Q10 Prove that $\sqrt{3}$ is irrational.

Before proving this, let us show the fact that will be used in the proof:

Fact: if k^2 is divisible by 3, then k must also be divisible by 3.

Proof of the Fact:

We know that after we divide any integer number by 3, we get a remainder of 1 or 2, or the number can be divided by 3 evenly.

For example, $7 = 2\times 3 + \boxed{1}$; $5 = 1\times 3 + \boxed{2}$; $6 = 2\times 3 + \boxed{0}$.

In general, any integer number can be written in the form

$3n$ or $3n+1$ or $3n+2$.

Consider $(3n+1)$. This number is not divisible by 3 since the remainder is 1.

Consider $(3n+1)^2$

$(3n+1)^2 = 9n^2 + 6n + 1 = 3(3n^2+2n) + \boxed{1}$

So $(3n+1)^2$ is not divisible by 3 either since the remainder is 1.

Consider $(3n+2)$. This number is not divisible by 3 since the remainder is $\boxed{2}$

Consider $(3n+2)^2$

$(3n+2)^2 = 9n^2 + 12n + 4 = 3(3n^2 + 4n + 1) + \boxed{1}$

So $(3n+2)^2$ is not divisible by 3 since the remainder is $\boxed{1}$

Consider $3n$

$(3n)^2 = 9n^2 = 3\times(3n^2) \rightarrow$ divisible by 3.

So, what we got:
If we consider an integral number k.

If k is divisible by 3, then k^2 is divisible by 3.
If k is not divisible by 3, then k^2 is not divisible by 3.

Let us look at the opposite direction.
If we have $k^2 \to$ a square of some number.
If we know that k^2 is divisible by 3, does it mean that k is divisible by ?

The answer is: YES! Because is k is not divisible by 3, then k^2 is not divisible by 3.

So, we proved the fact that if k^2 is divisible by 3, then k is divisible by 3.

Now suppose that $\sqrt{3}$ is rational.

Let $\sqrt{3} = p/q$ where p and q have no common factor. Then $3 = p^2/q^2$
Then $p^2 = 3q^2$ So p^2 is divisible by 3,
so p must be divisible by 3.
Thus $p = 3m$, m is integer
Then $(3m)^2 = 3q^2$
$9m^2 = 3q^2$
$3m^2 = q^2$, so q^2 divisible by 3
So, q must be divisible by 3.
Therefore, we assumed p and q have no common factor based on assumption that $\sqrt{3}$ is ration and got a contradiction since p and q have a common factor of 3.