, k such that $h = f \circ g \circ k$

Solution Answers can vary.

For example,

$f(x) = \dfrac{x}{x+4}$

$g(x) = (x+3)^2$

$k(x) = (x-3)$

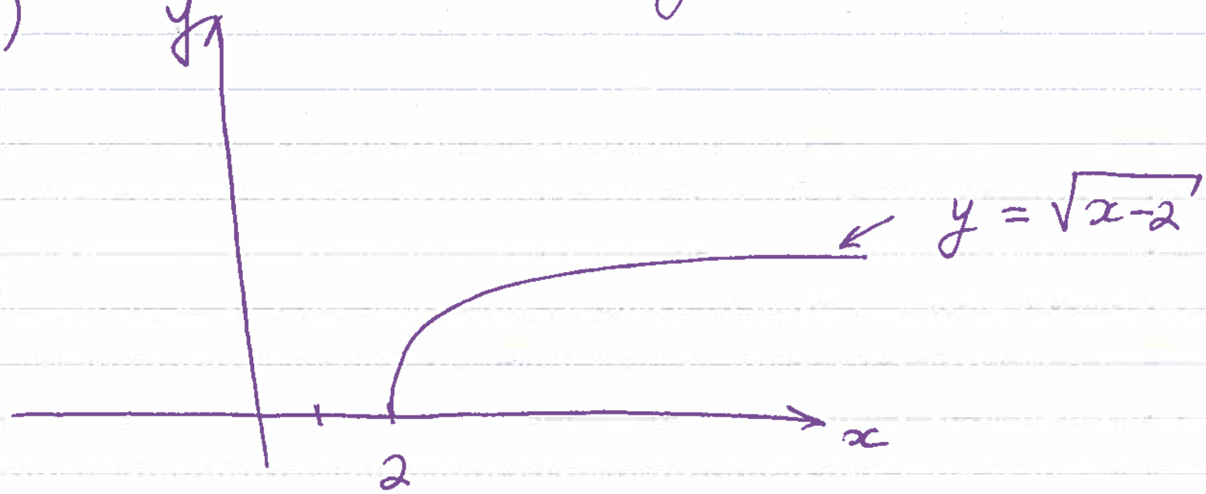
Check $f \circ g \circ k = f(g(k(x))) = f(g(x-3))$

$= f(x^2) = \dfrac{x^2}{x^2+4} = h.$

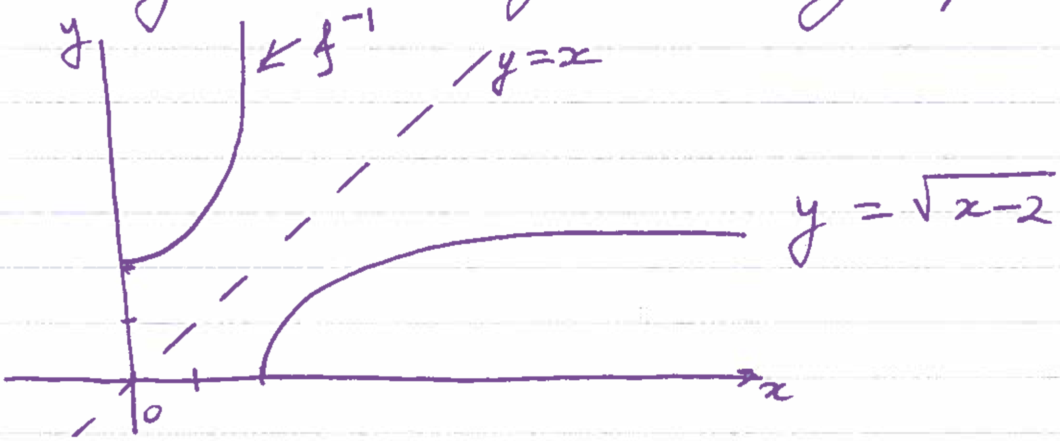
#8

$f(x) = \sqrt{x-2}$ horizontal shift by 2 units to the right.

a)



Since graphs of inverse functions are symmetrical about the line $y=x$, we reflect graph of $f(x)$ about line $y=x$ to get the graph of f^{-1}



Equation for f^{-1}:

$f(x)=\sqrt{x-2}$. Range for f: $\boxed{y \geq 0}$

Fact: Range for f becomes the domain for f^{-1}.

$y=\sqrt{x-2}$

$x=\sqrt{y-2}$

$x^2=y-2$

$\rightarrow y = x^2+2$, $\boxed{x \geq 0}$ ← Domain

$\therefore \boxed{f^{-1}(x) = x^2+2,\ x \geq 0}$

9

a) Sketch $f(x) = 4x - x^2$ using $g(x) = x^2$.

Solution

$f(x) = 4x - x^2 \rightsquigarrow$ complete the square

$$f(x) = -x^2 + 4x$$
$$= -(x^2 - 4x) = -(x^2 - 4x + 4 - 4)$$
$$= -(x^2 - 4x + 4) + 4$$
$$= -(x-2)^2 + 4.$$

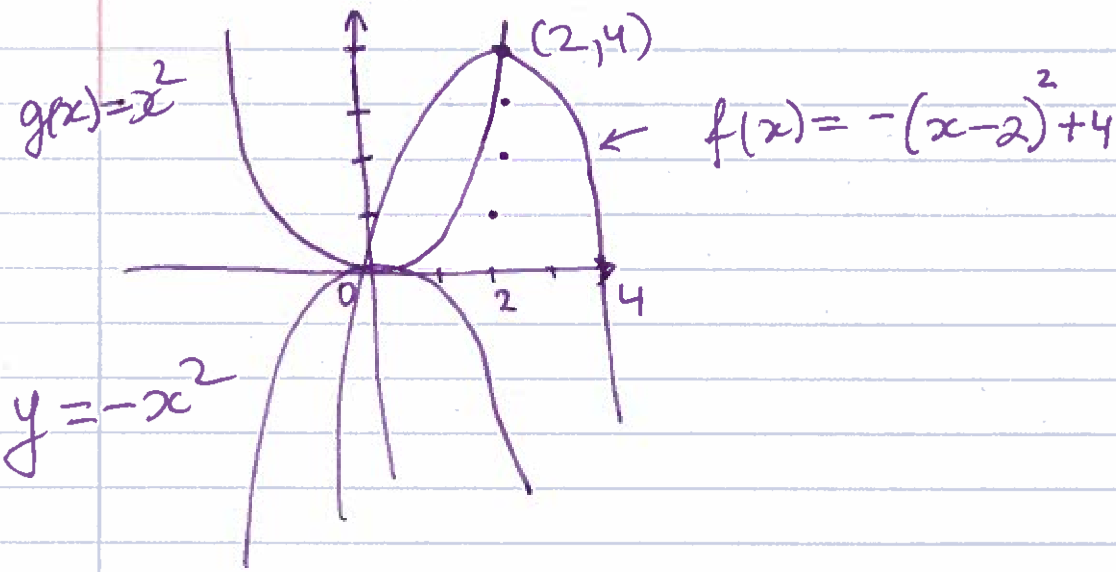
This is the vertex form of parabola

$$y = a(x-h)^2 + k$$
$$a = -1, \quad h = 2, \quad k = 4$$

$a = -1 \rightarrow$ refletion about the x-axis

$h = 2 \rightarrow$ horizontal shift 2 units to right

$k = 4 \rightarrow$ vertical shift 4 units up.



b) $h(x) = |4x - x^2| \rightarrow$ we reflect negative parts of the graph $f(x) = 4x - x^2$ about the x-axis