Assignment 3 Solutions

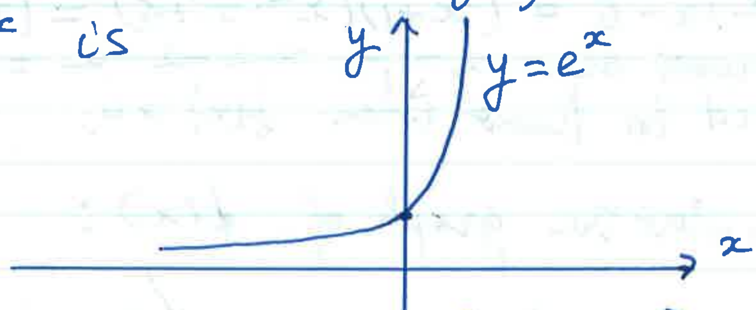
#1 Find domain

a) $f(x) = \dfrac{1}{\sqrt{x^3 e^x - 7xe^x - 6 e^x}}$

The domain for square root: must be non-negative ≥ 0 in general. Since here the root is the denominator, it cannot be zero, so just > 0.

$$x^3 e^x - 7x e^x - 6 e^x > 0$$
$$e^x (x^3 - 7x - 6) > 0$$

We know $e^x > 0$ always, since the graph of e^x is



The product of e^x and (x^3-7x-6) is positive if both factors are positive. We need to figure out when $x^3 - 7x - 6 > 0$.

Let $f(x) = x^3 - 7x - 6$.

Factor $f(x)$

By the Factor theorem:

The polynomial $f(x)$ has a factor $(x-k)$ if and only if $f(k)=0$

Meaning: k is a root of $f(x)$.

Possible values for k: $\pm 1, \pm 2, \pm 3, \pm 6 \rightarrow$ divisors of $\boxed{6}$

Recall: $f(k)$ gives us the remainder after dividing by $(x-k)$ [Remainder THEOREM]

$f(-1) = (-1)^3 - 7(-1) - 6 = 0 \Rightarrow (x+1)$ is a factor of $f(x)$

now you can do a long division OR try another values for k from the list above (divisors of 6).

long division:

$$\begin{array}{r} x^2 - x + 6 \quad \leftarrow \text{quotient} \\ \text{divisor} \rightarrow (x+1) \overline{) \; x^3 - 7x - 6} \quad \leftarrow \text{divident} \\ x^3 + x^2 \\ \hline -x^2 - 7x \\ - \quad -x^2 - x \\ \hline -6x - 6 \\ - \quad 6x + 6 \\ \hline 0 \quad \text{remainder.} \end{array}$$

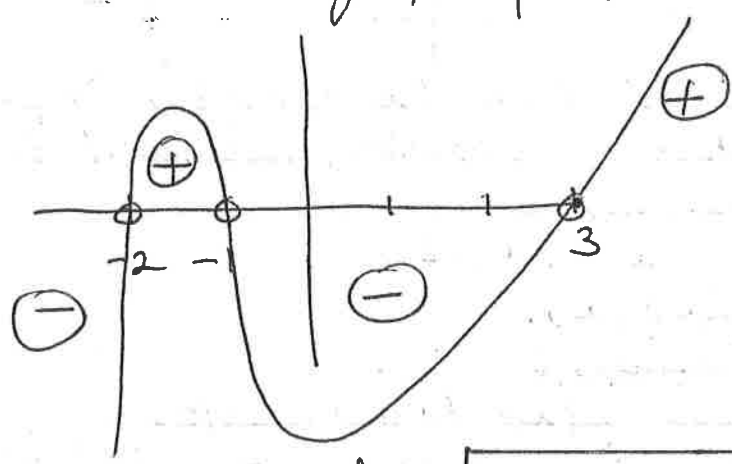
Division Statement:

$$x^3 - 7x - 6 = (x+1)(x^2 - x + 6) = (x+1)(x+2)(x-3)$$

The roots are $x = -1$ $x = -2$ $x = 3$

We need to know when $f(x) > 0$.

We can draw the graph of $f(x)$:



So $f(x) > 0$ for $\boxed{x \in (-2,-1) \cup (3,+\infty).}$

OR

we can make a chart

	$x < -2$	$-2 < x < -1$	$-1 < x < 3$	$x > 3$
$(x+1)$	−	−	+	+
$(x+2)$	−	+	+	+
$(x-3)$	−	−	−	+
$f(x) = (x+1)(x+2)(x-3)$	−	+	−	+

$\therefore$ Domain : $x \in (-2,-1) \cup (3,+\infty)$.

#1 b) $f(x) = \sqrt{x+5} - \log_3(3-x)$

The domain for root : nonnegetive ≥ 0

for log : > 0

$x+5 \geq 0$ and $3-x > 0$

$x \geq -5$ and $x < 3$

Place on a number line

$\therefore$ The domain is $\underbrace{\{x \in \mathbb{R} \mid -5 \leq x < 3\}}_{\text{set notation}}$

OR $[-5, 3) \rightarrow$ interval notation

#2 Solve the following equations:

a) $\log_2 3 + \log_2 x = \log_2 5 + \log_2 (x-2)$

$\log_2 (3x) = \log_2 [5(x-2)]$

$3x = 5(x-2)$

$3x = 5x - 10$

$10 = 2x$

$x = 5$

We always must check solutions for logarithmic equations:

$\log_2 3 + \log_2 5 = \log_2 5 + \log_2 (5-2)$

$\log_2 3 + \log_2 5 = \log_2 5 + \log_2 3$ ✓

$\therefore \boxed{x = 5}$

#2 b). $\log_2(x^2-x-2)=2$ Switch to exponential equation

$2^2 = x^2-x-2$

$x^2-x-6=0$

$(x-3)(x+2)=0$

$x=3\ ,\ x=-2.$

<u>Check</u>: $x=3$

$LHS = \log_2(3^2-3-2) = \log_2(4) = 2 = RHS$ ✓

$x=-2$: $LHS = \log_2[(-2)^2-(-2)-2] = \log_2(4) = 2 = RHS$ ✓

$\therefore\ x=3\ ,\ x=-2.$

c). $3^{4x} = 9^{x+1}$

Make the same base on both sides. $9=3^2$

$3^{4x} = (3^2)^{x+1}$

$3^{4x} = 3^{2(x+1)}$

Since the powers are equal, and the bases are equal, the exponents are equal as well

$4x = 2(x+1)$

$4x = 2x+2$

$2x = 2$

$x = 1$

Check: $\left.\begin{array}{l} LHS = 3^{4(1)} = 3^4 = 81 \\ RHS = 9^{1+1} = 9^2 = 81 \end{array}\right\}$ $LHS = RHS$

$\therefore\ x=1$

#3 Soln $Q(t) = 100e^{-0.035t}$

a) what is the relative rate of reduction of the substance?

> Def: The relative rate of change of a function $f(x)$ is $\frac{f'(x)}{f(x)} \cdot 100\%$

So, the relative rate of reduction of the substance

$$= \frac{Q'(t)}{Q(t)} \times 100\%$$

$$= \frac{(100)(-0.035)\,e^{-0.035t}}{(100)\,e^{-0.035t}} \cdot 100\%$$

$$= -3.5\%$$

We got a negative relative rate. It tells us that the substance is decreasing.

b) How many milligrams are present after 0 years?

$Q(0) = 100 \cdot e^{-0.035(0)} = 100 \cdot e^0 = 100 \cdot 1 = 100$

$\therefore$ 100 milligrams are present at $t = 0$.

(c) $20 = 100 \cdot e^{-0.035t}$

$$\frac{20}{100} = \frac{100 \cdot e^{-0.035t}}{100}$$

$$\frac{1}{5} = e^{-0.035t} = \frac{1}{e^{0.035t}}$$

$$5 = e^{0.035t} \Rightarrow \ln 5 = \ln e^{0.035t}$$

$\ln 5 = 0.035t \Rightarrow t = \dfrac{\ln 5}{0.035} \doteq 45.98$
≈ 46 years

$\therefore$ After 46 years it will be 20 milligrams present.

4 a).

$\lim\limits_{x \to 2} \dfrac{x^4 + x^3 - 24}{x^2 - 4}$ has the form $\dfrac{0}{0}$

(Do not apply $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$

since $\lim\limits_{x \to a} g(x) = 0$)

We should factor first

$$\lim_{x \to 2} \frac{x^4 + x^3 - 24}{x^2 - 4} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x^3 + 3x^2 + 6x + 12)}{\cancel{(x-2)}(x+2)}$$

$$= \lim_{x \to 2} \frac{x^3 + 3x^2 + 6x + 12}{x+2}$$

$$= \frac{\lim\limits_{x \to 2} x^3 + 3x^2 + 6x + 12}{\lim\limits_{x \to 2} x+2} = \frac{8 + 12 + 12 + 12}{2+2}$$

$$= \frac{44}{4} = 11$$

4 b) $\lim_{x\to 6} \dfrac{\sqrt{x-2}-2}{x-6}$ $\quad \frac{0}{0}$

Rationalize the numerator by multiplying by $\sqrt{x-2}+2$

$$= \lim_{x\to 6} \frac{\sqrt{x-2}-2}{x-6} \cdot \frac{\sqrt{x-2}+2}{\sqrt{x-2}+2}$$

$$= \lim_{x\to 6} \frac{(\sqrt{x-2}-2)\cdot\sqrt{x-2+2}}{(x-6)(\sqrt{x-2}+2)}$$ use the difference of squares formula $(a-b)(a+b)=a^2-b^2$

$$= \lim_{x\to 6} \frac{(\sqrt{x-2})^2-2^2}{(x-6)(\sqrt{x-2}+2)} = \lim_{x\to 6} \frac{\cancel{(x-2-4)}}{\cancel{(x-6)}(\sqrt{x-2}+2)}$$

$$= \lim_{x\to 6} \frac{1}{\sqrt{x-2}+2} = \frac{1}{\sqrt{6-2}+2} = \boxed{\frac{1}{4}}$$

c) $\lim_{x\to 2} f(x)$, where

$$f(x) = \begin{cases} \dfrac{x^3-8}{x-2}, & x<2 \quad (1) \\ 9x^2, & x>2 \quad (2) \end{cases}$$

<u>note</u>: the function is not defined at $x=2$, but limit can exist.

When we find the $\lim_{x\to 2} f(x)$, it doesn't matter if the function is defined at $x=2$ or not.

Also, note: the function changes its formula at $x=2$. Because of this change, we will find one-sided limits: $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$. If they are equal, then $\lim_{x\to 2} f(x)$ exists.

find $\lim\limits_{x \to 2^-} f(x)$. We use formula (1), since $x<2$

$= \lim\limits_{x \to 2^-} \frac{x^3-8}{x-2}$ Use formula for difference of cubes: $a^3-b^3=(a-b)(a^2+ab+b^2)$

$= \lim\limits_{x \to 2^-} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}} = \lim\limits_{x \to 2^-} (x^2+2x+4)$

$= 2^2+2(2)+4 = \boxed{12}$

find $\lim\limits_{x \to 2^+} f(x)$ We use formula (2), since $x>2$

$= \lim\limits_{x \to 2^+} 9x^2 = 9 \cdot 2^2 = \boxed{36}$

Since $\lim\limits_{x \to 2^+} f(x) \neq \lim\limits_{x \to 2^-} f(x)$,

$\lim\limits_{x \to 2} f(x)$ D.N.E. (does not exist).

4d) Find $\lim_{x \to 1^+} \frac{x^2-1}{|x-1|}$ and $\lim_{x \to 1^-} \frac{x^2-1}{|x-1|}$

Def. for abs. value: $|y| = \begin{cases} y, & \text{if } y \geq 0 \\ -y, & \text{if } y < 0. \end{cases}$

So $|x-1| = \begin{cases} (x-1), & \text{if } x-1 \geq 0 \\ -(x-1), & \text{if } x-1 < 0 \end{cases}$

$= \begin{cases} (x-1), & \text{if } x \geq 1 \\ -(x-1), & \text{if } x < 1 \end{cases}$

<u>when $x \to 1^+$</u>, $x > 1$, so $|x-1| = x-1$

$$\lim_{x \to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x \to 1^+} \frac{x^2-1}{(x-1)} = \lim_{x \to 1^+} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1^+} (x+1) = 1+1 = \boxed{2}$$

<u>when $x \to 1^-$</u>, $x < 1$, so $|x-1| = -(x-1)$

$$\lim_{x \to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x \to 1^-} \frac{x^2-1}{-(x-1)} = \lim_{x \to 1^-} \frac{\cancel{(x-1)}(x+1)}{-\cancel{(x-1)}}$$

$$= \lim_{x \to 1^-} -(x+1) = -(1+1) = \boxed{-2}$$

$$\therefore \lim_{x \to 1^+} \frac{x^2-1}{|x-1|} = \boxed{2}, \quad \lim_{x \to 1^-} \frac{x^2-1}{|x-1|} = \boxed{-2}$$

#5

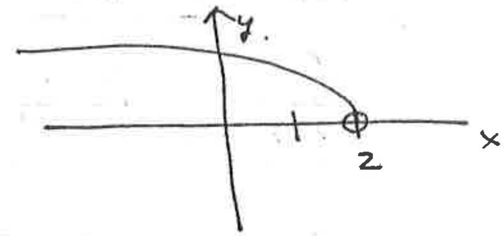
Let $f(x) = \begin{cases} \sqrt{2-x} & x < 2, \quad ① \\ x^3 + k(x+1) & x \geq 2. \quad ② \end{cases}$

Find k such that $f(x)$ is continuous everywhere.

1) Check $f(x)$ is continuous for $x < 2$.

When $x < 2$ $f(x) = \sqrt{2-x}$.
The Root function is continuous on its domain.
The domain is $2 - x \geq 0$
$x \leq 2$

So $f(x)$ is continuous on $x < 2$
and has the graph



2) Check $f(x)$ is continuous on $x > 2$.

When $x > 2$, $f(x) = x^3 + kx + k \rightsquigarrow$ polynomial.
polynomials are continuous everywhere

3) Since at $x = 2$, $f(x)$ changes its formula,
we need to find k so that $f(x)$ is continuous.
Function $f(x)$ is continuous at $x = 2$
if $\lim_{x \to 2} f(x) = f(2)$.

(i) $f(2) = 2^3 + k(2+1) \to$ work with line ②.

(ii) to find $\lim_{x \to 2} f(x)$, we need to find one-sided limits because $f(x)$ changes its formula.

$\lim_{x \to 2^-} f(x)$ use line 1 since $x < 2$

$= \lim_{x \to 2^-} \sqrt{2-x} = \sqrt{2-2} = \boxed{0}$

Note: I cannot sub $x = 2$ in the formula $\sqrt{2-x}$ because the line 1 is defined for $x < 2$, not $x = 2$.
That's why we need to find limit.

$\lim_{x \to 2^+} f(x)$ use line 2 since $x > 2$.

$= \lim_{x \to 2^+} x^3 + k(x+1) = \boxed{8 + k(2+1)}$

For $\lim_{x \to 2} f(x)$ to exist, we need

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$0 = 8 + k(3)$

$3k = -8$

$\boxed{k = -\frac{8}{3}}$

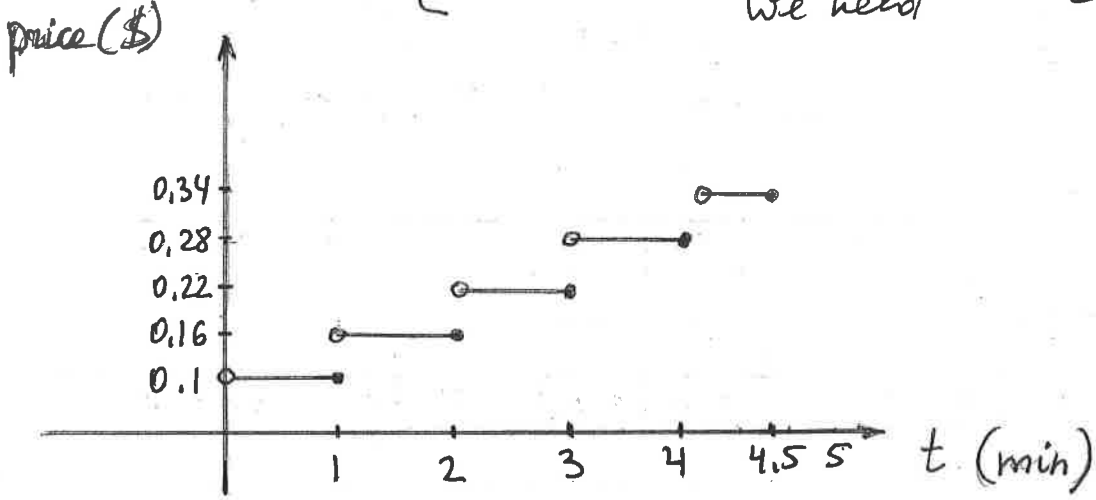
$\therefore$ for $f(x)$ to be continuous everywhere, we need $\boxed{k = -\frac{8}{3}}$

#6

a)

$$f(x) = \begin{cases} \$0.1, & 0 < t \le 1 \\ \$0.16, & 1 < t \le 2 \\ \$0.22, & 2 < t \le 3 \\ \$0.28, & 3 < t \le 4 \\ \$0.34, & 4 < t \le 5 \end{cases}$$

We need $4 < t \le 4\frac{1}{2}$.



b) From the graph, discontinuities occur at $t = 1, 2, 3, 4$.

#7 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

Note: $-1 \le \sin \frac{1}{x} \le 1$ because sine is always between -1 and 1.

To find the limit, we are going to use the Squeeze THEOREM.

$$-1 \le \sin \frac{1}{x} \le 1$$

Multiply all parts of the inequality by x^2; x^2 is positive, so we don't change signs of inequality.

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

$\lim_{x \to 0} (-x^2) = 0$ and $\lim_{x \to 0} x^2 = 0$

$\therefore \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$