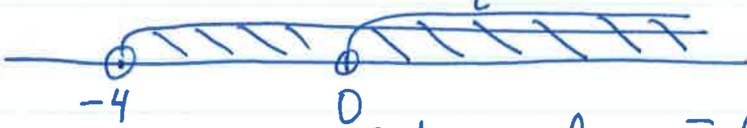
Final Exam Solutions.

#1 a) $\log_2(x) - 5 + \log_2(x+4)$

<u>Domain</u> $\text{dom}(f) = \{x \mid x > 0 \text{ and } x+4 > 0\}$

$$= \{x \mid x > 0 \text{ and } x > -4\}$$



"and" means intersection $= (0, +\infty)$

$$= \underbrace{\{x \mid x > 0\}}_{\text{set notation}} = \underbrace{(0, +\infty)}_{\text{interval notation}}$$

$\therefore$ Domain $= (0, +\infty)$.

<u>Roots</u> $\log_2(x) - 5 + \log_2(x+4) = 0$

$$\log_2(x) + \log_2(x+4) = 5$$

$$\log_2[x(x+4)] = 5$$

Switch to exponential form:

$$x(x+4) = 2^5$$

$$x^2 + 4x - 32 = 0$$

$$(x-4)(x+8) = 0$$

$$x = 4 \text{ or } x = -8$$

<u>Note</u>: for logarithm equations, we <u>must</u> check solutions.

<u>Check</u>: only $x = 4 \in$ Domain $= (0, +\infty)$
Still we have to substitute $x = 4$ into the original equation:

LHS: $\log_2(4) - 5 + \log_2(4+4) = 2 - 5 + 3 = 0 = \text{RHS}$

$\therefore \boxed{x = 4}$

#1 b) $\dfrac{2}{x^2-1} - \dfrac{1}{x(x-1)} - \dfrac{2}{x^2}$

__Domain__ : Denominators shouldn't be zeroes.

$x^2-1=0$ $\qquad$ $x(x-1)=0$ $\qquad$ $x^2=0$

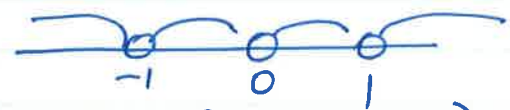
$(x-1)(x+1)=0$ $\qquad$ $x=0$ or $x=1$ $\qquad$ $x=0$

$x=1$ or $x=-1$

$\text{dom}(f) = \left\{ x \in \mathbb{R} \mid x^2-1 \neq 0 \text{ and } x(x-1) \neq 0 \text{ and } x^2 \neq 0 \right\}$

So, we take all real numbers $\mathbb{R}$ except for $x=1$, $x=-1$, $x=0$

$= \mathbb{R} \setminus \{-1, 0, 1\}$



$= (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, +\infty).$

__Roots__ : $f(x)=0$

$$\frac{2}{x^2-1} - \frac{1}{x(x-1)} - \frac{2}{x^2} = 0$$

$$\frac{2}{(x-1)(x+1)} - \frac{1}{x(x-1)} - \frac{2}{x^2} = 0$$

$$\frac{2x^2 - x(x+1) - 2\overbrace{(x+1)(x-1)}^{=(x^2-1)}}{x^2(x+1)(x-1)} = 0$$

$$\frac{2x^2 - x^2 - x - 2x^2 + 2}{x^2(x+1)(x-1)} = 0$$

$$\frac{-x^2 - x + 2}{x^2(x+1)(x-1)} = 0 \Rightarrow \frac{-(x^2+x-2)}{x^2(x+1)(x-1)} = 0$$

$$\frac{-(x-1)(x+2)}{x^2(x+1)(x-1)} = 0$$

We know a fraction equals 0 when the numerator $=0$ but denominator $\neq 0$

$x=1, x=-2$

$x=1$ is not on the domain.

$\boxed{x=-2}$

Check by substituting onto the original equation.

#1c). $2x-1+\sqrt{2-x}$

$$\text{Domain}(f) = \{x \mid x \in \mathbb{R},\ 2-x \geq 0\}$$
$$= \{x \in \mathbb{R} \mid x \leq 2\}$$
$$= (-\infty, 2]$$

Roots :

$$2x-1+\sqrt{2-x} = 0 \qquad (*)$$

$$\sqrt{2-x} = 1-2x$$ square both sides to get rid of $\sqrt{\ }$

$$(2-x) = (1-2x)^2$$
$$2-x = 1-4x+4x^2$$
$$4x^2-4x+1-2+x = 0$$
$$4x^2-3x-1 = 0$$ $\left.\begin{matrix} P=-4 \\ S=-3 \end{matrix}\right\} \Rightarrow -4, 1$
$$4x^2-4x+x-1 = 0$$
$$4x(x-1)+(x-1) = 0$$
$$(4x+1)(x-1) = 0$$
$$x = -\frac{1}{4} \text{ or } x = 1$$

Check: both $x=1$ and $x=-\frac{1}{4}$ are on the domain.

Please note, that we still must substitute onto the origional equation $(*)$.
So being on the domain is not enough to be a root.

$x=1$: $\text{LHS} = 2(1)-1+\sqrt{2-1}$
$= 2-1+1 = 2 \neq 0$

So $x=1$ is on the domain, but not a root.
This example shows that we always must check our solutions.

$x=-\frac{1}{4}$: $\text{LHS} = 2\cdot\left(-\frac{1}{4}\right)-1+\sqrt{2-\left(-\frac{1}{4}\right)}$
$= -\frac{1}{2}-1+\sqrt{\frac{9}{4}} = -\frac{3}{2}+\frac{3}{2} = 0 = \text{RHS}$

$\therefore \boxed{x = -\frac{1}{4}}$

#1 d). $|2x+3|-x$

domain: Since no restrictions, x can be any number.

$dom(f) = \mathbb{R}$.

Roots $\quad |2x+3|-x=0 \quad (1)$

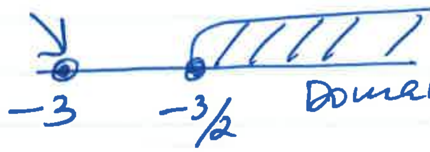
Case 1 $\quad 2x+3 \geq 0 \Rightarrow x \geq -\frac{3}{2}$

$|2x+3| = 2x+3$ since $2x+3 \geq 0$

$(1) \Rightarrow 2x+3-x=0$

$x+3=0$

$x=-3$



Domain for Case1

Since -3 is not in Case1, NO solutions for Case1

Case2 $\quad 2x+3<0 \Rightarrow x < -\frac{3}{2}$

domain for Case2

$|2x+3| = -(2x+3)$ since $2x+3<0$

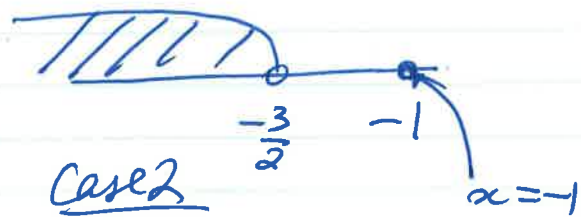
$(1) \Rightarrow -(2x+3)-x=0$

$-3-2x-x=0$

$-3-3x=0$

$3x=-3$

$x=-1$



Since $x=-1$ is not in Case 2 $(x<-\frac{3}{2})$, NO SOLUTIONS for Case2.

$\therefore$ NO SOLUTIONS.

#1 e) $2\cos^2(x) + \sin(x) - 1$.

domain: No restrictions for $\cos x$ and $\sin x$.

$\text{dom}(f) = \mathbb{R}$

roots $\quad 2\cos^2 x + \sin x - 1 = 0$.

Switch to $\sin x$: $\quad \cos^2 x = 1 - \sin^2 x$.

$2(1 - \sin^2 x) + \sin x - 1 = 0$.

$2 - 2\sin^2 x + \sin x - 1 = 0$

$-2\sin^2 x + \sin x + 1 = 0$.

Let $y = \sin x$

$-2y^2 + y + 1 = 0$

$2y^2 - y - 1 = 0 \qquad P = -2 \qquad -2, 1$

$2y^2 - 2y + y - 1 = 0 \qquad S = -1$

$2y(y-1) + (y-1) = 0$

$(2y+1)(y-1) = 0$

$2y = -1$ or $y = 1$

$y = -\frac{1}{2} \qquad \sin x = 1$

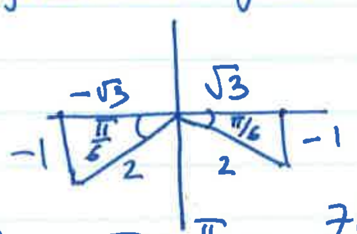
$\boxed{x = \frac{\pi}{2}}$

$\sin x = -\frac{1}{2}$

$\sin x < 0$

Q III, Q IV

reference angle is $\arcsin\left(\frac{1}{2}\right) = 30^\circ = \frac{\pi}{6}$



$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

$\theta = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$

Roots on $[0, 2\pi]$ are $\boxed{\frac{\pi}{2}; \frac{7\pi}{6}, \frac{11\pi}{6}}$

Question 2

a) $f(x) = (x^4 + x^2 - 1)\cos(x)$

$f(-x) = [(-x)^4 + (-x)^2 - 1]\cos(-x)$

$= (x^4 + x^2 - 1)\cos(x)$

$= f(x)$

We know $\cos(-x) = \cos(x)$

$\because f(-x) = f(x)$, $\boxed{f(x) \text{ is even}}$ and therefore symmetric about the y-axis.

b) $f(x) = x^3 + 3x - e^x$

$f(-x) = (-x)^3 + 3(-x) - e^{(-x)}$

$= -x^3 - 3x - \frac{1}{e^x}$ $\neq f(x)$

$\neq -f(-x)$

$\because f(-x) \neq f(x)$ and $f(-x) \neq -f(-x)$,

$\boxed{f(x) \text{ is neither}}$ even, nor odd

c) $g(x) = |x| \sin(x)$

$g(-x) = |-x| \sin(-x)$

we know $\sin(-x) = -\sin x$

$= |x|(-\sin x)$

$= -|x| \sin x$

$= -g(x)$

$\because g(-x) = -g(x)$, $\boxed{g(x) \text{ is odd}}$

Symmetry about the origin.

<u>Question 3</u> a) $x < \frac{2}{x-1}$ Put on one side.

$$x - \frac{2}{x-1} < 0 \Rightarrow \frac{x(x-1)-2}{x-1} < 0$$

$$\Rightarrow \frac{x^2-x-2}{x-1} < 0 \Rightarrow \frac{(x+1)(x-2)}{x-1} < 0$$

zeroes: $x=-1,\ x=1,\ x=2$

"Chart" method:

	$-\infty$	-1		1		2	$+\infty$
$(x+1)$	$-$	0	$+$		$+$		$+$
$(x-2)$	$-$		$-$		$-$	0	$+$
$(x-1)$	$-$		$-$	und	$+$		$+$
$\frac{(x+1)(x-2)}{(x-1)}$	$-$	0	$+$	undefined	$-$	0	$+$

<u>Solution:</u> $x \in (-\infty, -1) \cup (1, 2)$

b) $\left|\frac{x+1}{2} - \frac{x-1}{3}\right| > 1$

$$\left|\frac{3(x+1)}{2} - \frac{2(x-1)}{3}\right| > 1 \Rightarrow \left|\frac{x+5}{6}\right| > 1$$

$\frac{x+5}{6} < -1$ OR $\frac{x+5}{6} > 1$

$x+5 < -6$ $\qquad$ $x+5 > 6$

$x < -11$ $\qquad$ $x > 1$

$\therefore\ x \in (-\infty, -11) \cup (1, +\infty)$

Question 4 a). $f(x) = \sqrt{x-3}$

$g(x) = x^2$, $h(x) = x^3+2$

Find $f \circ g \circ h$.

Solution: $(f \circ g \circ h)(x) = f(g(h(x)))$

$= f(g(x^3+2)) = f((x^3+2)^2)$

$= \boxed{\sqrt{(x^3+2)^2-3}}$ Do not simplify.

b) Let $f(x) = 1-3^x$

$g(x) = x^2$

$k(x) = x$

$h(x) = 1-3^{x^2}$

$h(x) = (f \circ g \circ k)(x) = f(g(k(x)))$

$= f(g(x)) = f(x^2) = 1-3^{x^2}$

Question 5 Let x be the income; $f(x)$ be the tax

a) $f(x) = \begin{cases} 0.1x & 0 \le x \le 20{,}000 \quad ① \\ 2000 + 0.2(x-20{,}000), & x > 20{,}000 \quad ② \end{cases}$

b) find $f^{-1}(x)$

① line: $f(x) = 0.1x$

$y = 0.1x$

$x = 0.1y$

$y = 10x$

$f^{-1}(x) = 10x$

$0 \le x \le 2000$

$0 \le x \le 20{,}000$ Domain for f

$\downarrow$ Range for f^{-1}

$0 \le y \le 20{,}000$

$0 \le 10x \le 20{,}000$

$0 \le x \le 2000$

2 line $f(x) = 2000 + 0.2(x - 20{,}000)$, $x > 20{,}000$

$y = 2000 + 0.2(x - 20{,}000)$

$x = 2000 + 0.2(y - 20{,}000)$

$x - 2000 = 0.2(y - 20{,}000)$

$\frac{x - 2000}{0.2} = y - 20{,}000$

$5x - 10{,}000 + 20{,}000 = y$

$y = 5x + 10{,}000$

$f^{-1}(x) = 5x + 10{,}000$; $x > 2{,}000$

this is domain for f. = Range for f^{-1}

$y > 20{,}000$

$5x + 10{,}000 > 20{,}000$

$5x > 10{,}000$

$x > 2{,}000$

$$\boxed{\therefore f^{-1}(x) = \begin{cases} 10x, & 0 \le x \le 2{,}000 \quad ① \\ 10{,}000 + 5x, & x > 2000 \quad ② \end{cases}}$$

Since $f^{-1}(x)$ is the inverse of $f(x)$ and $f(x)$ represents the tax on income given $f^{-1}(x)$ represents the income on given tax.

meaning: If you pay $\$x$ for taxes, then $f^{-1}(x)$ is the income

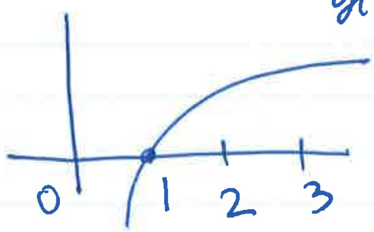
(c) tax = $10,000 Find income.

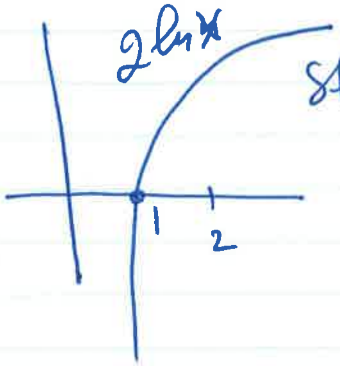
$f^{-1}(10{,}000)$ take line 2 of f^{-1} since $x > 2000$

$f^{-1}(10{,}000) = 10{,}000 + 5(10{,}000) = \boxed{\$60{,}000}$

Question #6

a) $g(x) = \ln x$





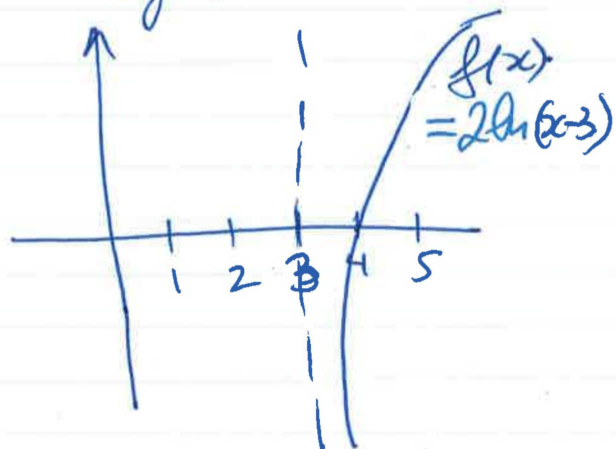
Stretch by 2

$\Longrightarrow$

$f(x) = 2\ln(x-3)$

(i) vertical stretch by a factor of 2

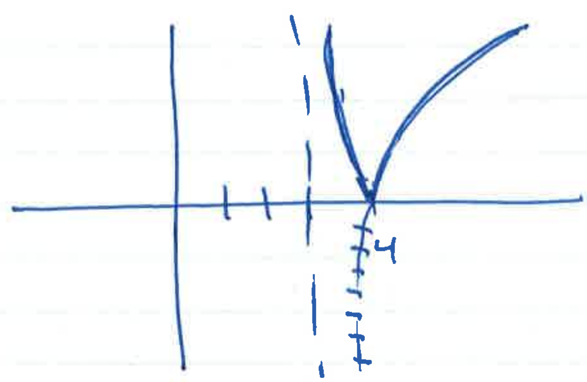
(ii) horizontal shift 3 units to the right.



VA: $x = 3$

b) $h(x) = 2|\ln(x-3)|$

reflect negative part about the x-axis



Question 7 (a) $\lim_{x \to 3} \dfrac{x^4+x^3-108}{x^2-9}$ has the form $\frac{0}{0}$

$\dfrac{3^4+3^2-108}{3^2-9} = \dfrac{0}{0}$

factor

Let $f(x) = x^4+x^3-108$

divisors of 108: $\pm 1, \pm 2, \pm 3, \ldots$

We already know

$f(3) = 0 \Rightarrow (x-3)$ is a factor

Perfor long division or synthetic division

Synthetic method:

	1	1	0	0	-108
3		3	12	36	108
	1	4	12	36	0

$x^3+4x^2+12x+36$

$f(x) = (x-3)(x^3+4x^2+12x+36)$

$= \lim_{x \to 3} \dfrac{\cancel{(x-3)}(x^3+4x^2+12x+36)}{\cancel{(x-3)}(x+3)}$

$= \lim_{x \to 3} \dfrac{x^3+4x^2+12x+36}{x+3} = \dfrac{\lim_{x \to 3}(x^3+4x^2+12x+36)}{\lim_{x \to 3} x+3}$

$= \dfrac{3^3+4 \cdot (3^2)+12(3)+36}{6}$

$= \boxed{\dfrac{135}{6}}$

(b) $\lim_{x \to 2} \dfrac{\sqrt{x-1}-1}{x-2}$ has the form $\frac{0}{0}$ and contains radicals

Rationalize the numerator

$= \lim_{x \to 2} \dfrac{\sqrt{x-1}-1}{(x-2)} \times \dfrac{\sqrt{x-1}+1}{\sqrt{x-1}+1}$

$= \lim_{x \to 2} \dfrac{(x-1)-1}{(x-2)(\sqrt{x-1}+1)} = \lim_{x \to 2} \dfrac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x-1}+1)}$

$$= \lim_{x\to 2} \frac{1}{\sqrt{x-1}+1} = \frac{1}{\lim_{x\to 2}(\sqrt{x-1}+1)} = \boxed{\frac{1}{2}}$$

e $\lim_{x\to 0^+} \frac{x}{|x|} + \lim_{x\to 0^-} \frac{x}{|x|}$

$(x>0)$ $(x<0)$

$|x| = x$ if $x>0$ $|x| = -x$ if $x<0$

$$= \lim_{x\to 0^+} \frac{x}{x} + \lim_{x\to 0^-} \frac{x}{(-x)}$$

$$= \lim_{x\to 0^+} 1 + \lim_{x\to 0^-} (-1) = (1)+(-1) = \boxed{0}$$

Question 8 $f(x) = \begin{cases} \sqrt{6-x} & x<6 \\ x^2+kx+1 & x \geq 6 \end{cases}$

$f(x) = \sqrt{6-x}$ $(x<6)$ is continuous on $(-\infty, 6)$

$f(x) = x^2+kx+1$ $(x>6)$ is continuous on $(6, +\infty)$

For $f(x)$ to be continuous everywhere in its domain, $f(x)$ must be continuous at $x=6$

$$\lim_{x\to 6^-} f(x) = \lim_{x\to 6^-} \sqrt{6-x} = 0$$

$$\lim_{x\to 6^+} f(x) = \lim_{x\to 6^+} x^2+kx+1 = 37+6k = f(6).$$

We must have $\boxed{\lim_{x\to 6} f(x) = f(6)}$

$$\lim_{x\to 6} f(x) = f(6) \iff \lim_{x\to 6^-} f(x) = \lim_{x\to 6^+} f(x) = f(6)$$

$$37+6k=0 \Rightarrow \boxed{k = -\frac{37}{6}}$$

Question 9 Show $\lim_{x \to 0} (x \cos \frac{1}{x}) = 0.$

Solution Let $f(x) = x \cos \frac{1}{x}$

We will show that $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0$

Using the squeeze theorem

(a) Show $\lim_{x \to 0^-} [x \cos(\frac{1}{x})] = 0.$

$-1 \leq \cos \frac{1}{x} \leq 1$ (Range of cosine function)

Multiply by x. Since $x \to 0^-$, $x < 0.$

$-x \geq x \cos \frac{1}{x} \geq x$

$\lim_{x \to 0^-} (-x) = 0$ $\quad\quad$ $\lim_{x \to 0^-} x = 0$

By Squeeze THEOREM $\lim_{x \to 0^-} [x \cos(\frac{1}{x})] = 0$

(b) Show $\lim_{x \to 0^+} [x \cos(\frac{1}{x})] = 0$

$-1 \leq \cos \frac{1}{x} \leq 1$ multiply by x
since $x \to 0^+$, $x > 0.$

$-x \leq x \cos \frac{1}{x} \leq x$

$\lim_{x \to 0^+} (-x) = 0$ $\quad\quad$ $\lim_{x \to 0^+} x = 0$

$\therefore \lim_{x \to 0^+} (x \cos \frac{1}{x}) = 0.$

Since $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0,$

$$\boxed{\lim_{x \to 0} (x \cos \frac{1}{x}) = 0}$$

#10 (a) Show that the sum of two irrational numbers is not necessarily irrational.
(Hint: give an example of two numbers $a, b \in I$ such that $(a+b) \in Q$.

Solution: When we want to show that some statement is false, it is enough to give just one example, showing that the statement is untrue.

Here the statement says that it is wrong to expect that the sum of two irrational numbers is always irrational.

An example would be:
If we take $a = \sqrt{2}$ and $b = -\sqrt{2}$, we know that $a, b \in I$.
But $a + b = (\sqrt{2}) + (-\sqrt{2}) = 0 \in Q$.

Another example:

$a = 2 + \sqrt{2}$, $b = 2 - \sqrt{2}$
Both a and $b \in I$. This would be shown in 10b.
But $a + b = (2+\sqrt{2}) + (2-\sqrt{2}) = 4 \in Q$.

Note in 10 b) we will prove that if c is a rational number and d is an irrational number, then $c+d$ is an irrational number.
In symbols:
If $c \in Q$ and $d \in I$, then $c+d \in I$.

A more general example: $r \in Q$, $i \in I$
Then $r - i \in I$. But $\underbrace{(r-i)}_{\in I} + \underbrace{i}_{\in I} = r \in Q$.

#10b) Consider the statement:
"The sum of a rational number and an irrational number must be rational."
Is it true or false?

Solution. The statement is false. On the contrary:
"The sum of a rational number and an irrational number must be irrational."
We can prove this new statement by contradiction.
We assume that the sum of a rational number and an irrational number must be rational.
So we assume: $\boxed{\text{if } a \in I,\ b \in \mathbb{Q},\ \text{then } a+b \in \mathbb{Q}.}$

Later we will encounter a contradiction while using this assumption, which means the assumption is wrong.

For our future reasoning, we need the fact that if two numbers are rational, then their sum (or difference) is rational as well.

Indeed if $c, d \in \mathbb{Q}$, then $c = \frac{m}{n}$ and $d = \frac{p}{q}$, where m, n, p, q are integers with $n \neq 0$ and $q \neq 0$. Then

$$c \pm d = \frac{m}{n} \pm \frac{p}{q} = \frac{mq \pm pn}{nq} \in \mathbb{Q} \text{ since}$$

$c \pm d$ is a fraction with $(nq) \neq 0$.

Since we assumed $a+b \in \mathbb{Q}$ ($a \in I, b \in \mathbb{Q}$), consider $\underbrace{(a+b)}_{\substack{\in \mathbb{Q} \\ \text{by assumption}}} - \underbrace{b}_{\in \mathbb{Q}} \in \mathbb{Q}$ as proven above (the difference of two rationals is rational)

But $(a+b)-b=a$. So, we got $\boxed{a \in \mathbb{Q}}$.
which is a contradiction to our choice of a.

Since we got a contradiction using the assumption $a+b \in \mathbb{Q}$ where $a \in I$, $b \in \mathbb{Q}$, it means our assumption $a+b \in \mathbb{Q}$ is false.

Therefore $a+b \notin \mathbb{Q} \Rightarrow a+b \in I$ (when $a \in I, b \in \mathbb{Q}$)