Introduction to Differential Equations (DE's)

September-10-12 2:01 PM

Algebraic equation: $x^2 - 3x + 1 = 0$
Two numbers x_1, x_2 make this an identity

DE for unknown function $y = \phi(t),\ t \in I \subseteq R$
- t is independent variable
- y is dependent variable

$y'' - 3y' + y = 0$ where $y' = \frac{dy}{dt} = \phi'(t)$ and $y'' = \frac{d^2y}{dt^2} = \phi''(t)$
when substitute $y = \phi(t)$ into DE, we get identities for $t \in I$

Example: Newton's Second Law

$$m\frac{d^2y}{dt^2} = F$$

F is force, may depend on $t, y, \frac{dy}{dt}$
For vertical motion of small object of mass m, known

Free fall:

$$F = -mg, \qquad g \approx 9.81\frac{m}{s^2}$$

Solve the DE

$$m\frac{d^2y}{dt^2} = -mg$$

$$y'' = -g$$

Solve by using antidifferentiation

$$\frac{dy}{dt} = -gt + C_1$$

$$y(t) = -\frac{1}{2}gt^2 + C_1t + C_2$$

This is a two-parameter family of solutions

State of motion in mechanics is given by $y(t)$ and $\frac{dy}{dt}(t)$
Given initial conditions at $t = 0$: $y(y) = y_0$ and $\frac{dy}{dt}(0) = v_0$ what is $y(t)$ at $t > 0$?
$C_1 = v_0$
$C_2 = y_0$

Modeling with DE's

September-12-12 1:39 PM

General Solution to DE

Contains arbitrary constants

Initial Conditions (IC)

What it sounds like: initial conditions that allow the general solution to the DE to be fixed to a single result

Autonomous DE

The independent variable does not appear.
In this situation we can introduce a new dependent variable for $\frac{dy}{dt}$, where t is the independent variable.

Classification of DE's

For some $F: \mathbb{R}^{n+2} \to \mathbb{R}$
(*) $F(t, y, y', \ldots, y^{(n)}) = 0$ is an n^{th} order differential equation. Generally non-linear

Where $y^{(n)} = \frac{d^n y}{dt^n}$, $\quad n = 0, 1, 2, \ldots$

Linear Differential Equation

$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t)$
Coefficients $a_i(t)$ are given functions of t as is the term $g(t)$

For $g(t) \neq 0$, this is a **linear, non-homogeneous, n^{th} order Differential Equation**

If $g(t) = 0 \; \forall \, t \in I$ then the DE is called **homogeneous.**

Example: Free Fall

$$\frac{d^2y}{dt^2} = -g$$

g is a given constant/parameter

General Solution

$$y = -g\frac{t^2}{2} + C_1 t + C_2$$

C_1 and C_2 are arbitrary constants and they make this a general solution
It is a two-parameter family of solutions

Initial Conditions (IC's) at $t = 0$

$$y(0) = y_0, \qquad \frac{dy}{dt}(0) = v_0, \qquad \text{given}$$

Look at the general solution and find C_1 and C_2 in general solution to accommodate the IC's

$$y(0) = C_2 = y_0$$

$$\frac{dy}{dt}(t) = -gt + C_1, \qquad \frac{dy}{dt}(0) = C_1 = v_0$$

Particular Solution

$$y = -g\frac{t^2}{2} + v_0 t + y_0$$

which satisfies both the original differential equation and the initial conditions.
As such, it predicts $y(t)$ at any $t > 0$ and $\frac{dy}{dt}(t)$ at $t > 0$
We solved an **Initial Value Problem** (IVP) — a DE plus IC's

Drag

Generalization of the model for motion in a field of gravity (2nd law of motion)

a) $m\frac{d^2y}{dt^2} = -mg - \gamma\frac{dy}{dt}$

γ is the friction/drag coefficient —always opposes motion

Changing Gravity

If y is "large", force of gravity is given by

$$-G\frac{Mm}{(R+y)^2}$$

b) $m\frac{d^2y}{dt^2} = -G\frac{Mm}{(R+y)^2} - \gamma y\frac{dy}{dt}$

This equation is called **autonomous** since the independent variable t does not appear, only its derivatives.

For a)
Introduce new dependent variable $v(t) = \frac{dy}{dt}$ and write

$$\begin{cases} m\frac{dv}{dt} = -mg - \gamma v \\ \frac{dy}{dt} = v \end{cases}$$

So we now have a system of two first order DE's

For b)

$$\begin{cases} m\frac{dv}{dt} = -G\frac{Mm}{(R+r)^2} - \gamma y v \\ \frac{dy}{dt} = v \end{cases}$$

Rocket burning fuel

Rocket burning fuel, expelled at $r\left(\frac{kg}{\text{sec}}\right)$ at velocity w relative to rocket
Mass of rocket:

$$m(t) = m_0 - rt$$

$$m(t)\frac{d^2y}{dt^2} = rw - G\frac{Mm(t)}{(R+y)^2} - \gamma y\frac{dy}{dt}$$

Solutions

September-14-12 1:34 PM

(*) $F(t, y, y', \ldots, y^{(n)}) = 0$

Explicit Solution

A function $\phi(t)$ that, when substituted for y in the DE (*), satisfies this equation for all $t \in I$ is called an **explicit solution.**

Implicit Solution

A relation $G(x, y) = 0$, for some $G: \mathbb{R}^2 \to \mathbb{R}$ is said to be an implicit solution to the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$, for some $F: \mathbb{R}^{n+2} \to \mathbb{R}$ on interval $x \in I$ if it defines one or more explicit solutions on I

Solutions of 1st order Differential Equations

(*) $\frac{dy}{dt} = f(t, y)$ for some $f: \mathbb{R}^2 \to \mathbb{R}$

Subject to the initial condition $y(t_0) = y_0$

The Existence and Uniqueness Theorem

for the Initial Value Problem (IVP) (*)

If f and $\frac{df}{dy}$ are continuous functions on $R = \{(t, y) | a < t < b, c < y < d\}$ that contains the point (t_0, y_0) then the IVP (*) has a unique solution $y = \phi(t)$ in some interval $a < t_0 - \delta < t < t_0 + \delta < b$

Explicit Solution Example

$\phi(t) = -g\frac{t^2}{2} + r_1 t + r_2, \frac{dy}{dt^2} = -g$

Show that

$v = -\frac{m}{\gamma}g + Ce^{-\frac{\gamma}{m}t}$

is a general solution for

$m\frac{dv}{dt} + \gamma v = -mg$

Implicit Solution Example

Show that relation $x^2 + y^2 = c$ for some positive constant c is the implicit solution to $y\frac{dy}{dx} + x = 0$ for $x \in I$

Use implicit differentiation assuming y is a function of x

$$\frac{d}{dx}(x^2 + y^2 - c) = 2x + 2y\frac{dy}{dx} - 0 = 2\left(x + y\frac{dy}{dx}\right) = 0$$

$x^2 + y^2 = c$ defines a circle so there are two explicit solutions:

$y = \pm\sqrt{c - x^2}$ for $x \in [-\sqrt{c}, \sqrt{c}]$

Initial-Value problem for n^{th} order DE

$F(t, y, \ldots, y^{(n)}) = 0, t \in I$

General solution $y = \phi(t) = \phi(t; C_1, C_2, \ldots, C_n)$

In general, there will be n integration constants

Find $C_1, \ldots, C_n$ to accommodate n initial conditions:

$y(t_0) = y_0$

$y'(t_0) = y_1$

…

$y^{(n-1)}(t_0) = y_{(n-1)}$

Example

Does the DE $\frac{dy}{dx} = -\frac{x}{y}$ have a unique solution such that

a) $y(1) = 2, \quad x_0 = 1, y_0 = 2$

   Yes, for $-\sqrt{5} < x < \sqrt{5}, \quad y = \sqrt{5 - x^2}$

b) $y(1) = 0, \quad x_0 = 1, y_0 = 0$

   No unique solution. Both $y = \pm\sqrt{1 - x^2}$ are solutions

General solution $x^2 + y^2 = C$

Solution Methods

Direction Fields for 1st order DE's

$\frac{dy}{dt} = f(t, y)$, we can generate a picture of the family of solution curves that correspond to general solutions.

Look for **isoclines** with constant slope $\frac{dy}{dx} = c$

Example

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y = -\frac{x}{c}$$

Rays exiting from the circle are isoclines.

Autonomous Equations

For autonomous DE $\frac{dy}{dt} = f(y)$

1) Look for equilibrium solutions such that $\frac{dy}{dt} = 0$. Solve $f(y) = 0$

Example

$$\frac{dv}{dt} = -g - \frac{\gamma}{m}v$$

$$\frac{dv}{dt} = 0 \Rightarrow -g - \frac{\gamma}{m}v = 0 \Rightarrow v_t = -\frac{mg}{\gamma} = \text{Terminal Velocity}$$

Euler's Method

September-17-12 1:34 PM

Numerical Approximation for 1st-order DE's

Euler's Method

IVP: $\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$

Find approximation for exact solution

$y = \phi(t), \qquad \phi'(t) = f\big(t, \phi(t)\big), \qquad \phi(t_0) = y_0$

Use partition of t-axis

$t_n = t_0 + nh, \qquad n = 0, 1, 2, \dots, \qquad h = step\ size$

Linear approximation:

$y = \phi(t_0) + \phi'(t_0)(t - t_0)$

$y = \phi(t_n) + \phi'(t_{n-1})(t_n - t_{n-1})$

Approximation:

$\phi(t_n) \sim y_n$

$\phi'(t_n) = f\big(t_n, \phi(t_n)\big) \sim f(t_n, y_n)$

$y_n = y_{n-1} + f(t_{n-1}, y_{n-1})(t_n - t_{n-1})$

Example

Euler's Method for

$y' = t\sqrt{y}, \qquad y(1) = 4, \qquad h = 0.1$

n	t_0	Euler's	Exact
0	1	4	4
1	1.1	4.200	4.213
2	1.2	4.425	4.452
3	1.3	4.678	4.720
4	1.4	4.959	5.018
5	1.5	5.271	5.318

Example

$y' = y, \qquad y(0) = 1$

Evaluate $y(1)$ using Euler's method

Exact: $y = e^t, \ y(1) = e = 2.718 \dots$

N	h	$y(1)$
1	1	2
2	0.5	2.250
4	0.25	2.441
8	0.125	2.566
16	0.0625	2.638

Dimensions of Physical Quantities

September-17-12 2:02 PM

In mechanics, all physical quantities have dimensions of form
$M^{\mu} \leftarrow$ Mass
$L^{\lambda} \leftarrow$ Length
$T^{\tau} \leftarrow$ Time

Consistency Requirements

1) Dimensional Homogeneity of equation:
   May only add, subtract or equate quantities that have the same dimension.
2) Quantities having different dimensions may only be combined by multiplication of division

Notation for quantity Q has dimension $[Q] = M^{\mu}L^{\lambda}T^{\tau}$

$[A+B] = [C] \Rightarrow [A] = [B] = [C]$
$[AB] = [A][B]$
$\left[\frac{A}{B}\right] = \frac{[A]}{[B]}$

Dimensionless time

$v' = -g - \frac{\gamma}{m}v$ notice
$\left[\frac{\gamma}{m}\right] = T^{-1}$

Define dimensionless time:
$\tau = \frac{\gamma}{m}t$

$[t] = \left[\frac{\gamma}{m}t\right] = 1$

Free Fall

$m\frac{d^2y}{dt^2} = F, \qquad v = \frac{dy}{dt}$
$[v] = \left[\frac{dy}{dt}\right] = \frac{[y]}{[t]} = LT^{-1}$
$\left[\frac{d^2y}{dt^2}\right] = \left[\frac{dv}{dt}\right] = \frac{[v]}{[t]} = LT^{-2}$
$[F] = [m]\left[\frac{d^2y}{dt^2}\right] = MLT^{-2}$

Work

$W = \int F dy$
$[W] = \left[\int F\, dy\right] = [F][y] = ML^2T^{-2}$

Free fall with Drag

$m\frac{dv}{dt} = -mg - \gamma v =$ force
Dimension of drag coefficient
$[\gamma] = \frac{[F]}{[v]} = \frac{MLT^{-2}}{LT^{-1}} = MT^{-1}$

Using dimensionless time:
$v(t) = u(\tau)$
$\frac{dv}{dt} = \frac{du}{d\tau}\frac{d\tau}{dt} = \frac{\gamma}{m}u'(\tau)$
$\frac{\gamma}{m}u'(\tau) = -g - \frac{\gamma}{m}u(\tau)$
$\frac{du}{d\tau} = u = -\frac{m}{\gamma}g$

Solving First Order DEs

September-19-12 1:25 PM

Types of 1st Order DE's

That can be solved analytically

$$\frac{dy}{dx} = f(x,y)$$

1) Separable Equation
   $f(x,y) = g(x)p(y)$
2) Linear Equation
   $\frac{dy}{dx} + P(x)y = Q(x)$
3) Exact Equation
   $M(x,y)dx + N(x,y)dy = 0$

Separable DE

Let $p(y) = \frac{1}{h(y)}$

$f(x,y) = g(x)p(y) \leftrightarrow \boxed{h(y)dy = g(x)dx}$

Let $H(y) = \int h(y)dy, \quad G(y) = \int g(x)dx$

Solution is given by

$H(y) = G(y) + C$

In general this is implicit

Example

$xdx + ydy = 0 \leftrightarrow xdx = -ydy$

$\frac{1}{2}x^2 = -\frac{1}{2}y^2 + K$

$x^2 + y^2 = C$

Example

Recall example of a body experiencing gravity and drag force

$\frac{dv}{dt} = -g - \frac{\gamma}{m}v, \quad v(t_0) = 0$

Define dimensionless time $\tau = \frac{\gamma}{m}t$

$v(t) = u(\tau), \quad \text{where } \frac{du}{d\tau} = \frac{du}{dt}\frac{dt}{d\tau} = \left(-g - \frac{\gamma}{m}v\right)\left(\frac{m}{\gamma}\right) = -\frac{mg}{\gamma} - u$

Define terminal velocity

$v_t = \frac{mg}{\gamma}$

so that

$\frac{du}{d\tau} = -v_t - u$

$\int \frac{du}{u + v_t} = -\int d\tau$

$\ln|u + v_t| = -\tau + K$

$|u + v_t| = e^K e^{-\tau}$

$u = \begin{cases} -v_t + e^K e^{-\tau}, & u > -v_t \\ -v_t - e^K e^{-\tau}, & u < -v_t \end{cases}$

Define $C = \pm Ce^K, C \in \mathbb{R}$

$u(\tau) = -v_t + Ce^{-\tau}$

$\boxed{v(t) = -v_t + Ce^{\frac{\gamma}{m}t}}$

$v(t) = v_0 e^{-\frac{\gamma}{m}t} - v_t\left(1 - e^{-\frac{\gamma}{m}t}\right), \quad \lim_{t\to\infty} v(t) = -v_t$

Stable equilibrium solution where $\frac{dv}{dt} = 0 \Rightarrow v = -v_t$

Population Growth

y = number of species at time $t \geq 0$ with $y(0) = y_0 > 0$

Simple equation of growth with rate $r \geq 0$

$\frac{dy}{dt} = ry$

$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$

K is the carrying capacity of the system

This is an autonomous equation, there is no t in $f(y) = r\left(1 - \frac{y}{K}\right)y$

Equilibrium solution:

$r\left(1 - \frac{y}{K}\right)y = 0 \rightarrow \begin{cases} y_1 = 0 \\ y_2 = K \end{cases}$

$f'(K) = -r < 0 \Rightarrow Stable$

$f'(0) = r > 0 \Rightarrow Unstable$

Stable equilibriums have $f(y) < 0$ unstable have $f(y) > 0$

Solve $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$ by separable variables

$$\int \frac{dy}{y\left(1-\frac{y}{K}\right)} = r\int dt$$

Partial fraction decomposition

$$\frac{1}{y\left(1-\frac{y}{K}\right)} = \frac{1}{y} - \frac{1}{y-K}$$

$$\ln|y| - \ln|y-K| = rt + C_1$$

$$\ln\left|\frac{y}{y-K}\right| = rt + C_1$$

$$\frac{y}{y-K} = \pm e^{C_1}e^{rt} = C_2 e^{rt}$$

$$\boxed{y = \frac{K}{1 - C_3 e^{-rt}}}$$ general solution

where $C_3 = 1/C_2$

Initial Condition $y(0) = y_0 \Rightarrow$

$$y = \frac{Ky_0}{y_0 - (y_0 - K)e^{-rt}}$$

Second Order Des with missing independent variable

$$\frac{d^2y}{dt^2} = F\left(y, \frac{dy}{dt}\right), \qquad \text{no } t \text{ so let } v(t) = \frac{dy}{dt}$$

$$\frac{dv}{dt} = F(y, v)$$

Let $v(t) = v\big(y(t)\big)$ and use chain rule.

Chain rule:

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy} \Rightarrow \frac{dv}{dy} = \frac{1}{v}F(y, v), \qquad y \text{ independent}$$

In mechanics, $F(y, v) = f(y)$ separable DE

$$\frac{dv}{dy} = \frac{f(y)}{v}$$

More Solving

September-21-12 1:37 PM

Existence and Uniqueness Theorem

$\frac{dy}{dt} = f(t,y) = -P(t)y + Q(t)$

$f(t,y)$ and $\frac{df}{dy} = -P(t)$

If $P(t)$ and $Q(t)$ are continuous in interval I containing the initial point t_0 then there exists a unique solution to IVP for all $t \in I$

In mechanics, 2^{nd} Newton in 1-D

$m\frac{d^2y}{dt^2} = F(y)$, missing independent variable t

$\frac{dy}{dt} = v$

$m\frac{dv}{dt} = F(y), assume\ v(t) = v\big(y(t)\big)$

$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$, y "independent"

$mv\frac{dv}{dy} = F(y)$, separable eq

$m\int v\,dv = \int F(y)dy$

$\frac{mv^2}{2} + U(y) = C = const = m\frac{v_0^2}{2} + U(y_0)$

$U(y) = -\int F(y)dy$

Conservation of energy

IC at $t = 0, y(0) = y_0, v(t=0) = v(y = y_0) = v_0$

Solve for $v(y) = W(y)$

Sub into 1

$\frac{dy}{dt} = W(y)$

Solve as separable DE

$\int \frac{dy}{W(y)} = t + C_2$

Escape velocity

$m\frac{d^2y}{dt^2} = -G\frac{Mm}{(R+y)^2} = F(y)$

$\Rightarrow U(y) = -\frac{GmM}{R+y}$

$\frac{mv^2}{2} - \frac{GmM}{R+y} = \frac{mv_0^2}{2} = \frac{GmM}{R}$

Minimum speed to send rocket to space

escape velocity:

$v_e = \sqrt{26\frac{M}{R}} \approx m\frac{\text{km}}{\text{sec}}$

Cases of DE's that may be converted to separable DE's

Homogeneous

1) Example

   $\frac{dy}{dx} = f(x,y) = g\left(\frac{y}{x}\right)$

   Use substitution

   $\mathcal{Z}(x) = \frac{y(x)}{x}$

   Assignment Q1

2) $\frac{dy}{dx} = g(ax+by), a, b = const$

   Define $\mathcal{Z}(x) = ax + by(x)$

Linear Equations

$\frac{dy}{dt} = P(t)y + Q(t)$

given function of t

IC $y(t_0) = t_0$

Solve by Method of Integrating Factors

(see problem 39 in Sec.1.2, page 26 for Method of Variation of Parameters)

Bernoulli's equation

$$\frac{dy}{dt} = p(t)y = 2^t y^r, r \neq 1$$

Define $z(t) = y^{1-r}(t)$,

$$\frac{dz}{dt} + (1-r)p(t)z = (1-r)y(t)$$

Assignment Q2

Exact Differential Equations

September-24-12 1:28 PM

Exact Equation

(*) $M(x,y)dx + N(x,y)dy = 0$

This differential equation is called an **exact equation** if there exists a $F(x,y)$ that is C^1 such that

$M(x,y) = \dfrac{\partial F(x,y)}{\partial x}$ and $N(x,y) = \dfrac{\partial F(x,y)}{\partial y}$

Its general solution is given by $F(x,y) = C$

Test of Exactness

Let $M(x,y), N(x,y), \frac{\partial M}{\partial y}(x,y)$, and $\frac{\partial N}{\partial x}(x,y)$ be continuous functions on a simply-connected domain D in $\mathbb{R}^2$

Then $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if and only if $\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$ in D

Exact Solution Proof

If $y = \phi(x)$ is a solution to (*) then upon substitution into $F(x,y) = C$, use Chain rule

$F\big(x, \phi(x)\big) = C$

$\dfrac{d}{dx}F(x,y) = 0$

$\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial x} = 0$

$M(x,y) + N(x,y)\dfrac{dy}{dx} = 0 \Leftrightarrow \dfrac{dy}{dx} = -\dfrac{M(x,y)}{N(x,y)}$

Proof of Test of Exactness

Show f exact, then there exist F, $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$

Differentiate

$\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 F}{\partial y \partial x}$

$\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 F}{\partial x \partial y}$

Since $\frac{\partial^2 F}{\partial y \partial x}$ and $\frac{\partial^2 F}{\partial x \partial y}$ are continuous

Other way

Assume $\frac{\partial F}{\partial x}(x,y) = M(x,y)$ and $\frac{\partial F}{\partial y}(x,y) = N(x,y)$

Integrate with respect to x, keep y fixed.

$F(x,y) = \displaystyle\int M(x,y)dx + K(y)$

Define $Q(x,y)$ such that $\frac{\partial Q}{\partial x}(x,y) = M(x,y)$

$\dfrac{\partial F}{\partial y}(x,y) = \dfrac{\partial Q}{\partial y}(x,y) + K'(y) = N(x,y)$

Use this to determine $K(y)$

$K'(y) = N(x,y) - \dfrac{\partial Q}{\partial y}(x,y)$

This must be independent of x, derivative with respect to x must vanish.

$\dfrac{\partial}{\partial x}\left[N(x,y) - \dfrac{\partial Q}{\partial y}(x,y)\right] = \dfrac{\partial N}{\partial x}(x,y) - \dfrac{\partial^2 Q}{\partial x \partial y}(x,y) = \dfrac{\partial N}{\partial x}(x,y) - \dfrac{\partial^2 Q}{\partial y \partial x}(x,y) = \dfrac{\partial N}{\partial x}(x,y) - \dfrac{\partial}{\partial y}\left(\dfrac{\partial Q}{\partial x}\right)$

$= \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 0$

Example

Solve the differentiation equation

$\dfrac{\partial y}{\partial x} = \dfrac{x^2 - y}{y^2 + x} \Leftrightarrow (y - x^2)dx + (y^2 + x)dy = 0$

$M = y - x^2, \qquad N = y^2 + x$

Test for exactness

$\dfrac{\partial}{\partial y}(y - x^2) = 1 = \dfrac{\partial}{\partial x}M(y^2 + x)$

Integrate $\frac{\partial F(x,y)}{\partial x} = y - x^2$

$F(x,y) = xy - \dfrac{1}{3}x^3 + K(y)$

$\dfrac{\partial F}{\partial y}(x,y) = \dfrac{\partial}{\partial y}\left(xy - \dfrac{1}{3}x^3\right) + K'(y) = x + K'(y) = y^2 + x$

$K(y) = \dfrac{y^3}{3} + C$

Putting together

$F(x,y) = \dfrac{y^3}{3} + xy - \dfrac{x^3}{3} + C$

Initial condition: $(x,y) = (0,0)$ gives $C = 0$

Second Order Differential Equations

September-24-12 2:10 PM

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right), IC\ y(t_0) = t_0, \frac{dy}{dt}(t_0) = y_1$$

Standard Form of Linear

(Non-Homogeneous DE)

$y'' + p(t)y + g(t)y = f(t)$

Given the three functions p, g, f of t

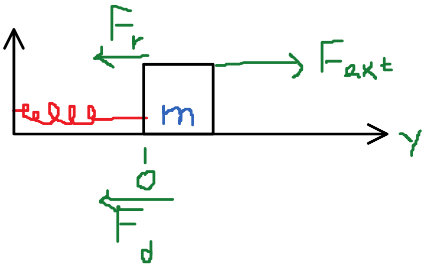
Standard Form of Linear With Constant Coefficients

$ay'' + by' + cy = f(t)$

a, b, c are constant

Modelling with 2nd -order DE's with constant coefficients

Mechanical sprint -mass oscillator



F_{ext} external force
F_r restoring force of spring
F_d drag force
Origin is the equilibrium position (unstretched spring)

$F_r = -ky,\qquad k = \text{spring constant}$

$F_d = -\gamma\frac{dy}{dt},\qquad \gamma = \text{drag coefficient}$

Displacement of the mass m from equilibrium position follows 2^{nd} Newton

$m\frac{d^2y}{dt^2} = F_r + F_d + F_{ext}$

$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + ky = F_{ext}(t)$

$y'' + \delta y' + \omega^2 y = g(t)$

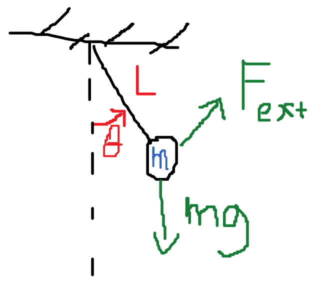
$\delta = \frac{\gamma}{m} = \text{damping factor}$

$\omega = \sqrt{\frac{k}{m}} = \text{frequency of oscillator}$

$[\delta] = [\omega] = T^{-1}$

$g(t) = \frac{F_{ext}(t)}{m}$

Linearization of Pendulum



Linear Displacement

$y = L\theta$

Forces acting tangentially

$F_d = -\gamma\frac{dy}{dt} = -\gamma\,L\frac{d\theta}{dt}$

$F_r = -mg\sin\theta$

$m\frac{d^2y}{dt^2} = F_r + F_d + F_{ext}$

$mL\frac{d\theta}{dt^2} + \gamma L\frac{d\theta}{dt} + mg\sin\theta + F_{ext}(t)$

But this is not linear $\rightarrow \sin\theta$

For small angular displacements: $|y| \ll L,\ |\theta| \ll 1$

$\sin\theta = \theta + \frac{\ell^{3}}{5} + \cdots$

Drop all powers but the leading one

$\sin\theta \approx \theta$

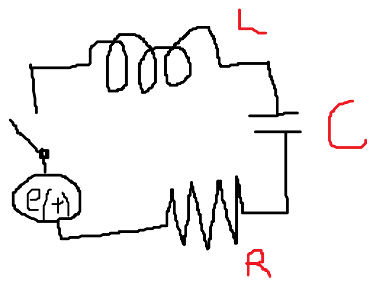
$\theta'' + \delta\theta' + \omega\theta = \alpha_{ext}(t)$

$\delta = \frac{}{m},\qquad \omega = \sqrt{\frac{}{L}},\qquad \alpha_{ext}(t) = \frac{F_{\epsilon}\quad t)}{mL}$

Initial Conditions

$y(0) = 0, y'(0) = v_0$

Electrical Oscillator: Series RLC Circuit



L = inductance
C = capacitance
R = resistance

State of the RLC circuit is defined by
q = charge on plates of capacitor
i = current through circuit
$[q] = C$, units Coulomb

$$i = \frac{dq}{dt}$$

$$[i] = \left[\frac{dq}{dt}\right] = \frac{[q]}{[t]} = CT^{-1}$$

$e(t)$ = given source of voltage

$$[e] = \left[\frac{U}{q}\right] = \frac{[energy]}{[q]} = \frac{ML^2T^{-2}}{C} = ML^2T^{-2}C^{-1}$$

Kirchhoff's Voltage Law

$$e(t) = v_L + v_C + v_R = L\frac{di}{dt} + \frac{q}{C} + Ri$$

$$L\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{q}{C} = e(t)$$

$$q'' + \delta q' + \omega^2 q = g(t) \quad (*)$$

$$\delta = \frac{R}{L}, \qquad \omega = \frac{1}{\sqrt{LC}}, \qquad g(t) = \frac{e(t)}{L}$$

Initial Conditions

$$q(0) = q_0, \qquad \frac{dq}{dt}(t) = i_0$$

Equivalent DE for Current

$$i(t) = \frac{dq}{dt}(t)$$

Differentiate $(*)$

$$i'' + \delta i' + \omega^2 i = g'(t)$$

Initial Conditions

$$i(0) = \frac{dq}{dt}(0) = i_0$$

$$i'(0) = \frac{di}{dt}(0) = \frac{d^2 i}{dt^2}(0) = ?$$

To get $i'(0)$, set $t = 0$ in $(*)$

$$i'(0) = q''(0) = q(0) - \delta q'(0) - \omega^2 q(0) = \frac{e(0)}{L} - \frac{R}{L}i_0 - \omega^2 q_0$$

Theory of 2^{nd} order linear DE's

September-26-12 2:08 PM

$$\frac{d^2y}{dt^2} = p(t)\frac{dy}{dt} + q(t)y = f(t) \quad (*)$$

Initial Conditions $y(t_0) = Y_0,\ y'(t_0) = Y_1$ $(**)$

Existence and Uniqueness Theorem

If $p(t), q(t)$, and $f(t)$ are continuous functions on some interval I, which contains t_0, then there exists a unique solution to the differential equation $(*)$ for all $t \in I$; which satisfies the initial conditions.

Linear Operator

Define linear operator

$\hat{L} = \frac{d^2}{dt} + p(t)\frac{d}{dt} + q(t)$

$(*) \Leftrightarrow \hat{L}[y] = f(t) \leftarrow$ Non-homogeneous equation

$\hat{L}[y] = 0 \leftarrow$ Homogeneous differential equation, when $f(t) = 0\ \forall t \in I$

Theorem

If $y_1(t)$ and $y_2(t)$ are sufficiently differentiable on the interval I and C_1 and C_2 are any constants then

$\hat{L}[C_1y_1 + C_2y_2] = C_1\hat{L}[y_1] + C_2\hat{L}[y_2]$

Corollary (Superposition Principle)

If y_1 and y_2 are any solutions of the homogeneous DE $\hat{L}[y] = 0$ then the linear combination $y = C_1y_1 + C_2y_2$ is also a solution to that same DE.

Linear Independence

Two functions y_1 and y_2 are said to be linearly independent on interval I iff neither is a constant multiple of the other.

Fundamental Set (Basis)

If y_1 and y_2 are solutions of the homogeneous DE (*) that are linearly independent on I then they are said to form a fundamental set, or basis, of solutions.

Theorem: Representation of General Solution to (*)

Representation of general solutions to homogeneous, linear, second-order differential equation.

If $y_1(t)$ and $y_2(t)$ are linearly independent solutions of linear DE $(*)$ on I, then every solution of that equation is give by

$y(t) = C_1y_1(t) + C_2y_2(t)$

where C_1 and C_2 are arbitrary constants that may be determined from the ICs $(**)$

Comment

$t_0 \in I, Y_0, Y_1$ are arbitrary.

Wronskian

The Wronskian of two functions y_1 and y_2 is defined by

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

Abel's Theorem for Wronskian

For any two solutions of the DE (*), $y_1(t)$ and $y_2(t)$

$\hat{L} = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + g(t), \qquad \hat{L}[y_1] = 0, \qquad \hat{L}[y_2] = 0$

then

$W[y_1, y_2](t) = Ce^{-\int p(t)dt}, \qquad C = const$

Corollary

$W[y_1, y_2](t)$ is either never zero or always zero on I

so the value of $C = W[y_1, y_2](t_0)$ depends on point t_0

Theorem

If $y_1(t)$ and $y_2(t)$ are any two solutions of $(*)$ on I, then they are linearly dependent on I iff their Wronskian is identically zero on I.

How to represent every (general) solution to 2nd order homogeneous linear DE

A) Need y_1 and y_2 linearly independent

Forget solutions to (*) which are identically zero on I (trivial solutions)

Note

If $y(t) = u(t) + iv(t)$ is a complex-valued solution of DE (*) with real-valued coefficients, so are its real $u(t)$, and imaginary $iv(t)$ parts.

Proof of Theorem (Solution to IVP)

$y(t_0) = C_1y_1(t_0) + C_2y_2(t_0) = Y_0$

$y'(t_0) = C_1y_1'(t_0) + C_2y_2'(t_0) = Y_1$

$$C_1 = \frac{\begin{vmatrix} Y_0 & y_2(t_0) \\ Y_1 & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} = \frac{Y_0y_2'(t_0) - Y_1y_2(t_0)}{W[y_1, y_2](t_0)}$$

$$C_2 = \frac{|\cdot|}{|\cdot|} = \frac{Y_1y_1(t_0) - Y_2y_1'(t_0)}{W[y_1, y_2](t_0)}$$

Proof of Theorem

1) Assume $y_2(t) = Ky_1(t)$
   $W[y_1, y_2] = y_1(t)Ky_1'(t) - y_1'(t)Ky_1(t) = 0$
2) $W[y_1, y_2] = 0$. Assume that $y_1(t_0) \neq 0$ for some $t_0 I$
   Then $\frac{W[y_1, y_2](t)}{y_1^2(t)} = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{d}{dt}\left(\frac{y_2}{y_1}\right)$
   $\Rightarrow \frac{y_2(t)}{y_{1(t)}} = \text{const} = K$
   $\Rightarrow y_2(t) = K\, y_1(t)$

Proof of Abel's Theorem

For $\hat{L}[y] = 0$

$\hat{L} = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$

If y_1 and y_2 are solutions to $\hat{L}[y] = 0$ then

$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$

is given by

$W(t) = ce^{-\int p(t)dt} = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}$

Proof

$y_1'' + py_1' + qy_1 = 0$

$y_2'' + py_2' + qy_2 = 0$

$\frac{dW}{dt} = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2'$

$= y_1(y_2'' + py_2' + qy_2 - py_2' - qy_2) - (y_1'' + py_1' + qy_1 - py_1' - qy_1)y_2$

$W' = -py_1y_2' - qy_1y_2 + py_1'y_2 + qy_1y_2 = -p(y_1y_2' - y_1'y_2) = -pW(t)$

$\frac{dW}{dt} = -p(t)W(t) \Rightarrow W(t) = ce^{-\int p(s)ds}$

Solving 2^{nd} order DEs with constant coefficients

October-01-12 1:32 PM

Fundamental Solutions

Find the fundamental solutions to
$ay'' + by' + cy = 0, \quad a \neq 0, b, c$ are constants
$\rightarrow y_1(t)$ and $y_2(t)$

Assume solution in form $y(t) = e^{\lambda t}$ with λ being a parameter
$y'(t) = \lambda e^{\lambda t}, \quad y''(t) = \lambda^2 e^{\lambda t}$
$\Rightarrow (a\lambda^2 + b\lambda + c)e^{\lambda t} = 0 \Rightarrow a\lambda^2 + b\lambda + c = 0$
Solve the characteristic equation (auxiliary equation):
$a\lambda^2 + b\lambda + c = 0 \Rightarrow \lambda_{1,2} = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$
These two roots give us $y_1(t) = e^{\lambda_1 t}, \ y_2(t) = e^{\lambda_2 t}$
These are linearly independent iff $\lambda_1 \neq \lambda_2 \Leftrightarrow b^2 \neq 4ac$

1) Distinct real roots, $b^2 > 4ac$
2) Distinct complex roots $b^2 < 4ac \Rightarrow \lambda_{1,2} = \mu \pm iv$
   $y_{1,2} = e^{\mu t}e^{ivt} = e^{\mu t}(\cos(vt) \pm i\sin(vt))$
   Use $y_1 = e^{\mu t}\cos vt, \quad y_2 = e^{\mu t}\sin vt$
3) Equal roots, $\lambda_1 = \lambda_2, \ b^2 = 4ac$
   We only have $y_1(t) = e^{\lambda_1 t}$
   Use reduction of order, assuming
   $y_2(t) = K(t)y_1(t) \Rightarrow K(t) = t + C$

   Alternate method
   Assume $\lambda_2 = \lambda_1 + \epsilon$, and let $\epsilon \to 0$
   General solution:
   $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = (c_1 + c_2 e^{\epsilon t})e^{\lambda_1 t}$
   Expand with Maclaurin series
   $y(t) = \left(C_1 + C_2 + C_2\epsilon t + C_2\frac{\epsilon^2}{2}t^2 + C_2\frac{\epsilon^3}{6}t^3 + \cdots\right)e^{\lambda_1 t}$
   C_1 and C_2 depend on ϵ so that
   $\lim_{\epsilon\to 0} C_2\epsilon = K_2 =$ finite const
   $\lim_{\epsilon\to 0}(C_1 + C_2) = K_1 =$ finite const
   $y(t) \rightarrow_{\epsilon\to 0} \ K_1 e^{\lambda_1 t} + K_2 t e^{\lambda_2 t}$

   Conclusion
   If $y_2(t)$ repeats $y_1(t)$ then just multiply $y_1(t)$ by a single factor of t

Non-Homogeneous Linear Equation

$\hat{L}[y] = f(t), \quad \hat{L} = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$
Let $y_h(t) = C_1y_1(t) + C_2y_2(t)$ be general solution of associated linear equation: e.g. $\hat{L}[y] = 0$
and let $y_p(t)$be any particular solution of the non-homogeneous equation
$\hat{L}[y_p] = f(t)$

Then the general solution of the equation is
$y(t) = y_h(t) + y_p(t)$

Example

Solve the IVP for
$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$
Characteristic equation
$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -1$
General solution:
$y = C_1e^{2t} + C_2e^{-t}$

$y(0) = C_1 + C_2 = 1$
$y'(t) = 2C_1e^{2t} - C_2e^{-t} \Rightarrow y'(0) = 2C_1 - C_2 = 0$
$C_1 = \frac{1}{3}, C_2 = \frac{2}{3}$
and the solution is
$y(t) = \frac{1}{3}(e^{2t} + 2e^{-t})$

Example

$y'' - 2y' + 5y = 0, \quad y(0) = 1, y'(0) = 0$
$\lambda^2 - 2\lambda + 5 = 0, \quad \lambda_{1,2} = 1 \pm i2$
$y(t) = C_1e^t\cos(2t) + C_2e^t\sin(2t)$
$y(0) = C_1 = 1$
$y'(t) = (C_1\cos(2t) + C_2\sin(2t))e^t + (-2C_1\sin(2t) + 2C_2\cos(2t))e^t$
$y'(0) = C_1 + 2C_2 = 0$
$C_2 = -\frac{1}{2}$

$y(t) = e^t\left(\cos(2t) - \frac{1}{2}\sin(2t)\right)$

How can we write this as
$y(t) = Ae^t\cos(2t + \alpha)$

$\cos(2t + \alpha) = \cos(2t)\cos\alpha - \sin(2t)\sin\alpha$

$y(t) = e^t\frac{\sqrt{5}}{2}\left(\cos(2t)\frac{2}{\sqrt{5}} - \sin(2t)\ \frac{1}{\sqrt{5}}\right) = \frac{\sqrt{5}}{2}e^t\cos\left(2t + \arctan\left(\frac{1}{2}\right)\right)$

Example

$y'' + 2y' + y = 0, \quad y(0) = 1, y'(0) = 0$
$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$

$y(t) = C_1e^{-t} + C_2te^{-t}$
$y(0) = C_1 = 1$
$y'(t) = -C_1e^{-t} + C_2e^{-t} - tC_2e^{-t}$
$y'(0) = -C_1 + C_2 = 0 \Rightarrow C_2 = C_1 = 1$
$y(t) = (1 + t)e^{-t}$

Superposition Principle for Non-Homogenous Linear DEs

October-03-12 1:26 PM

General Solution to Linear Second Order Non-Homogeneous DEs

$\hat{L}[y] = f(t)$

$\hat{L} = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$

$y_h(t) = y_h(t; c_1, c_2) =$ general solution of associated homogeneous equation
$\hat{L}[y_h] = 0$
$y_p(t) =$ particular solution of $\hat{L}[y_p] = f(t)$

Then the general solution of $\hat{L}[y] = f(t)$ is
$y(t) = y_h(t; c_1, c_2) + y_p(t)$

Comment

If $f(t) = f_1(t) + f_2(t)$ and $y_{p_1}(t)$ and $y_{p_2}(t)$ solve $\hat{L}[y_{p_1}] = f_1(t)$ and $\hat{L}[y_{p_2}] = f_2(t)$ then $y_{p(t)} = y_{p_1}(t) + y_{p_2}(t)$ solves $\hat{L}[y_p] = f(t)$

Method of Undetermined Coefficients

DE with constant coefficient
$\alpha y'' + by' + cy = f(t), \qquad a \neq 0, b, c = \text{const}$

If $f(t)$ is

- polynomial in t
- exponential $e^{\alpha t}$
- $\sin(\beta t)$ or $\cos(\beta t)$
- or product of the above

Use **Method of Undetermined Coefficients**

1) $f(t) = t^n$, assume $y_p(t) = A_0 + A_1 t + \cdots + A_n t^n$
2) $f(t) = e^{\alpha t}$ assume $y_p(t) = Ae^{\alpha t}$
3) $f(t) = \sin(\beta t)$ or $f(t) = \cos(\beta t)$ assume
   $y_p(t) = A\cos(\beta t) + B\sin(\beta t)$

Exception

If $f(t)$ reproduces any of the functions in the basis of solutions $y_1(t), y_2(t)$ to the homogeneous DE, then just multiply your assumption for $y_p(t)$ by a single factor of t.

Method of Variation of Parameters (or Constants)

for $y'' + p(t)y' + q(t)y = f(t)$
If we have a fundamental set of solutions $y_1(t), y_2(t)$
Recall $y_h(t) = c_1 y_1(t) + +c_2 y_2(t)$

To find particular solution
Assume $y_p(t) = K_1(t)y_1(t) + K_2(t)y_2(t)$
Need to determine functions K_1, K_2

$y_p'(t) = K_1'(t)y_1(t) + K_2'(t)y_2(t) + K_1(t)y_1'(t) + K_2(t)y_2'(t)$
Assume $K_1'(t)y_1(t) + K_2'(t)y_2(t) = 0$
$y_p''(t) = K_1'(t)y_1'(t) + K_2'(t)y_2'(t) + K_1(t)y_1''(t) + K_2(t)y_2''(t)$

Sub into initial DE
$y_p'' + py_p' + qy_p = f(t)$
$K_1(y_1'' + py_1' + qy_1) + K_2(y_2'' + py_2' + qy_2) + K_1'(t)y_1'(t) + K_2'(t)y_2'(t) = f(t)$
$K_1'(t)y_1'(t) + K_2'(t)y_2'(t) = f(t)$

1) $K_1'(t)y_1(t) + K_2'(t)y_2(t) = 0$
2) $K_1'(t)y_1'(t) + K_2'(t)y_2'(t) = f(t)$

Wronskian $W[y_1, y_2] = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

$K_1'(t) = \frac{y_2(t)f(t)}{W[y_1, y_2](t)}$

$K_2'(t) = \frac{y_1(t)f(t)}{W[y_1, y_2](t)}$

Quiescent Initial Conditions

Compare solutions to the IVPs

1) $y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$
   $\Rightarrow y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$
2) $y'' - y - 2y = e^{-t}, \qquad y(0) = -1, \qquad y'(0) = 0$
   $\Rightarrow y(t) = \frac{4}{9}e^{2t} + \frac{5}{9}e^{-t} - \frac{t}{3}e^{-t}$

Example

Find $y_p(t)$ to $y'' - y' - 2y = 3t$

Let $y_p(t) = At + B$, A, B are undetermined coefficients
$y_p'(t) = A_1, \qquad y_p''(t) = 0$
$y_p'' - y_p' - 2y = -A - 2(At + B) = 3t$
$t(2A + 3) + A + 2B = 0$

Since t and 1 are linearly independent functions
$2A + 3 = 0, \qquad A + 2B = 0 \Rightarrow A = -\frac{3}{2}, \qquad B = \frac{3}{4}$
$\therefore y_p(t) = -\frac{3}{2}t + \frac{3}{4}$

Example

$y'' - y' - 2y = e^t$

$y_p(t) = Ae^t, \qquad y_p' = Ae^t, \qquad y_p'' = Ae^t$
DE: $(A - A - 2A)e^t = e^t$
$(2A + 1)e^t = 0 \Rightarrow A = -\frac{1}{2}$
$y_p(t) = -\frac{1}{2}e^t$

Example

$y'' - y' - 2y = e^{-t}$
$y_p(t) = Ae^{-t}, \qquad y'(t) = -Ae^{-t}, \qquad y''(t) = Ae^{-t}$
$(A + A - 2A)e^{-t} = e^{-t}$

What went wrong?
Recall: linearly independent solutions of the associated homogeneous DE were $y_1(t) = e^{2t}$, $y_2(t) = e^{-t}$

For $y'' - y' - 2y = e^{-t}$, assume $y_p(t) = Ate^{-t}$
$y_p'(t) = A(1 - t)e^{-t}, \qquad y_p''(t) = A(-2 + t)e^{-t}$

$A(-2 + t - 1 + t - 2t)e^{-t} = e^{-t} \Rightarrow A = -\frac{1}{3}$
$y_p(t) = -\frac{t}{3}e^{-t}$

Example

Solve IVP for
$y'' - y' - 2y = e^{-t}, \qquad y(0) = 1, \qquad y'(0) = 0$
$y_h(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{2t} + c_2 e^{-t}$
$y_p(t) = -\frac{t}{3}e^{-t}$

$y(t) = y_h(t) + y_p(t)$
$y(0) = c_1 + c_2 = 1$
$y'(t) = 2c_1 e^{2t} - c_2 e^{-t} - \frac{1 - t}{3}e^{-t}$
$y'(0) = 2c_1 - c_2 - \frac{1}{3} = 0$
$c_1 = \frac{4}{9}, c_2 = \frac{5}{9}$

$\therefore y(t) = \frac{4}{9}e^{2t} + \frac{5}{9}e^{-t} + \frac{t}{3}e^{-t}$

Example

$y'' + 2y' + y = e^{-t}$
Recall $\lambda_1 = \lambda_2 = -1, y_1(t) = e^{-t}$, $y_2(t) = te^{-t}$
$y_p(t) = At^2 e^{-t}$

Example

$y'' - y' - 2y = \sin t$
$y_p(t) = A\sin t + B\cos t$
$y_p'(t) = A\cos t - B\sin t$
$y_p''(t) = -A\sin t - B\cos t$

$(-A + B - 2A)\sin t + (-B - A - 2B)\cos t = \sin t$
$-3A + B = 1, -3B - A = 0$
$A = -\frac{3}{10}, \qquad B = \frac{1}{10}$

$\therefore y_p(t) = \frac{1}{10}(\cos t - 3\sin t)$

2) $y'' - y - 2y = e^{-t}, \qquad y(0) = -1, \qquad y'(0) = 0$

$$\Rightarrow y(t) = \frac{4}{9}e^{2t} + \frac{5}{9}e^{-t} - \frac{t}{3}e^{-t}$$

Difference is solutions of the IVP $y'' - y' - 2y = e^{-t},\ y(0) = 0, y'(0) = 0$
With 0 for initial conditions, called **Quiescent** initial conditions.

$$y_Q(t) = \left(\frac{4}{9} - \frac{1}{5}\right)e^{2t} + \left(\frac{5}{9} - \frac{2}{3}\right)e^{-t} - \frac{t}{3}e^{-t}$$
$$y_Q(t) = \frac{1}{9}e^{2t} - \frac{1}{9}e^{-t} - \frac{t}{3}e^{-t}$$

$$y_Q(0) = 0, \qquad y_Q'(0) = 0$$

Solution to the IVP with Quiescent initial conditions. $y(t_0) = 0,\ y'(t_0) = 0$

Integrate formulas for $K_1',\ K_2'$ such that $K_1(t_0) = 0, K_2(t_0) = 0$

$$K_1(t) = \int_{t_0}^{t} \frac{y_2(\tau)f(\tau)}{W(\tau)}d\tau$$
$$K_2(t) = \int_{t_0}^{t} \frac{y_1(\tau)f(\tau)}{W(\tau)}d\tau$$

Recall $y(t) = K_1(t)y_1(t) + K_2(t)y_2(t)$

Green's Propagator

$$y(t) = \int_{t_0}^{t} G(t,\tau)f(\tau)d\tau$$
$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau)}$$

$$\hat{L}[y] = f(t), \qquad \text{where } \hat{L} = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$$

$G(t,\tau)$ serves in some sense as an inverse of $\hat{L}[y]$

$$A = -\frac{\quad}{10}, \qquad B = \frac{\quad}{10}$$

$$\therefore y_p(t) = \frac{1}{10}(\cos t - 3\sin t)$$

Example of Method of Variation of Parameters

Find $y_p(t)$ for $y'' + y = \csc(t) = \frac{1}{\sin(t)}$
From $y'' + y = 0,\ \lambda^2 + 1 = 0,\ \lambda_{1,2} = \pm i$
$y_1(t) = \sin t\,, y_2(t) = \cos t$

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1$$

$$K_1'(t) = -\frac{y_2 f}{W} = \frac{\cos t}{\sin t}$$
$$K_1(t) = \int \frac{\cos t}{\sin t}dt = -\ln|\sin t|$$

$$K_2'(t) = -\frac{y_1 f}{W} = -\frac{\sin t}{\sin t} = -1$$
$$K_2(t) = -t$$

$$y_p(t) = K_1(t)y_1(t) + K_2(t)y_2(t) = \sin t \ln|\sin t| - t\cos t$$

Example - Multiplication by t for constant coefficients

Find $y_p(t)$ for $y'' - y' - 2y = e^{-t}$
Recall $y_1(t) = e^{2t},\ y_2 = e^{-t}$

Use method of variation of parameters
$W(t) = -e^{2t}e^{-t} - 2e^{2t}e^{-t} = -3e^{-t}$

$$K_1'(t) = \frac{e^{-t}}{-3e^{t}}e^{-t} = \frac{1}{3}e^{-3t}$$
$$K_2'(t) = \frac{e^{2t}}{-3e^{t}}e^{-t} = -\frac{1}{3}$$
$$K_1(t) = -\frac{1}{9}e^{-3t}, \qquad K_2(t) = -\frac{t}{3}$$

$$y_p(t) = -\frac{1}{9}e^{-3t}e^{2t} - \frac{t}{3}e^{-t} = -\frac{1}{9}e^{-t} - \frac{t}{3}e^{-t}$$
but e^{-t} is linearly dependent on y_2 so
$$y_p(t) = -\frac{t}{3}e^{-t}$$

$$y_h(t) = c_1 e^{2t} + \left(c_1 - \frac{1}{9}\right)e^{-t}$$

Oscillator DE and Resonance

October-10-12 1:31 PM

Recall: mechanical and electrical oscillators

- Mass on spring, possibly damped
  $$m\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+ky=F_{\text{ext}}$$
- RLC circuit
  $$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{de(t)}{dt}$$

Both systems are described by
$y''+2\zeta\omega y'+\omega^2y=f(t)$
where $y(t)=y(t)$ or $i(t)$
$f(t)=\frac{F_e}{m}$ or $\frac{1}{L}\frac{de(t)}{dt}$
$\omega=\sqrt{\frac{k}{m}}$ or $\frac{1}{\sqrt{LC}}$
$\omega=$ natural frequency

$\zeta=\frac{\gamma}{2\sqrt{km}}$ or $\frac{R}{2}\frac{1}{\sqrt{LC}}$
$\zeta=$ damping parameter. Normally $\zeta>0$

Assume harmonic forcing
$f(t)=F\cos(\Omega t)$
F = amplitude, $F>0$
$\Omega=$ frequency

Free Oscillations

First consider free oscillations with $f(t)=0$
The associated homogeneous DE
$y''+2\zeta\omega y'-\omega^2=0$

> Aside: we could use dimensionless time $\tau=\omega t$
> $\frac{d^2y}{d\tau^2}+2\zeta\frac{dy}{d\tau}+y=0$

$\lambda^2+2\zeta\omega\lambda+\omega^2=0$
$\lambda_{1,2}=-\omega\left(\zeta\mp\sqrt{\zeta^2-1}\right)$

Determinant cases:

1) $\zeta>1$: overdamped motion of oscillator
   $y_h(t)=c_1e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega t}+c_2e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega t}$

2) $\zeta=1$: critically damped oscillator
   $y_h(t)=c_1e^{-\omega t}M+c_2te^{-\omega t}$
3) $\zeta<1$: underdamped oscillator
   $y_h(t)=e^{-\omega t}\left(c_1\cos\left(\sqrt{1-\zeta^2}\omega t\right)+c_2\sin\left(\sqrt{1-\zeta^2}\omega t\right)\right)$
   $=ce^{-\omega t}\cos\left(\sqrt{1-\zeta^2}\omega t+\Gamma\right)$

c_1 and c_2 are determined by initial conditions: $y(0)=y_0,\ y'(0)=v_0$

Forced Oscillations

$f(t)=F\cos(\Omega t)$
$y''+2\zeta\omega y'+\omega^2y=f(t)$

General solution: $y(t)=y_h(t)+y_p(t)$
$y_p(t)=A_1\cos(\Omega t)+A_2\sin(\Omega t)=A\cos(\Omega t-\phi)=A\cos\left(\Omega(t-t_0)\right)$
Undetermined amplitude, phase shift

Steady-State Response

Since $y_p(t)$ persists, $y_p(t)$ is called steady-state response.

Transient Response

Since $y_h(t)\to 0$, it is called the transient response of the system.
The initial conditions are "forgotten".

Solving

To determine A and ϕ assume complex-valued solution $Y(t)=y(t)+iz(t)$

$y'' + 2\zeta\omega y' + \omega^2 y = F\cos(\Omega t)$
$iz'' + i2\zeta\omega z' + i\omega^2 z = iF\sin(\Omega t)$
$Y'' + 2\zeta\omega Y' + \omega^2 Y = Fe^{i\Omega t}$

Find particular solution $Y_p(t)$ and take $y_p(t) = Re[Y_p(t)] = Re[y_p(t) + izp(t)]$
Assume $Y_p(t) = \alpha e^{i\Omega t}$ where $\alpha = |\alpha|e^{i\,arg(\alpha)} = Ae^{-i\phi} \Rightarrow Y_p(t) = Ae^{i(\Omega t-\phi)}$
$Re[Y_p(t)] = A\cos(\Omega t - \phi)$

$Y_p'(t) = i\Omega Ae^{i(\Omega t-\phi)}$
$Y_p''(t) = \Omega^2 Ae^{i(\Omega t-\phi)}$
$(-\Omega^2 + 2i\zeta\omega\Omega + \omega^2)Ae^{i(\Omega t-\phi)} = Fe^{i\Omega t}$
$(\omega^2 - \Omega^2 + 2i\zeta\Omega)A = Fe^{it} = F\cos(\phi) + iF\sin(\phi)$
$\Leftrightarrow$
$(\omega^2 - \Omega^2)A = F \cdot \cos(\phi)$
$2\zeta\omega\Omega A = F \cdot \sin(\phi)$

$[(\omega^2 - \Omega^2)^2 + 4\zeta^2\omega^2\Omega^2]A^2 = F^2$

Solution to Forced Oscillations

$$A = \frac{F}{\sqrt{(\omega^2 - \Omega^2)^2 + 4\zeta^2\omega^2\Omega^2}}$$
$$\phi = \operatorname{acos}\left(\frac{\omega^2 - \Omega^2}{\sqrt{(\omega^2 - \Omega^2)^2 + 4\zeta^2\omega^2\Omega^2}}\right)$$

Remark: Notation in textbook

$\alpha = F \cdot G(i\Omega)$
$A = |\alpha| = F|G(i\Omega)|$
$$|G(i\Omega)| = \frac{A}{F} = \frac{1}{\omega^2}\frac{1}{\sqrt{\left(1 - \left(\frac{\Omega}{\omega}\right)^2\right)^2 + 4\zeta^2\left(\frac{\Omega}{\omega}\right)^2}}$$
Analyze this as function of "reduced" frequency $\frac{\Omega}{\omega}$

Resonance

For $0 < \zeta < \frac{1}{\sqrt{2}}$, there is a local maximum in the plot of $|G(i\Omega)|$.
This is the resonant frequency.
If $\zeta > \frac{1}{\sqrt{2}}$

Zero Damping

$\zeta = 0$
$y'' + \omega^2 y = F \cdot \cos(\Omega t)$
$y_h(t) = c_1\cos(\omega t) + c_2\sin(\omega t) \not\to 0$
Persists for all t

$$y_p(t) = A\cos(\Omega t - \phi) = \frac{F}{\omega^2 - \Omega^2}\cos(\Omega t)$$
$A(\omega^2 - \Omega^2) = F\cos(\phi) = A$

Recall

$2\zeta\omega\Omega A = F\sin\phi$

General solution
$y(t) = y_h(t) + y_p(t)$

What happens as $\Omega \to \omega$
"Borrow" part from $y_h(t)$ and consider quiescent system
$y(0) = 0, y'(0) = 0$
$$y_q(t) = \frac{F}{\omega^2 - \Omega^2}\cos(\Omega t) - \frac{F}{\omega^2 - \Omega^2}\cos(\omega t)$$
$$y_q(t) = 2F\frac{\sin\left(\frac{\Omega - \omega}{2}t\right)}{\Omega - \omega} \cdot \frac{\sin\left(\frac{\Omega + \omega}{2}t\right)}{\Omega + \omega}$$

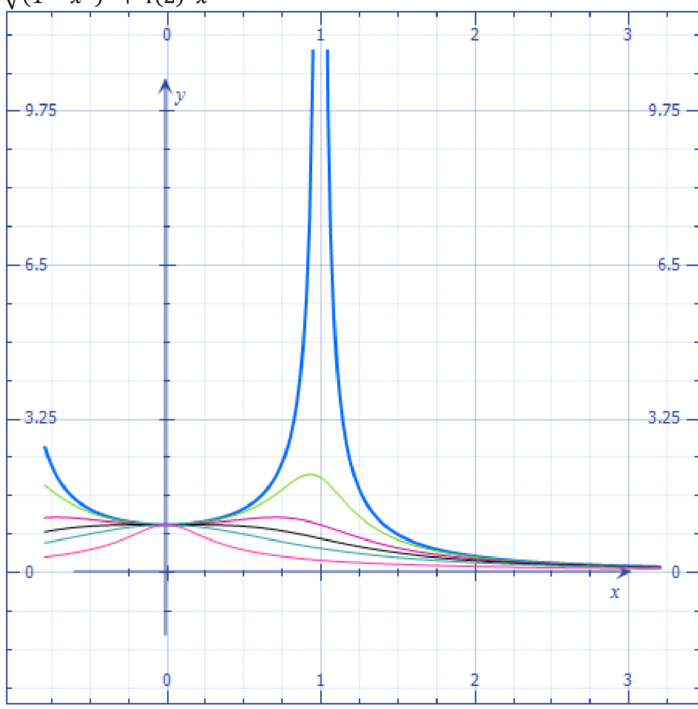
Beats

$y_q(t)$ consists of a large amplitude and large period sine wave filled with a small amplitude wave. These are known as **beats**

$$y_q(t) \to_{\Omega\to\omega} \frac{F}{2\omega}t\sin(\omega t)$$
This is a linearly-increasing amplitude sine wave.

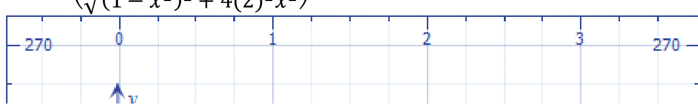
Resonance Amplitude

$$\frac{1}{\sqrt{(1-x^2)^2 + 4(0)x^2}}$$
$$\frac{1}{\sqrt{(1-x^2)^2 + 4(0.25)^2x^2}}$$
$$\frac{1}{\sqrt{(1-x^2)^2 + 4(0.5)^2x^2}}$$
$$\frac{1}{\sqrt{(1-x^2)^2 + 4\left(\frac{1}{\sqrt{2}}\right)^2x^2}}$$
$$\frac{1}{\sqrt{(1-x^2)^2 + 4(1)^2x^2}}$$
$$\frac{1}{\sqrt{(1-x^2)^2 + 4(2)^2x^2}}$$

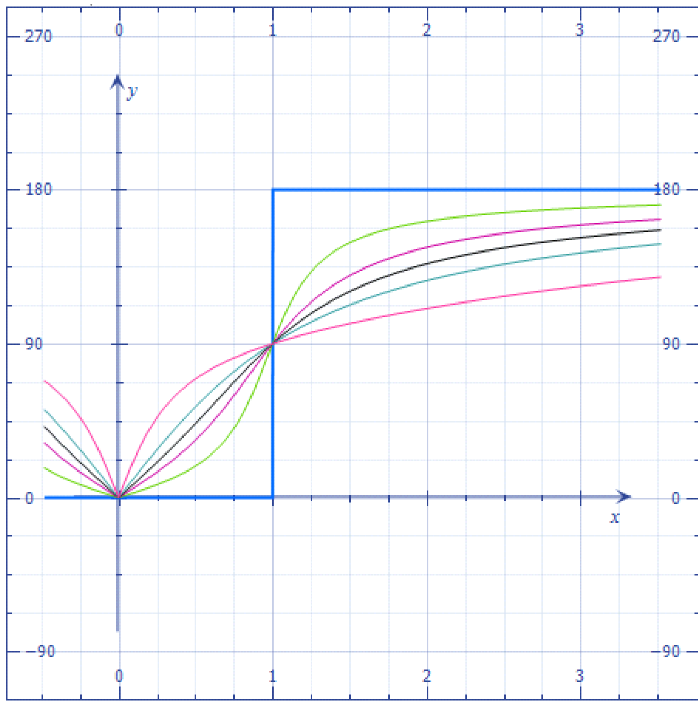


Resonance Phase

$$\arccos\left(\frac{(1-x^2)}{\sqrt{(1-x^2)^2 + 4(0)^2x^2}}\right)$$
$$\arccos\left(\frac{(1-x^2)}{\sqrt{(1-x^2)^2 + 4(0.25)^2x^2}}\right)$$
$$\arccos\left(\frac{(1-x^2)}{\sqrt{(1-x^2)^2 + 4(0.5)^2x^2}}\right)$$
$$\arccos\left(\frac{(1-x^2)}{\sqrt{(1-x^2)^2 + 4(\frac{1}{\sqrt{2}})^2x^2}}\right)$$
$$\arccos\left(\frac{(1-x^2)}{\sqrt{(1-x^2)^2 + 4(1)^2x^2}}\right)$$
$$\arccos\left(\frac{(1-x^2)}{\sqrt{(1-x^2)^2 + 4(2)^2x^2}}\right)$$



$y_q(t) \rightarrow_{\Omega \rightarrow \omega} \frac{}{2\omega} t \sin(\omega t)$
This is a linearly-increasing amplitude sine wave.



Systems of First Order DEs

October-15-12 1:55 PM

Phase Portrait

The phase portrait of the solution is the description of a solution to the DE as a circle on the y and $\frac{v}{\omega}$ plane.

Homogeneous Undamped Oscillator

Solve IVM for homogeneous equation

$y'' + \omega^2 y = 0\,, \qquad y(0) = y_0, y'(0) = v_0$

Characteristic equation

$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda_{1,2} = \pm i\omega$

$y_h(t) = y(t)c_1\cos(\omega t) + c_2\sin(\omega t)$

$y_0 = c_1 = y_0$

$y'(t) = -\omega c_1 \sin(\omega t) + \omega c_2 \cos(\omega t)$

$y'(0) = \omega c_2 = v_0$

$y(t) = y_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$

$v(t) = v_0\cos(\omega t) - \omega y_0 \sin(\omega t)$

The vector $\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$ defines the state of the system at time $t > 0$

$\Rightarrow \begin{bmatrix} y(t) \\ \frac{v(t)}{\omega} \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} y_0 \\ \frac{v_0}{\omega} \end{bmatrix}$

Describe the state for the system by a curve in the $\left(y, \frac{v}{\omega}\right)$ plane.

Rewrite $y'' + \omega y = 0$ as a system of equations for state variables

$\frac{dy}{dt} = v$

$\frac{dv}{dt} = -\omega^2 y \Leftrightarrow m\frac{dv}{dt} = -ky$

$\frac{d}{dt}\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix} \quad (**)$

Define the vector

$\vec{x}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$

so the system $(**)$ becomes

$\frac{d\vec{x}(t)}{dt} = M\,\vec{x}(t)$ where $M = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$

This looks like a separable equation

$\vec{x}(t) = e^{Mt}\vec{x}(0)$

Matrix exponential:

$e^{\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}t} = \begin{bmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix}$

$y(t) = \sqrt{y_0^2 + \left(\frac{v_0}{\omega}\right)^2}\,\sin(\omega t + \delta)$

$\frac{v(t)}{\omega} = \sqrt{y_0^2 + \left(\frac{v_0}{\omega}\right)^2}\,\cos(\omega t + \delta)$

where $\delta = \arcsin\left(\frac{y_0}{\sqrt{y_0^2 + \left(\frac{v_0}{\omega}\right)^2}}\right)$

$\Rightarrow y^2 + \left(\frac{v}{\omega}\right)^2 = y_0 + \left(\frac{v_0}{\omega}\right)^2$

This is a circle on the $y_0 \;- \frac{v_0}{\omega}$ plane. **Phase portrait**

Recall $\omega = \sqrt{\frac{k}{m}}$

Multiply by $\frac{k}{2}$ and obtain

$k\frac{y^2}{2} + m\frac{v^2}{2} = k\frac{y_0^2}{2} + m\frac{v_0^2}{2}$

This equation represents conservation of energy.

Laplace Transform

October-17-12 1:38 PM

Integral Transform

An Integral Transform is a linear operator that maps functions $y(t) \in \mathbb{V}_y$ to functions $Y(t) \in \mathbb{V}_Y$, defined by

$$Y(s) = \int_\alpha^\beta K(s,t) \cdot y(t)dt$$

α, β

$K(s,t)$ is called the **Kernel**

Laplace Transform

Laplace transform of $f(t)$, defined on $t \in [0, \infty)$ is the function $F(s)$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

Note

In this course, s is real but in general it is possible that $s \in \mathbb{C}$

Notation

$F(s) = \mathcal{L}\{f(t)\}$

The domain of definition of $F(s)$ is the set of all s values for which the integral exists (converges).

Note: since $F(s)$ involves an improper integral

$$F(s) = \lim_{A \to \infty} \int_0^A e^{-st} f(t)dt$$

Note

$\mathcal{L}$ is a linear operator

$\mathcal{L}\{c_1 f(t) + +c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$

Piece-Wise Continuous

A function $f(t)$ is piece-wise continuous (PWC) on finite (bounded) interval $I \subset \mathbb{R}$ if it is continuous at every point of I except possibly for a finite number of points $t_j \in I$, where $f(t)$ has (finite) jump discontinuities.

That is,

$\lim_{t \to t_j^-} f(t)$, $\lim_{t \to t_j^+} f(t)$ exist but $\lim_{t \to t_j^-} f(t) \neq \lim_{t \to t_j^+} f(t)$

Exponential Order

A function $f(t)$ is said to be of **exponential order α** if there exist constants $\alpha, M > 0, T > 0$, such that

$|f(t)| \leq Me^{\alpha t}$ $for\ t \geq T$

Equivalently, $|f(t)| \in O(e^{\alpha t})$

Theorem: Existence of $\mathcal{L}$

If $f(t)$ is piecewise continuous on some finite interval $[0, T]$ for any $T > 0$ and $f(t)$ is of exponential order α, then $\mathcal{L}\{f(t)\}$ exists for all $s > \alpha$

Existence of Laplace Transform

1) $f(t)$ is piecewise continuous (PWC) on $[0, T]$, and
2) $f(t)$ is of exponential order α

then

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

exists for $s > \alpha$

Aside: Triangle Inequality

$$\sum_{j=1}^N c_j \leq \left|\sum_{j=1}^N c_j\right| \leq \sum_{j=1}^N |c_j|$$

Corollary

$|F(s)| < \frac{L}{s}$

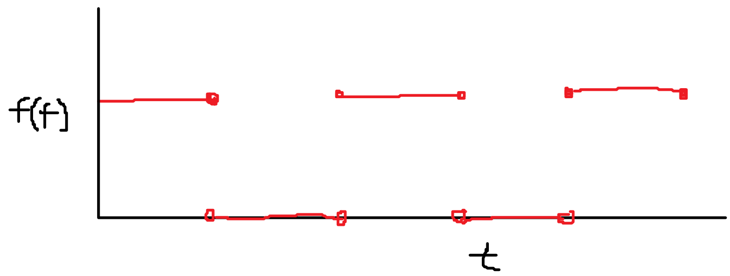
for some $L > 0$

$\Rightarrow \lim_{s \to \infty} F(s) = 0$

Consider the oscillator DE $y'' + \omega^2 y = f(t)$

$f(t) =$ periodic forcing

Example:



How do we solve this?

1) Undetermined coefficients
   for solutions over intervals when $f(t)$ is continuous
2) The Green's function
   $y_q(t) = \int_0^t G(t-s) \cdot f(s)ds$

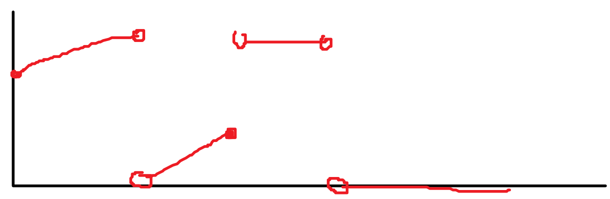
What about higher-order linear DE's with constant coefficients ?
These arise with coupled oscillators.

$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$

3) Can use undetermined coefficients
   $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$
   This has n complex roots

4) The Green's function, but what is $G(t-s)$ for the n-th order DE?

Example PWC Function with Jump Discontinuities



Exponential Order

Example

$f(t) = e^{7t} \cdot \cos(2t)$ is of exponential order 7

$\frac{|f(t)|}{e^{\alpha t}} = \frac{e^{7t}}{e^{\alpha t}} |\cos(2t)| \leq e^{7-\alpha} \leq 1$ for $\alpha \geq 7$

Example

$f(t) = t^7$ is of exponential order

$\frac{|f(t)|}{e^{\alpha t}} = t^7 e^{-\alpha t} \leq t_{\max}^7 e^{-\alpha t_{\max}}$

Example

$f(t) = e^{t^2}$ is not of exponential order

$\frac{|f(t)|}{e^{\alpha t}} = e^{t^2 - \alpha t}$

This is unbounded

Proof of Theorem

Sketch

$$\int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt$$

^ exists

Need to show

$$\lim_{A \to \infty} \int_T^A e^{-st} f(t)dt$$

converges.

$$\lim_{A \to \infty} \int_T^A e^{-st} f(t)dt \leq \lim_{A \to \infty} \int_T^A e^{-st} |f(t)|dt \leq \lim_{A \to \infty} \int_T^A e^{-(s-\alpha)t}dt = \lim_{A \to \infty} M \frac{e^{-(s-\alpha)T} - e^{-(s-\alpha)A}}{s - \alpha}$$

$s - \alpha > 0$ so $e^{-(s-\alpha)A} \to 0$

Examples of Laplace Transform

1) Unit Function

$f(t) = 1, \qquad \mathcal{L}(1) = \int_0^\infty e^{-st} dt = \lim_{A\to\infty} \frac{1}{s}(1 - e^{-sA}) = \frac{1}{s}, \qquad s > 0$

2) Heaviside unit step function

$$H(t) = U(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

a) Shifted $U(t-c) = U_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$

for $c > 0$,

$$\mathcal{L}\{U_c(t)\} = \int_0^\infty e^{-st} U(t-c) dt = \int_c^\infty e^{-st} dt = \lim_{A\to\infty} \int_c^A e^{-st} dt = \frac{e^{-sc}}{s}, \qquad s > 0$$

b) Indicator function for interval $t \in [c, d],\ d \geq c > 0$

$U_{cd}(t) = U(t-c) - U(t-d)$

$$\mathcal{L}\{U_{cd}(t)\} = \frac{e^{-sc} - e^{-sd}}{s}$$

3) $f(t) = e^{kt}, \qquad k = \text{const}$

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st} e^{kt} dt = \lim_{A\to\infty} \int_0^A e^{-(s-k)t} dt = \frac{1}{s-k}, \qquad s > k$$

$f(t) = \cos(\omega t), \sin(\omega t)$

$$\mathcal{L}\{\cos t\} = \int_0^\infty e^{-st} \cos(\omega t)\, dt$$

or write using previous result $f(t) = e^{i\omega t}$

$$L\{e^{i\omega t}\} = \frac{1}{s - i\omega}, \qquad s > Re(i\omega) = 0$$

4) $\mathcal{L}\{\cos t\} = \mathcal{L}\{Re(e^{i\omega t})\} = Re[\mathcal{L}\{e^{i\omega t}\}] = Re\left(\frac{1}{s - i\omega}\right) = Re\left(\frac{s + i\omega}{s^2 + \omega^2}\right) = \frac{s}{s^2 + \omega^2}$

5) $\mathcal{L}\{\sin t\} = Im\left[\frac{s + i\omega}{s^2 + \omega^2}\right] = \frac{\omega}{s^2 + \omega^2}$

6) $f(t) = t^n e^{kt}, \qquad n \in \mathbb{Z}^+, \qquad k = const$

$$\mathcal{L}\{t^n e^{kt}\} = \int_0^\infty t^n e^{-(s-k)t}\, dt = \text{magic} = \frac{n!}{s^{n+1}}, \qquad s > k$$

$$f(t) = \frac{\partial^n}{\partial k^n} e^{kt} = t^n e^{kt}$$

$$\mathcal{L}\{t^n e^{kt}\} = \frac{\partial^n}{\partial k^n} \mathcal{L}\{e^{kt}\} = \frac{\partial^n}{\partial k^n}\left(\frac{1}{s-k}\right)$$

Laplace & Inverse Laplace Transform

October-24-12 1:59 PM

Common Functions

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{kt}	$\frac{1}{s-k}, \quad s > k$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}, \quad s > 0$
$\cos(\omega t)$	$\frac{s}{x^2+\omega^2}, \quad s > 0$

Inverse Laplace Transform

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

Linear operator

$\mathcal{L}^{-1}\{C_1F(s) + C_2G(s)\} = C_1\mathcal{L}^{-1}\{F(s)\} + C_2\mathcal{L}^{-1}\{G(s)\}$

$\mathcal{L}^{-1}$ of Proper Rational Functions

$F(s) = \frac{P(s)}{Q(s)} = \frac{\text{polynomial}}{\text{polynomial}}, \deg P < \deg Q$

Use partial fraction decomposition

Properties (Theorems) of $\mathcal{L}$

1. **First Shift Theorem**

   If $F(s) = \mathcal{L}\{f(t)\}$ exists, then $\mathcal{L}\{e^{kt}f(t)\} = F(s-k)$

Results of Laplace Transform

$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}, \qquad s > Re(k)$

Let $k = i\omega,\ \omega \in \mathbb{R}$

$\mathcal{L}\{\cos(\omega t)\} = Re[\mathcal{L}\{e^{i\omega t}\}] = Re\left(\frac{1}{s-i\omega}\right) = \frac{s}{s^2+\omega^2}$

$\mathcal{L}\{\sin(\omega t)\} = Im[\mathcal{L}\{e^{i\omega t}\}] = Im\left(\frac{1}{s-i\omega}\right) = \frac{\omega}{s^2+\omega^2}$

Let k =real-valued parameter

Since $\frac{\partial^n}{\partial k^n}e^{kt} = t^n e^{kt}$

$\mathcal{L}\{t^n e^{kt}\} = \frac{\partial^n}{\partial k^n \mathcal{L}\{e^{kt}\}} = \frac{\partial^n}{\partial k^n}\left(\frac{1}{s-k}\right) = \frac{n!}{(s-k)^{n+1}}, \qquad s > k$

Setting $k = 0$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad s > 0$

Example of Inverse Laplace Transform

Find $\mathcal{L}^{-1}\{F(s)\}$ for

$F(s) = \frac{14+7s-3s^2}{s^2(s+2)} = \frac{7}{s^2} - \frac{3}{s+2}$

$\mathcal{L}^{-1}\{F(s)\} = 7\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = 7t - 3e^{-2t}$

$F(s) = \frac{1}{(s-1)(s^2+1)} = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{s}{s^2+1} - \frac{1}{2}\cdot\frac{1}{s^2+1}$

$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$

Example

Show that

$f(t) = \int_0^\infty \frac{\sin(tx)}{x}dx = \frac{\pi}{2}, \qquad \forall t \neq 0$

$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\left[\int_0^\infty \frac{\sin(tx)}{x}dx\right]dt = \int_0^\infty \frac{1}{x}\left[\int_0^\infty e^{-st}\sin(tx)\,dt\right]dx$

$= \int_0^\infty \frac{1}{x}\cdot\frac{x}{s^2+x^2}dx, \qquad s > 0$

$= \int_0^\infty \frac{dx}{x^2+s^2} = \frac{1}{s}\arctan(\xi)\Big|_{\xi=0}^{\xi=\infty} = \frac{\pi}{2}\cdot\frac{1}{s}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{\pi}{2}\cdot\frac{1}{s}\right\} = \frac{\pi}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = \frac{\pi}{2}$

Proof of First Shift Theorem

$\mathcal{L}\{e^{kt}f(t)\} = \int_0^\infty e^{-st}e^{kt}f(t)dt = \int_0^\infty e^{-(s-k)t}f(t)dt$

Example

$\mathcal{L}\{e^{-2t}\sin(3t)\} = \mathcal{L}\{\sin(3t)\}_{\{s\to s+2\}} = \left(\frac{3}{s^2+3^2}\right)\Big|_{\{s\to s+2\}} = \frac{3}{(s+2)^2+9} = \frac{3}{s^2+4s+13}$

Example

$\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-6s+25}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s-3)+9}{(s-3)^2+16}\right\} = e^{3t}\mathcal{L}^{-1}\left\{\frac{2s+9}{s^2+4^2}\right\}$

$= e^{3t}\left(2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} - \frac{9}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\}\right) = e^{3t}\left[2\cos(4t) + \frac{9}{4}\sin(4t)\right]$

Laplace Transform and DEs

October-26-12 1:33 PM

★ Theorem (Laplace Transform of Derivative)

If $f(t)$ is continuous and $f'(t)$ is PWC on $0 \le t \le A$ (any A) and $f(t)$ and $f'(t)$ are of exponential order α, then

$\mathcal{L}\{f'(t)\} = s \cdot \mathcal{L}\{f(t)\} - f(0), \qquad s > \alpha$

Generalization

$f(t), f'(t), \dots, f^{(n-1)}(t)$ are continuous and $f^{(n)}(t)$ is PWC and are all of exponential order α then

$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$

Solving Differential Equations with Laplace Transform

DE in t

↓

Algebraic equation in s

↓

Solve algebraic equation in s

↓

Solution of DE in t

★ Solving Secord Order

$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0)$

Theorem (Derivative of Laplace Transform)

If $f(t)$ is PWC on $0 \le t \le A$, and is of exponential order α then

$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(0)$

where $F(s) = \mathcal{L}\{f(t)\}$

Non-Constant Coefficients

Can handle DEs with non-constant coefficients

$\mathcal{L}\{y'' - ty] = 0$, **Airy function**

$s^2 Y(s) - sy(0) - y'(0) + Y'(s) = 0$

Proof of Theorem

$f'(t)$ has finite jumps at $t_1, t_2, \dots, t_n \in (0, A)$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = \lim_{a\to\infty} \int_0^A e^{-st} f'(t)dt$$

$$= \int_0^{t_1} \dots + \int_{t_1}^{t_2} \dots + \dots + \int_{t_n}^{A} \dots$$

Integrating by parts in each

$$= e^{-st} f(t) \Big|_{t=0}^{t=t_1} + e^{-st} f(t) \Big|_{t=t_1}^{t=t_2} + \dots + e^{-st} f(t) \Big|_{t=t_n}^{t=A}$$

$$- \int_0^{t_1} f(t) \frac{\partial e^{-st}}{\partial t} dt - \int_{t_1}^{t_2} f(t) \frac{\partial e^{-st}}{\partial t} dt - \dots - \int_{t_n}^{A} f(t) \frac{\partial e^{-st}}{\partial t}$$

$\frac{\partial e^{-st}}{\partial t} = -se^{-st}$ so

$$= -f(0) + e^{-sA} f(A) + s \int_0^A e^{-st} f(t)dt$$

$$\mathcal{L}\{f'(t)\} = \lim_{A\to\infty} \left[-f(0) + e^{-sA} f(A) + s \int_0^\infty e^{-st} f(t)dt \right] = s\mathcal{L}\{f(t)\} - f(0), \qquad s > \alpha$$

Example

Use $\mathcal{L}$ to solve the IVP

$y'' + 2y' + y = 4e^t, \; y(0) = 1, y'(0) = 2$

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{4e^t\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = \frac{4}{s-1}$$

$$Y(s)(s^2 + 2s + 1) - s - 4 = \frac{4}{s-1}$$

$$Y(s) = \frac{s+4}{s^2 + 2s + 1} + \frac{4}{(s-1)(s^2 + 2s + 1)} = \frac{s^2 + 3s}{(s-1)(s+1)^2}$$

Partial Fraction Decomposition

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2} = \frac{(A+B)s^2 + (2A+C)s + A - B - C}{(s-1)(s+1)^2}$$

$A + B = 1, \quad 2A + C = 3, \quad A - B - C = 0$

$A = 1, \quad B = 0, \quad C = 1$

$$Y(s) = \frac{1}{s-1} + \frac{1}{(s+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^t + t \cdot e^{-t}$$

Proof of Theorem

$$F^{(s)}(s) = \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial^n e^{-st}}{\partial s^n} f(t)dt = \int_0^\infty (-t)^n e^{-st} f(t)dt$$

$$= (-1)^n \int_0^\infty t^n f(t)dt = (-1)^n \mathcal{L}\{t^n f(t)\}$$

$\mathcal{L}$ of an Integral

October-29-12 1:27 PM

$\mathcal{L}$ of an Integral

If $f(t)$ is PWC and of exponential order α then

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

Second Shift Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ exists for some $s > \alpha$, and c is a positive constant, then

$\mathcal{L}\{U_c(t)\cdot f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = U(t-c)f(t-c)$

Recall $U_c(t) = U(t-c)$ is the shifted Heaviside function

Remark

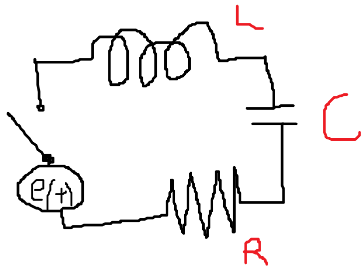
$\mathcal{L}\{U(t-c)g(t)\} = e^{-cs}\mathcal{L}\{g(t+c)\}$

Proof of $\mathcal{L}$ of an Integral

Let $g(t) = \int_0^t f(\tau)d\tau$, $\quad g'(t) = f(t), \quad g(0) = 0$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}$ ■

Example: RLC Current



L = inductance
C = capacitance
R = resistance

Solve for $i(t)$ with IC's $i(0) = i_0,\ q(0) = q_0$

Kirchhoff's Voltage Law

$L\frac{di}{dt} + Ri + \frac{q}{C} = e(t)\ \ (*)$

$i(t) = \frac{dq}{dt}$

Write $q(t) = q_0 + \int_0^t i(\tau)d\tau$

Take $\mathcal{L}$ of eq. (*)

$L\mathcal{L}\{i'(t)\} + +R\mathcal{L}\{i(t)\} + \frac{1}{C}\mathcal{L}\{q(t)\} = \mathcal{L}\{e(t)\}$

$L[s\cdot I(s) - i] + R\cdot I(s) + \frac{1}{C}\left[\mathcal{L}\{q_0\} + \mathcal{L}\left\{\int_0^t i(\tau)d\tau\right\}\right] = E(s)$

$LsI(s) - Li_0 + RI(s) + \frac{1}{C}\cdot\frac{q_0}{s} + \frac{1}{C}\cdot\frac{I(s)}{s} = E(s)$

Solve for $I(s)$

Proof of Second Shift Theorem

$\mathcal{L}\{U(t-c)f(t-c)\} = \int_0^\infty U(t-c)f(t-c)dt = \int_c^\infty e^{-st}f(t-c)dt\ , \qquad let\ \tau = t-c$

$= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau = e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)d\tau = e^{-cs}F(s)$

Example

$\mathcal{L}\{t^2U(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\} = e^{-s}(\mathcal{L}\{t^2\} + 2\mathcal{L}\{t\} + \mathcal{L}\{t\}) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$

$= \frac{e^{-s}}{s^3}(s^2 + 2s + 2)$

Example

$\mathcal{L}\left\{\frac{e^{-2s}}{s^2}\right\} = U(t-2)\mathcal{L}\left\{\frac{1}{s^2}\right\}_{t\to t-2} = U(t-2)(t)\Big|_{t\to t-2} = (t-2)\cdot U(t-2)$

Example

Second shift theorem:

$\mathcal{L}^{-1}\left\{\frac{5e^{-2s}(s-4)}{s^2 - 8s + 25}\right\} = U(t-2)\mathcal{L}^{-1}\left\{\frac{5(s-4)}{(s-4)^2 + 9}\right\}_{t\to t-2}$

With first shift theorem:

$= U(t-2)\cdot\left(e^{4t}\mathcal{L}\left\{\frac{5s}{s^2 + 3^2}\right\}\right)_{t\to t-2} = U(t-2)\cdot(e^{4t}5\cos(3t))_{t\to t-2}$

$= U(t-2)5e^{4(t-2)}\cdot\cos\big(3(t-2)\big)$

Example

Solve IVP

$y'' + y = f(t)$ where

$$f(t) = \begin{cases} t, & 0 \le t < \pi \\ 2\pi - t, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}, \qquad y(0) = 0, \qquad y'(0) = 0$$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$

$s^2Y(s) - sy(0) - y'(0) + Y(s) = F(s)$

$Y(s) = \frac{1}{s^2+1}F(s)$

$f(t) = t[U(t) - U(t-\pi)] + (2\pi - t)[U(t-\pi) - U(t-2\pi)]$

$f(t) = t\cdot U(t) - 2(t-\pi)\cdot U(t-\pi) + (t-2\pi)U(t-2\pi)$

$F(s) = \mathcal{L}\{t\} - 2e^{-\pi s}\mathcal{L}\{t\} + e^{-2\pi s}\mathcal{L}\{t\} = \frac{1 - 2e^{-\pi s} + e^{-2\pi s}}{s^2} = \frac{(1 - e^{-\pi s})^2}{s^2}$

$$Y(s) = \frac{1 - 2e^{-\pi s} + e^{-2\pi s}}{s^2(s^2+1)}$$

$y(t)$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} - 2U(t-\pi)\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}_{t\to t-\pi} + U(t-2\pi)\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}_{t\to t-2\pi}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin t$$

$$y(t) = t \cdot \sin t - 2U(t-\pi)\cdot[t - \pi - \sin(t-\pi)] + U(t-2\pi)[t - 2\pi - \sin(t-2\pi)]$$

$$y(t) = \begin{cases} t - \sin t, & 0 < t < \pi \\ -t + 2\pi + 3\sin t, & \pi < t < 2\pi \\ -4\sin t, & t > 2\pi \end{cases}$$

Periodic Functions

October-31-12 1:34 PM

Periodic Function

A function is periodic with period $T > 0$ iff $f(t+T) = f(t)$

Theorem

If $f(t)$ is periodic with period T and is piecewise continuous on $[0, T]$, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

Proof of Theorem

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt$$

Let $\tau = t - T$

$$F(s) = \int_0^T e^{-st} f(t)dt + \int_0^\infty e^{-s(\tau+T)} f(\tau+T)d\tau = \int_0^T e^{-st} f(t)dt + \int_0^\infty e^{-s(\tau+T)} f(\tau)\, d\tau$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT} \int_0^\infty e^{-s\tau} f(\tau)d\tau = \int_0^T e^{-st} f(t)dt \; + e^{-sT} F(s)$$

Alternate proof:

Define window function

$$f_T(t) = \begin{cases} f(t), & 0 \le t \le T \\ 0, & t > T \end{cases}$$

$$\int_0^T e^{-sT} f(t)dt = \int_0^\infty e^{-st} f_T(t)dt = F_T(s)$$

Represent entire $f(t)$ as series

$$f(t) = \sum_{k=0}^\infty f_T(t-kT)U(t-kT)$$

$$\mathcal{L}\{f(t)\} = \lim_{N\to\infty} \sum_{k=0}^{N-1} \mathcal{L}\{f_T(t-kT)U(t-kT)\} = \lim_{N\to\infty} \sum_{k=0}^{N-1} F_T(s)e^{-kTs} = F_T(s) \lim_{N\to\infty} \frac{1-e^{-Nst}}{1-e^{-sT}}$$

$$= \frac{F_T(s)}{1-e^{-sT}}$$

Example

Square-shaped "sine" function of period T. Let $c = \frac{T}{2}$

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{T}{2} = c \\ -1, & \frac{T}{2} \le t < T = 2c \end{cases}$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{F_T(s)}{1-e^{-s2c}}$$

$$F_T(s) = \int_0^{\frac{T}{2}=c} e^{-st}dt - \int_{\frac{T}{2}=c}^{T=2c} e^{-st}dt = \frac{1-e^{-cs}}{s} - \frac{e^{-cs}-e^{-2cs}}{s} = \frac{(1-e^{-cs})^2}{s}$$

$$F(s) = \frac{1}{s} \cdot \frac{(1-e^{-cs})^2}{(1-e^{-cs})(1+e^{-cs})} = \frac{1}{s} \cdot \frac{1-e^{-cs}}{1+e^{-cs}}$$

Now find inverse

$\mathcal{L}^{-1}(F(s))$

where $F(s)$ is in the above form

$$F(s) = \frac{1}{s} \cdot \frac{1-e^{-cs}}{1+e^{-cs}} = \frac{1-e^{-cs}}{s} \sum_{k=0}^\infty (-1)^k e^{-kcs} = \frac{1}{s} \sum_{k=0}^\infty (-1)^k \left[e^{-kcs} - e^{-(k+1)cs}\right]$$

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=0}^\infty (-1)^k \mathcal{L}^{-1}\left\{\frac{e^{-kcs}-e^{-(k+1)cs}}{s}\right\} = \sum_{k=0}^\infty (-1)^k \big(U(t-kc) - U(t-(k+1)c)\big)$$

Alternately

$$F(s) = \frac{1}{s}\left[1 + \sum_{k=1}^\infty (-1)^k e^{-kcs} + \sum_{k=0}^\infty (-1)^{k+1} e^{-(k+1)cs}\right] = \frac{1}{s}\left(1 + \sum_{k=1}^\infty (-1)^k e^{-kcs}\right)$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 1 + 2\sum_{k=1}^\infty (-1)^k U(t-ck)$$

Example

Solve the IVP

$$\frac{d^2y}{dt^2} + y = f(t), \qquad y(0) = 0, \qquad y'(0) = 0$$

where $f(t) = 1 + 2\sum_{k=1}^\infty (-1)^k U(t-kc)$, $\qquad c = \frac{T}{2}$

Square wave forcing with period $T = 2c$ and frequency $\Omega = \frac{2\pi}{T} = \frac{\pi}{c}$

Recall: The natural frequency of $\frac{d^2y}{dt^2} + y$ is 1 with period 2π

Resonance occurs when $\Omega \to 1$

$\mathcal{L}$ on DE:

$s^2Y(s) + Y(s) = F(s)$

$$Y(s) = \frac{F(s)}{s^2+1}, \text{ where } F(s) = \frac{1}{s}\left[1 + 2\sum_{k=1}^\infty (-1)^k e^{-kcs}\right]$$

$$Y(s) = \frac{1}{s(s^2+1)}\left[1 + 2\sum_{k=1}^\infty (-1)^k e^{-kcs}\right]$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} + 2\sum_{k=1}^\infty (-1)^k \mathcal{L}^{-1}\left\{\frac{e^{-kcs}}{s^2+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-kcs}}{s(s^2+1)}\right\} = U(t-c)[1-\cos(t-kc)]$$

$$y(t) = 1 - \cos t + 2\sum_{k=1}^{\infty}(-1)^k U(t-kcs)\,[1-\cos(t-kc)]$$

What happens when $\Omega \to 1, or\ c = \pi$

Recall Midterm Question #4

$y'' + y = \sin(\Omega t)$

$$y(t) = \frac{\sin(\Omega t) - \Omega\sin(t)}{1-\Omega^2}$$

L'Hôpital's rule as $\Omega \to 1$

$$\frac{1}{2}\sin(t) - \frac{1}{2}\mathrm{t}\cdot\cos t$$

$$y(t) = 1 + 2\sum_{k=1}^{\infty}(-1)^k U(t-ck) - \cos t - 2\sum_{k=1}^{\infty}(-1)^k U(t-ck)\cos(t-ck)$$

$$= f(t) - \cos(t)\left(1 + 2\sum_{k=1}^{\infty}U(t-k\pi)\right)$$

What happens when $c = (2l+1)\pi,\ \ l = 0,1,2,3,\ldots$

$\cos(t - k(2l+1)\pi) = (-1)^{k(2l+1)}\cos(t)$

$$y(t) = f(t) - \cos(t)\left(1 + 2\sum_{k-1}^{\infty}(-1)^{2k(l+1)}U(t-k(2l+1)\pi)\right)$$

Fourier Series

$$f(t) = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{1}{2l+1}\cdot\sin\big((2l+1)\Omega\mathrm{t}\big)$$

Convolution

November-02-12 1:58 PM

Convolution Function

If $f(t)$ and $g(t)$ are PWC on $t \in [0,\infty)$ then their convolution is function $h(t)$ define on $t \in [0,\infty)$ by

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = (f*g)(t)$$

Notation and Properties

1) Commutativity: $f*g = g*f$
2) Distributivity: $f*(g_1+g_2) = f*g_1 + f*g_2$
3) Associativity: $(f*g)*h = f*(g*h)$

Convolution Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ and they exist for $s > \alpha$, then

$\mathcal{L}\{(f*g)(t)\} = F(s)G(s), \quad for\ \ s > \alpha$

$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$

Using Convolution Theorem to solve n^{th} order linear DE with constant coefficients

$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$

with IC's

using $\mathcal{L}$:

$$Y(s) = \frac{\text{Initial Conditions}}{a_n s^n + a_{n-1}s^{n+1} + \cdots + a_0} + \frac{F(s)}{a_n s^n + \cdots + a_0}$$

Assuming quiescent state, all IC's = 0

$Y_q(s) \equiv Y(s) = G(s)F(s)$

where

$F(s) = \mathcal{L}\{f(t)\}$

$$G(s) = \frac{1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \text{Transfer function}$$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \text{Green's function}$

$$y_q(t) = y(t) = \mathcal{L}^{-1}\{G(s)F(s)\} = \int_0^t g(t-\tau)f(\tau)d\tau$$

Pulse (Impulse Response)

What happens when $f(t)$ is a narrow pulse at say, $c > 0$

$$y(t) \approx g(t-c)\int_0^t f(\tau)dt \approx g(t-c)Area(\text{pulse})$$

Consider $c \to 0^+$, then $y(t) \approx g(t)$

Another name for Green's function is the **Impulse Response**

Alternately, if

$y(t) = \mathcal{L}^{-1}\{G(s)F(s)\}$ then

$y(s) \sim g(t) = \mathcal{L}^{-1}\{G(s)\}$ when $F(s) = 1$

Proof of Commutativity

Let $u = t-\tau, \quad du = -d\tau$

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)dt = -\int_t^0 f(u)g(t-u)\,du = \int_0^t g(t-u)f(u)du \ = (g*f)(t)$$

Proof of Convolution Theorem

$$\mathcal{L}\{(f*g)(t)\} = \int_0^\infty e^{-st}\int_0^t f(t-\tau)g(\tau)d\tau\,dt = \int_0^\infty e^{-st}\int_0^\infty U(t-\tau)f(t-\tau)g(\tau)d\tau\,dt$$

$$= \int_0^\infty \left(\int_0^\infty e^{-st}U(t-\tau)f(t-\tau)\,dt\right)g(\tau)d\tau = \int_0^\infty \mathcal{L}\{U(t-\tau)f(t-\tau\}g(t-\tau)d\tau$$

$$= F(s)\int_0^\infty e^{-s\tau}g(\tau)d\tau = F(s)G(s)$$

Solving IVP

$ay'' + by' + cy = f(t), \qquad y(0) = y_0, y'(0) = y_1, \qquad a,b,c$ const

$a(s^2Y(s) - sy_0 - y_1) + b(sY(s) - y_0) + cY(s) = F(s)$

$$Y(s) = \frac{(as+b)y_0 + ay_1}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}$$

Define $G(s) = \frac{1}{as^2+bs+c}$ with $g(t) = \mathcal{L}^{-1}\{G(s)\} = \text{Green's function}$, then

$$y(s) = \mathcal{L}^{-1}\left\{\frac{(as+b)y_0 + ay_1}{as^2+bs+c}\right\} + \mathcal{L}^{-1}\{G(s)F(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(as+b)y_0 + ay_1}{as^2+bs+c}\right\} + \int_0^\infty g(t-\tau)f(\tau)d\tau$$

$$G(s) = \frac{1}{as^2+bs+c} \quad \text{Is the transfer function}$$

Dirac Delta Function

November-05-12 1:48 PM

Dirac's Delta "function" (distribution)

Consider rectangular pulse at some $t = c > 0$

$$\delta_\epsilon(t-c) = \begin{cases} \frac{1}{\epsilon}, & c - \frac{\epsilon}{2} < t < c + \frac{\epsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Area} = \int_{-\infty}^{\infty} \delta_\epsilon(t-c)dt = \frac{1}{\epsilon}\int_{c-\frac{\epsilon}{2}}^{c+\frac{\epsilon}{2}} dt = 1$$

Sampling Function

Let $f(t)$ be continuous on interval containing c. Define the **sampling function**

$$S_\epsilon[f(t), c] = \int_{-\infty}^{\infty} f(t)\delta_\epsilon(t-c)dt$$

$$S[f(t),c] = \lim_{\epsilon\to 0^+} S_\epsilon[f(t),c] = \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_{c-\frac{\epsilon}{2}}^{c+\frac{\epsilon}{2}} f(t)dt = \lim_{\epsilon\to 0^+}\frac{f(\theta)}{\epsilon}\int_{c-\frac{\epsilon}{2}}^{c+\frac{\epsilon}{2}} dt = f(c)$$

$c - \frac{\epsilon}{2} < \theta < c + \frac{\epsilon}{2}$ is given by the Mean Value Theorem

Using $\delta_\epsilon(t-c)$ in $S_\epsilon[f(t),c]$ gives representation of Dirac's function, that is its "Sifting" property.

$$\int_{-\infty}^{\infty} f(t)\delta(t-c)\,dt = \int_{c^-}^{c^+} f(t)\delta(t-c)dt = f(c)\int_{c^-}^{c^+}\delta(t-c)dt = f(c)$$

Definition of $\delta(t)$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)\ \forall \text{ functions } f$$

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases}$$

Properties of $\delta(t)$

1) Even (symmetric)
$$\delta(-t) = \delta(t)$$
2) Scaling
$$\delta(Kt) = \frac{1}{|K|}\delta(t)$$
3) $\int_{-\infty}^{\infty}\delta(t)dt = 1$
4) Sifting
$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$
5) $\int_{-\infty}^{\infty} f(t)\delta'(t)dt = -f'(0)$

Laplace Transform of Dirac Function

Need to modify definition of $\mathcal{L}$

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st}f(t)dt$$

$$\mathcal{L}^{-1}\{f'(t)\} = sF(s) - f(0^-)$$

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} e^{-st}\delta(t)dt = e^{-s0}\int_{0^-}^{\infty}\delta(t)dt = 1$$

Convolution

$$\int_{0^-}^{t} g(t-\tau)f(\tau)d\tau$$

$$y(t) = \int_{0^-}^{\infty} g(t-\tau)\delta(\tau)d\tau = g(t), \quad \text{for } t > 0$$

Note: Relation of $\delta(t)$ to $U(t)$

$$\int_{-\infty}^{t}\delta(\tau)d\tau = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Example: Newton's law with pulse force

$F(t) = f_0 \cdot \delta(t)$

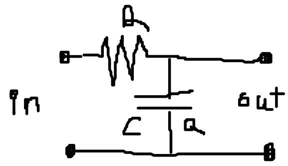
$m\frac{dv}{dt} = F(t) \Rightarrow$ Integrate from $\int_{0^-}^{0^+} \square$

$$m\int_{0^-}^{0^+}\frac{dv}{dt}dt = \int_{0^-}^{0^+} F(t)dt$$

$$mv(0^+) - mv(0^-) = f_0$$

Instantaneous change of momentum

Transfer Function and Impulse Response for RC-filter



Input voltage $e(t)$
Output voltage $v(t)$
$q(t) = Cv(t)$
KVL: $e(t) = Ri(t) + v(t)$

$$RC\frac{dv}{dt} + v = e(t)$$

For IC: $v(0^-) = 0$
use $\mathcal{L}$:

$$RCsV(s) + V(s) = E(s)$$

$$G(s) = \frac{V(s)}{E(s)} = \frac{1}{RCs+1}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{RC}e^{-\frac{t}{RC}}, \quad t > 0$$

Example

Drug absorption in body
A pill is taken at time $t_0 = 0, t_1, t_2, \ldots$ and amount of drug is given by

$$\frac{dy}{dt} = -ry + f(t)$$

r is release rate, f(t) is drug intake

$$f(t) = \delta(t) + \delta(t-t_1) + \delta(t-t_2) + \cdots = \sum_{j=0}^{N}\delta(t-t_j)$$

IC $y(0^-) = 0$

Using $\mathcal{L}$

$$sY(s) + rY(s) = F(s)$$

$$Y(s) = \frac{F(s)}{s+r}$$

$$F(s) = \mathcal{L}\{f(t)\} = \sum_{j=0}^{N}\mathcal{L}\{\delta(t-t_j)\} = \sum_{j=0}^{N} e^{-st}\delta(t-t_j)dt = \sum_{j=0}^{N} e^{-st_j}$$

Transfer function $G(s) = \frac{1}{s+r}$

$$y(t) = \mathcal{L}^{-1}\{G(s)F(s)\} = \sum_{j=0}^{N}\mathcal{L}^{-1}\left\{\frac{e^{-st_j}}{s+r}\right\} = \sum_{j=0}^{N} e^{-r(t-t_j)}U(t-t_j)$$

Using Convolution Theorem

$$y(t) = \int_{0^-}^{t} g(t-\tau)f(\tau)d\tau = \sum_{j=0}^{N}\int_{0^-}^{t} e^{-r(t-\tau)}\delta(\tau-t_j)d\tau$$

$$= \sum_{j=0}^{N} e^{-r(t-t_j)}\int_{0^-}^{t}\delta(\tau-t_j)d\tau = \sum_{j=0}^{N} e^{-r(t-t_j)}U(t-t_j)$$

After the N^{th} interval,

$$y(t_N) = e^{-rT}\frac{1-e^{-NrT}}{1-e^{-rT}}$$

Steady state:

$$\frac{e^{-rT}}{1-e^{-rT}}$$

Stability of a System

November-07-12 1:59 PM

Zeroes and Poles

In general,

$$G(s) = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})} = \frac{P(s)}{Q(s)}, \qquad n' = \deg(P) < \deg(Q) = n$$

$Q(s)$ is the characteristic equation, polynomial of degree n

$$G(s) = K \cdot \frac{(s-Z_1)(s-Z_2)\dots(s-Z_n)}{(s-p_1)^{m_1}(s-p_2)^{m_2}\dots(s-p_k)_k^{m_k}}$$

Aside: if $p_j^{\pm} = \sigma_j \pm i\omega_j$

$$(s-p_j^+)(s+p_j^-) = (s-\sigma_j - i\omega_j)(s-\sigma_j + i\omega_j) = (s-\sigma_j)^2 + \omega_j^2$$

All poles of $G(s)$ are either real or complex-conjugate pairs.

$m_j =$ multiplicity of j^{th} pole p_j so that $m_1 + m_2 + \dots + m_k = n$

Roots of $P(s)$ are zeroes of $G(s)$
Roots of $Q(s)$ are poles of $G(s)$

By partial fraction decomposition

$$G(s) = \sum_{j=1}^{k}\sum_{l_j=1}^{m_k} \frac{A_j^{(l_j)}}{(s-p_j)^{l_j}}$$

Green's function = impulse response
He have terms of the form

$$\mathcal{L}^{-1}\left\{\frac{A_j^{(l_j)}}{(s-p_j)^{l_j}}\right\} = A_j^{(t_j)} \frac{t^{l_j - 1}}{(l_j - 1)!} e^{t\sigma_j} e^{it\omega_j}$$

Stability

A system is asymptotically stable iff
$\lim_{t\to\infty} g(t) = 0$

Theorem

A system is asymptotically stable iff all the poles of the transfer function $G(s)$ are located in the left half of the complex s-plane. $(Re(s) < 0)$

If any poles are on the imaginary axis then the system will not be asymptotically stable.

BIBO Stability

Bounded Input Bounded Output

If the forcing $|f(t)| < M$ then the response is also $|y(t)| < M_e$

Proof of Theorem (using BIBO Stability)

$$y(t) = \int_0^t g(t-\tau) f(\tau) d\tau$$

$|y(t)| = \left|\int_0^t g(t-\tau)f(\tau)d\tau\right| \le \int_0^t |g(t-\tau)||f(\tau)|d\tau \le M_1 \int_0^t |g(t-\tau)|d\tau = M_1 \int_0^t |g(u)|du < M_2$ iff $\sigma_j < 0$ for all j

Harmonic Forcing

$$f(t) = F_0 \cos(\Omega t) = F_0 Re(e^{-i\Omega t}) \Rightarrow$$

$$F(s) = \mathcal{L}\{f(t)\} = F_0 \frac{s}{s^2+\Omega^2} = F_0 \frac{s}{(s+i\Omega)(s-i\Omega)}$$

Laplace transform of response is

$$Y(s) = G(s)F(s) = F_0 \frac{s}{(s+i\Omega)(s-i\Omega)} G(s)$$

$$G(s) = \sum_j \sum_{l_j} \frac{A_j^{(l_j)}}{(s-p_j)^{l_j}}$$

$$Y(s) = \frac{C_+}{s-i\Omega} + \frac{C_-}{s+i\Omega} + \sum_j \sum_{l_j} \frac{B_j^{(l_j)}}{(s-p_j)^{l_j}}$$

Find $C_\pm$ using "Cover-up Rule"

$$C_\pm = \lim_{s\to i\Omega}(s_\mp - i\Omega)Y(s) = F_0 \pm \frac{i\Omega}{\pm 2i\Omega} G(\pm i\Omega) = \frac{F_0}{2} G(\pm i\Omega)$$

$$Y(s) = \frac{F_0}{2}\frac{G(i\Omega)}{s-i\Omega} + \frac{F_0}{2}\frac{G(-i\Omega)}{s+i\Omega} + \sum_j \sum_{l_j} \frac{B_j^{(l_j)}}{(s-p_j)^{l_j}}$$

$$\frac{F_0}{2}\frac{G(i\Omega)}{s-i\Omega} + \frac{F_0}{2}\frac{G(-i\Omega)}{s+i\Omega} = F_0 Re\left[\frac{G(i\Omega)}{s-i\Omega}\right] = Y_{ss}(s) = \text{Steady state}$$

$$\sum_j \sum_{l_j} \frac{B_j^{(l_j)}}{(s-p_j)^{l_j}} = Y_{ts}(s) = \text{Transient State}$$

$$y(t) = \mathcal{L}^{-1}\{Y_{ss}(s)\} + \mathcal{L}^{-1}\{Y_{ts}(s)\} = F_0 Re\left[G(i\Omega)\cdot e^{i\Omega t}\right] + y_{tr}(t)$$

$$y_{ss}(t) = F_0 Re\left[|G(i\Omega)| \cdot e^{i\, arg[G(i\Omega)]}\, e^{i\Omega t}\right] = F_0 |G(i\Omega)| \cos[\Omega t + \arg[G(i\Omega)]]$$

$$= F_0 A(\Omega)\cos[\Omega t - \phi(\Omega)]$$

$$\mathcal{L}^{-1}\left\{\frac{G_j^{l_j}}{(s-p_j)^{l_j}}\right\} =$$

Systems of DEs

November-09-12 2:06 PM

Methods of Solving Linear System of DEs

1. Using $\mathcal{L}$
2. Deduce 2^{nd} order decoupled DEs for $m_1(t)$ and $m_2(t)$ (by eliminating state variables)
3. Use matrix or vector DEs

Vector / Matrix Calculus

Just integrate / differentiate each element individually.

Equilibrium Solutions

If just trying to find the steady state solutions you can set the derivatives to 0 and solve the system.

Mixing Problem

Tank contains chemical.
Contents are well-mixed and concentration is uniform in V.
Inflow is constant rate f_{in} at concentration c_{in}

$[f_{in}] = L^3T^{-1}$
$[c_n] = ML^{-3}$
$[m] = M$

Mass of chemical at time t determined by mass-balance

$$\frac{dm}{dt} = f_{in}c_{in} - f_{out}c_{out}$$

Find $c_{out(t)} = \frac{m(t)}{V(t)}$

Volume of fluid in tank is

$$\frac{dV}{dt} = f_{in} - f_{out}$$

Consider $f_{in} = f_{out} = f$ so that $V = const$

Coupled Tanks

Two tanks have flow into each other
$f_1: \to V_1$
$f_2: V_2 \to V_1$
$(f_1 + f_2): V_1 \to V_2$
$f_1: V_2 \to$

State variables:
amount of chemical in tanks 1 and 2
$m_1(t)$ and $m_2(t)$

$$\frac{dm_1}{dt} = f_1c_{in} + f_2\frac{m_2}{V_2} - (f_1+f_2)\frac{m_1}{V_1}$$
$$\frac{dm_2}{dt} = (f_1+f_2)\frac{m_1}{V_1} - (f_1+f_2)\frac{m_2}{V_2} \qquad (*)$$

ICs $m_1(0) = 0,\ \ m_2(0) = 0$

"forcing" term is f_1c_{in}

Matrix/Vector Method (for Example)

Let $\vec{x}(t) = \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} m_1(0) \\ m_2(0) \end{bmatrix}$

Forcing

$$\vec{f} = \begin{bmatrix} f_1c_{in} \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{f_1+f_2}{V_1} & \frac{f_2}{V_2} \\ \frac{f_1+f_2}{V_2} & -\frac{f_1+f_2}{V_2} \end{bmatrix}$$

Now (*) may be rewritten as

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + \vec{f}$$

Using Laplace Transform term-by-term

To simplify things, let $V_1 = V_2 = 1,\ f_1 = 3,\ f_2 = 1$

$$sM_1(s) - M_1(0) = \frac{3c_{\text{in}}}{s} + M_2(s) - 4M_1(s)$$
$$sM_2(s) - m_2(0) = 4M_1(s) - 4M_2(s)$$
$$(s+4)M_1 - M_2 = \frac{3c_{\text{in}}}{s}$$
$$-4M_1 + (s+4)M_2 = 0$$

By Crammer's Rule

$$M_1 = \frac{\begin{vmatrix} \frac{3c_{\text{in}}}{s} & -1 \\ 0 & s+4 \end{vmatrix}}{\begin{vmatrix} s+4 & -1 \\ -4 & s+4 \end{vmatrix}} = \frac{(s+4)\frac{3c_{\text{in}}}{s}}{(s+4)^2 - 4} = \frac{s+4}{(s+2)(s+6)} \cdot \frac{3c_{\text{in}}}{s}$$

$\frac{3c_{\text{in}}}{s}$ is the Laplace of the forcing

$\frac{s+4}{(s+2)(s+6)}$ is the Transfer function $G_1(s)$

$$\frac{(s+4)3c_{\text{in}}}{s(s+2)(s+6)} \Rightarrow m_1(t) = \mathcal{L}^{-1}\{M_1(s)\} = \cdots$$

$$M_2(s) = \frac{\begin{vmatrix} s+4 & \frac{3c_{\text{in}}}{s} \\ -4 & 0 \end{vmatrix}}{\begin{vmatrix} s+4 & -1 \\ -4 & s+4 \end{vmatrix}} = \frac{4}{(s+2)(s+6)} \cdot \frac{3c_{\text{in}}}{s} = \frac{12c_{\text{in}}}{s(s+2)(s+6)} \quad (**)$$

$$G_2(s) = \frac{4}{(s+2)(s+6)}$$

$$m_2(t) = \mathcal{L}^{-1}\{M_2(s)\} = 12c_{\text{in}}\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)(s+6)}\right\}$$

$$\frac{1}{s(s+2)(s+6)} = \frac{1}{12} \cdot \frac{1}{s} - \frac{1}{8} \cdot \frac{1}{s+2} + \frac{1}{24} \cdot \frac{1}{s+6}$$

$$m_2(t) = c_{\text{in}} - \frac{3}{2}c_{\text{in}}e^{-2t} + \frac{1}{2}c_{\text{in}}e^{-6t}$$

The $\frac{1}{s}$ term becomes the steady state solution, while the other poles become the transient part.

Recall

$c_{\text{out}}(t) = \dfrac{m_2(t)}{V_2} = M_2(t)$
$\lim_{t\to\infty} m_2(t) = c_{\text{in}}$

The terms
$-\dfrac{3}{2}c_{\text{in}}e^{-2t} + \dfrac{1}{2}c_{\text{in}}e^{-6t}$
are the transients ($\to 0$ as $t \to \infty$)

Obtaining Just Steady-State

We could obtain steady-state values for $m_1(t)$ and $m_2(t)$ by setting $m_1'(t) = 0$ and $m_2'(t) = 0$ in the original system (*)

These are called the equilibrium solutions.

$0 = 3c_{\text{in}} + m_2^{\text{eq}} - 4m_1^{\text{eq}}$
$0 = 4m_1^{\text{eq}} - 4m_2^{\text{eq}}$
$\Rightarrow m_1^{\text{eq}} = m_2^{\text{eq}} = c_{\text{in}}$

Decoupling Equations

Method 1

From (**)

$(s+2)(s+6)M_2(s) = (s^2+8s+12)M_2(s) = \dfrac{12c_{\text{in}}}{s}$

By initial conditions, $m_2(0) = 0,\ m_2'(0) = 0$

$\mathcal{L}^{-1}\{s^2M_2\} + 8\mathcal{L}^{-1}\{sM_2\} + 12\mathcal{L}^{-1}\{M_2\} = 12c_{\text{in}}\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\}$

$m_2'' + 8m_2' + 12\,m_2 = 12\,c_{\text{in}}U(t)$

Method 2

Using equation 2 from (*)
$\dfrac{d}{dt}m_2' = \dfrac{d}{dt}4m_1 - 4m_2$
$m_2'' = 4m_1' - 4m_2'$
From equation 1
$m_2'' = 4(3c_{\text{in}} + m_2 - 4m_1) - 4m_2'$
From equation 2: $4m_1 = m_2' + 4m_2$
$m_2'' = 12c_{\text{in}} + 4m_2 - 4m_2' - 16m_2 - 4m_2'$
$m_2'' + 8m_2' + 12\,m_2 = 12\,c_{\text{in}}$

2nd Order Systems of DEs

1:28 PM

2^{nd} Order system of 1^{st} order DEs

for state variables $x(t)$ and $y(t)$

$$(*)\begin{cases}\dfrac{dx}{dt}=p(t,x,y) & x(t_0)=x_0\\ \dfrac{dy}{dt}=q(t,x,y) & y(t_0)=y_0\end{cases}$$

p, q are generally non-linearly

Matrix Notation

Define the vector-valued function $\vec{x}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$

$\vec{p}(t,\vec{x})=\begin{bmatrix}p(t,x,y)\\q(t,x,y)\end{bmatrix}$

IC: $\vec{x}(t_0)=\vec{x}_0=\begin{bmatrix}x_0\\y_0\end{bmatrix}$

So that system $(*)$ is written as

$$\frac{d\vec{x}}{dt}=\vec{p}(t,\vec{x})$$

If t is missing in $\vec{p}$ then the system is autonomous.

Linear Systems

$\vec{p}(t,\vec{x})=A(t)\vec{x}+\vec{f}(t)$

$A=\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix},\qquad \vec{f}=\begin{bmatrix}f_1\\f_2\end{bmatrix}$

Normal form of a linear system

$$\frac{d\vec{x}}{dt}=A\vec{x}+\vec{f}$$

Generally non-homogeneous system.

If $\vec{f}=\vec{0}$, then homogeneous system

$$\frac{d\vec{x}}{dt}=A\vec{x}$$

Calculus of Matrices

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix}$$

In general, $A=[a_{ij}]\ \Rightarrow\ \dfrac{dA}{dt}=\left[\dfrac{da_{ij}}{dt}(t)\right]$

$$\frac{d}{dt}(AB)=A'B+B'A$$

$$\int A(t)dt=\left[\int a_{ij}(t)dt\right]$$

Basic Theory For linear Systems

Existence and Uniqueness Theorem

If $a_{ij}(t)$ and $f_i(t)$ are continuous on I and contain t_0, then for any IC $\vec{x}(t_0)=\vec{x}_0$, there exists a unique solution $\vec{x}(t)$ $\vec{x}(t)$ to the IVP

$$\frac{d\vec{x}}{dt}=A\vec{x}+\vec{f},\qquad \vec{x}(t_0)=\vec{x}_0$$

on the whole interval I.

Superposition Theorem

Write $\vec{x}=A\vec{x}+\vec{f}$ as $\hat{L}[\vec{x}]=\vec{f}$ where $\hat{L}=I\frac{d}{dt}-A$, I = Identity Matrix is **linear operator**

Development of Superposition

If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions to the homogeneous system $\hat{L}[\vec{x}]=\vec{0}$ then any linear combination of $c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$ is also a solution to $\hat{L}[\vec{x}]=\vec{0}$ where c_1 and c_2 are arbitrary **scalar constants.**

Given $\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$

what do we require of

$\vec{x}_1(t)=\begin{bmatrix}x_{11}(t)\\x_{21}(t)\end{bmatrix}$ and $\vec{x}_2(t)=\begin{bmatrix}x_{12}(t)\\x_{22}(t)\end{bmatrix}$ to be able to solve the IVP at t_0?

Example: Predator-Prey Model

(Lotka - Volterra eqs.)

Let

x = # of prey species

y = # of predator species

$$\frac{dx}{dt}=ax-bxy$$
$$\frac{dy}{dt}=cxy-dy$$

Linear Systems Examples

Coupled Mixing Tanks

$$(**)\begin{cases}\dfrac{dm_1}{dt}=-4m_1+m_2+3c_{\text{in}}\\ \dfrac{dm_2}{dt}=4m_1-4m_2\end{cases}$$

$\vec{x}=\begin{bmatrix}m_1\\m_2\end{bmatrix},\qquad A=\begin{bmatrix}-4 & 1\\4 & -4\end{bmatrix},\qquad \vec{f}=\begin{bmatrix}3c_{\text{in}}\\0\end{bmatrix}$

1. Solved $(**)$ by $\mathcal{L}$
2. Decoupled 2^{nd} order DEs for $m_1(t)$ and $m_2(t)$
   $m_2''+8m_2'+12m_2=12c_{\text{in}}$
   $m_2(0)=0,\qquad m_2'(0)=0$

   2^{nd} order DEs may always be written as 2^{nd} order system

Example: Mechanical Oscillator

$$m\frac{dv}{dt}=-ky-\gamma v+F(t)$$

State variables: $y(t)$ and $v(t)=\frac{dy}{dt}$

$$\frac{d}{dt}\begin{bmatrix}y\\v\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{\gamma}{m}\end{bmatrix}\begin{bmatrix}y\\v\end{bmatrix}+\begin{bmatrix}0\\ \frac{F(t)}{m}\end{bmatrix}$$

RLC Circuit

$$L\frac{di}{dt}+\frac{q}{C}+Ri=e(t),\qquad i(t)=\frac{dq}{dt}$$

State variables: $v(t)=\frac{q(t)}{C}$, $i(t)=C\frac{dv}{dt}$

$$\frac{d}{dt}\begin{bmatrix}v\\i\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{C}\\ -\frac{1}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}v\\i\end{bmatrix}+\begin{bmatrix}0\\ \frac{e(t)}{L}\end{bmatrix}$$

Product Rule for AB

$$AB=\left[\sum_k a_{ik}b_{kj}\right]$$

$$\frac{d}{dt}(AB)=\left[\sum_k\frac{d}{dt}(a_{ik}b_{kj})\right]=\left[\sum_k\frac{da_{ik}}{dt}b_{kj}+\sum_k a_{ik}\frac{db_{kj}}{dt}\right]=\frac{dA}{dt}B+A\frac{dB}{dt}$$

Note, $\frac{dA}{dt}B+A\frac{dB}{dt}\neq B\frac{dA}{dt}+\frac{dB}{dt}A$

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = \begin{bmatrix} x_{11}c_1 \\ x_{21}c_2 \end{bmatrix} + \begin{bmatrix} x_{12}c_2 \\ x_{22}c_2 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\vec{x}_1 \quad \vec{x}_2]\vec{c} = X\vec{c}$
where $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and $X(t) = [\vec{x}_1(t) \quad \vec{x}_2(t)]$

$X(t)$ is solution matrix generated by A, or
$\frac{d\vec{x}}{dt} = A\vec{x}$

$\vec{x}(t) = X(t)\vec{c}$
At $t = t_0$: $X(t_0)\vec{c} = \vec{x}_0$

Need: $X(t_0)$ is invertible at t_0
$\Leftrightarrow \det\big(X(t_0)\big) \neq 0$
$\Leftrightarrow$ columns $\vec{x}_1(t_0)$ and $\vec{x}_2(t_0)$ are linearly independent.

Theory of Linear Systems

November-16-12 1:29 PM

Properties from Linear Algebra

For square matrix X and vector $\vec{c}$, the following are equivalent:

- X is invertible
- $\det(X) \neq 0$
- columns of X are linearly independent
- $X\vec{c} = \vec{0}$ has only trivial solution for $\vec{c}$

Notation

For two vector functions

$$\vec{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}, \qquad \vec{x}_2(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}$$

and scalar constants $c_1, c_2 \rightarrow \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Define matrix function $X(t) = [\vec{x}_1(t) \quad \vec{x}_2(t)]$ so that linear combinations read

$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = X(t)\vec{c}$

Linear Independence

Two vector functions $x_1(t)$ and $x_2(t)$ are:

a) **linearly independent** on I iff eq. $X(t)\vec{c} = \vec{0}$ has only trivial solution $\vec{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $t \in I$

b) **linearly dependent** on I iff eq. $X(t)\vec{c} = \vec{0}$ has nontrivial solutions $\vec{c} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $t \in I$

Caution

We combine two concepts of linear (in)dependence

$\vec{x}_1(t) = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

clearly $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are linearly independent

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = \begin{bmatrix} c_1t + c_2 \\ c_1t + c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow c_1 = c_2 = 0$$

Fundamental Set

A set of solutions $\{\vec{x}_1(t), \vec{x}_2(t)\}$ to a homogeneous system $\vec{x}'(t) = A\vec{x}(t)$ that are linearly independent on I is called a **fundamental set** of solutions, and the solution matrix $X(t) = [\vec{x}_1(t), \quad \vec{x}_2(t)]$ is called the **fundamental matrix**.

Use determinant of $X(t)$ to test for linear independence.

Wronskian

Wronskian of any two vector function $\vec{x}_1(t),\ \vec{x}_2(t)$ is

$$W[\vec{x}_1, \vec{x}_2] = \det\big(X(t)\big) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} = x_{11}(t)x_{22}(t) - x_{12}(t)x_{21}(t)$$

Theorem

If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions to $\vec{x}' = A\vec{x}$ on I, then their Wronskian is either identically 0, or never 0 for all $t \in I$.

Can be proved with Abel's Formula

Abel Formula

$$W[\vec{x}_1, \vec{x}_2](t) = W[\,\vec{x}_1, \vec{x}_2]e^{\int_{t_0}^{t} tr(A)\,d\tau}$$

Comment

For a 2^{nd} order DE $y'' + p(t)y' + q(t)y = 0$

Let $\vec{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}, \qquad A(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}$

$$\frac{d\vec{x}}{dt} = A\vec{x}, \qquad y_{1,2} \rightarrow \vec{x}_{1,2} = \begin{bmatrix} y_{1,2} \\ y'_{1,2} \end{bmatrix}$$

$$W[\vec{x}_1, \vec{x}_2] = W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$

Theorem

Let $\vec{x}_1(t),\ \vec{x}_2(t)$ be solutions to $\vec{x}' = A\vec{x}$ on I

Then $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are linearly independent on I, iff $W[\vec{x}_1, \vec{x}_2](t_0) \neq 0$ for some $t_o \in I$

Example

Midterm Q5

$p(t) = \frac{1}{t}, \qquad g(t) = \frac{1}{t^2}$

$\vec{x}_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \qquad \vec{x}_2(t) = \begin{bmatrix} t\ln(t) \\ 1 + \ln(t) \end{bmatrix}$

$$\vec{x}_i'(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{1}{t} \end{bmatrix}\vec{x}_i(t), \qquad i = 1,2$$

$$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} t & t\ln t \\ 1 & 1 + \ln t \end{vmatrix} = t \neq 0 \text{ for } t \neq 0$$

Theorem

If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are any two linearly independent solutions to $\vec{x}' = A\vec{x}$ on interval I, then every (i.e. general) solution to $\vec{x}' = A\vec{x}$ may be written as $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = X(t)\vec{c}$

We may determine $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for arbitrary IC $\vec{x}_0$ at $t_0 \in I$

$\vec{x}(t_0) = \vec{x}_0 = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$

$\vec{x}_0 = X(t_0)\vec{c} \Rightarrow \vec{c} = X^{-1}(t_0)\vec{x}_0$

Non-Homogeneous System

$\vec{x}' = A\vec{x} + \vec{f}(t)$

The general solution $\vec{x}(t) = X(t)\vec{c} + \vec{x}_p(t)$ where $\vec{x}_p$ is a particular solution to the non-homogenous equation.

Solve IVP $\vec{x}(t_0) = \vec{x}_0 = X(t_0)\vec{c} + \vec{x}_p(t_0)$

Solve for $\vec{c} = X^{-1}(t_0)\left(\vec{x}_0 - \vec{x}_p(t_0)\right)$

Linear Systems With Constant Coefficients

November-16-12 2:12 PM

Homogeneous Case

$\vec{x}' = A\vec{x}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, const

Assume $\vec{x}(t) = e^{\lambda t}\vec{v}$

where $\vec{v}$ is a constant vector

$\vec{x}'(t) = \lambda e^{\lambda t}\vec{v} = \lambda\vec{x}(t)$

$\lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v}, \quad e^{\lambda t} \neq 0$

Eigenvalue problem for A:

$A\vec{v} = \lambda\vec{v}, \quad$ or $(A - \lambda I)\vec{v} = 0$

For 2×2 matrix we have two eigenpairs $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$

where $\lambda_{1,2}$ are solutions to characteristic equation

$\chi(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$

$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1, \quad \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2$

$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} e^{\lambda_1 t}v_{11} & e^{\lambda_2 t}v_{12} \\ e^{\lambda_1 t}v_{21} & e^{\lambda_2 t}v_{22} \end{vmatrix} = e^{(\lambda_1+\lambda_2)t}\begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} = e^{tr(A)t}\det(V)$

where $V = [\vec{v}_1 \quad \vec{v}_2]$

2nd Order Systems, homogenous, const. matrix

(Review of above)

Constant matrix A, $x' = A\vec{x}$

Assume $\vec{x}(t) = e^{\lambda t}\vec{v}$, $\vec{v}$ const.

Eigenvalue Problem $(A - \lambda I)\vec{v} = 0$

eigenpairs $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$

Solution

$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1, \quad \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2$

$W[\vec{x}_1, \vec{x}_2](t) = e^{(\lambda_1+\lambda_2)t}\det(V), \quad V = [\vec{v}_1, \vec{v}_2]$

Result Linear Algebra

Eigenvectors corresponding to distinct eigenvalues are linearly independent, giving $\det(V) \neq 0$, $W(t) \neq 0 \ \forall t$

Complex Eigenvalues

If $\lambda = \alpha + i\beta$ is an eigenvalue of A with eigenvectors $\vec{v} = \vec{u} + i\vec{w}$ then $\bar{\lambda} = \lambda^* = \alpha - i\beta$ is also an eigenvalue wand with eigenvector $\bar{\vec{v}} = \vec{v}^* = \vec{u} - i\vec{w}$

Let $\vec{x}(t) = e^{\lambda t}\vec{v} = e^{(\alpha+i\beta)t}(\vec{y} + i\vec{w}) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{u} + i\vec{w}) = e^{\alpha t}(\vec{u}\cos\beta t - \vec{w}\sin\beta t) + ie^{\alpha t}(\vec{u}\sin\beta t + \vec{w}\cos\beta t)$

Linear independent solutions are

$\vec{x}_1(t) = Re(\vec{x}(t)) = \frac{1}{2}(\vec{x}(t) + \vec{x}^*(t)) = e^{\alpha t}(\vec{u}\cos\beta t - \vec{w}\sin\beta t)$

$\vec{x}_2(t) = Im(\vec{x}(t)) = \frac{1}{2i}(\vec{x}(t) - \vec{x}^*(t)) = e^{\alpha t}(\vec{u}\sin\beta t + \vec{w}\cos\beta t)$

So, solution matrix

$X(t) = [\vec{x}_1(t) \quad \vec{x}_2(t)] = e^{\alpha t}VR(t)$, where

$V[\vec{u} \quad \vec{v}], \quad R(t) = \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix}$

Equal Eigenvalues

Two cases

a) Two linearly independent eigenvectors
b) 1 eigenvector

Mixing Tanks Example

Mixing tanks $V_1 = V_2 = 1$

$m_1' = -(f_1 + f_2)m_1 + f_2 m_2$

$m_2' = (f_1 + f_2)m_1 - (f_1 + f_2)m_1$

$\vec{x}(t) = \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}$

Example with $(\lambda_1 \neq \lambda_2)$

$f_1 = 3, \quad f_2 = 1$

$\vec{x}' = A\vec{x}$

$A = \begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 \\ 4 & -4-\lambda \end{vmatrix} = (\lambda + 4)^2 - 4 = (\lambda + 2)(\lambda + 6) = 0$

$\lambda_1 = -2, \quad \lambda_2 = -6$

$A - \lambda_1 I = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A - \lambda_2 I = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\vec{x}_1(t) = e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{x}_2(t) = e^{-6t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad W[\vec{x}_1, \vec{x}_2](t) = -4e^{-8t}$

Example with Complex Eigenvalues

$A = \begin{bmatrix} -2 & 6 \\ -3 & 4 \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 - 2\lambda + 10 = 0$

$\lambda_1 = 1 + 3i, \quad \lambda_2 = 1 - 3i$

$A - \lambda_1 I = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}$

$(-1 - i)v_{11} + 2v_{12} = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\vec{x}_1(t) = e^t\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 3t - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin 3t\right) = e^t\begin{bmatrix} \cos 3t + \sin 3t \\ \cos 3t \end{bmatrix}$

$\vec{x}_2(t) = e^t\begin{bmatrix} \sin 3t - \cos 3t \\ \sin 3t \end{bmatrix}$

Check the Wronskian

$V = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad R(t) = \begin{bmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{bmatrix}$

$W[\vec{x}_1, \vec{x}_2] = e^{2\lambda}\det(V)\det(R(t)) = e^{2t}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{vmatrix} = e^{2t}$

Example

Type a)

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \vec{x}' = A\vec{x}$

$m_1' = -m_1, \quad m_2' = -m_n$

$\det(A - \lambda I) = (\lambda + 1)^2 = 0$

$\lambda_1 = \lambda_2 = -1$

$A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Pick Standard Basis

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}_1(t) = e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{x}_2(t) = e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Type b)

Coupled tanks with $f_1 = 1$, $f_2 = 0$

$m_1' = -m_1, \quad m_2' = m_1 - m_2$

$\det(A - \lambda I) = (\lambda + 1)^2$

$\lambda_1 = \lambda_2 = -1$

$(A - \lambda_1 I) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$(A - \lambda_1)1\vec{v} = 0$

$\vec{v}' = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = ?$

$v_2 = \text{anything} = 1$

$\vec{v}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}_1(t) = e^{\lambda t}\vec{v} = e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$

Try $\vec{x}_2(t) = t\vec{x}_1(t)$

Does not work

Repeated Eigenvalues

In general, for repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$ assume the solution to $\vec{x}' = Ax$ in the form

$\vec{x}(t) = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}, \quad \vec{v}, \vec{w}, \text{const}$

$\vec{x}'(t) = e^{\lambda t}\vec{v} + \lambda t\, e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{w}$

$(*)\ A\vec{x}(t) = te^{\lambda t}A\vec{v} + e^{\lambda t}A\vec{w}$

Multiply $(*)$ by $e^{-\lambda t} \neq 0$ and get

$t(A - \lambda I)\vec{v} + (A - \lambda I)\vec{w} - \vec{v} = \vec{0}$

First we solve $t(A - \lambda I)\vec{v} = \vec{0}$, then solve $(A - \lambda I)\vec{w} = \vec{v}$

$(A - \lambda I) \Rightarrow$
$(A - \lambda I)^2 \vec{w} = (A - \lambda I)\vec{v} = \vec{0}$
$(A - \lambda I)^2$ is the zero matrix by the Cayley Hamilton Theorem

So $\vec{x}_1(t) = e^{-t}\vec{v} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$
$\vec{x}_2(t) = t\vec{x}_1(t) + e^{-t}\vec{w}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow 0 = 0, \qquad w_1 = 1$
$w_2 = \text{anything} = 0$

$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
Check Wronskian
$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} 0 & e^{-t} \\ e^{t} & te^{-t} \end{vmatrix} = -e^{-2t} < 0$

Phase Plane Analysis

November-23-12 1:31 PM

Phase plane analysis for autonomous systems

$\frac{dx}{dt} = f(x,w), \frac{dy}{dt} = g(x,y)$

Solutions $x = x(t)$, $y = y(t)$ represent a parametric curve in the x-y-plane = phase plane

Parametric curve = phase portrait

Critical Point

A point (x_*, y_*) defined by $f(x_*, y_*) = 0$ and $g(x_*, y_*) = 0$ is called critical, equilibrium, or stationary point.

Stability

A critical point (x_*, y_*) is called

a) Asymptotically stable iff
   $\lim_{t\to\infty} x(t) = x_*, \lim_{t\to\infty} y(t) = y_*$
b) Stable iff
   $\sqrt{(x(t)-x_*)^2 + (y(t)-y_*)^2} < m \ \forall t > 0$
c) Unstable

Linear autonomous Systems (2^{nd} order)

$f(x,y) = a_{11}x + a_{12}y + b_1$

$g(x,y) = a_{21}x + a_{22}y + b_2$

$\frac{d\vec{x}}{dt} = A\vec{x} + b, \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\frac{d\vec{x}}{dt} = \vec{0} = A\vec{x} + \vec{b} \Rightarrow \vec{x}_* = -A^{-1}b$

Let $\vec{\zeta}(t\ 0 = \vec{x}(t) = \vec{x}_*, \ \frac{d\vec{\zeta}}{dt} = A\vec{\zeta}$

Critical Point at the origin of the phase plane

$(A - \lambda_{1,2}I)\vec{v}_{1,2} = \vec{0}, \qquad \vec{x}_{1,2}(t) = e^{\lambda_{1,2}t}\vec{v}_{1,2}$

If $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$, the origin is stable critical point

Linear Isomorphism

Two sets of points $\Omega_1, \Omega_2 \subseteq \mathbb{R}^2$ are linearly isomorphic if there exists an invertible matrix V such that $\Omega_2 = V\Omega_1$

$\Omega_2 = \{Vx : x \in \Omega_1\}$

Note

$V = [\vec{v}_1 \quad \vec{v}_2] \Rightarrow AV = [A\vec{v}_1 \quad A\vec{v}_2] = [\lambda_1\vec{v}_1 \quad \lambda_2\vec{v}_2] = [\vec{v}_1 \quad \vec{v}_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = VD$

$AV = VD \Rightarrow A = VDV^{-1}$

$\vec{x}' = A\vec{x} = VDV^{-1}\vec{x} \Rightarrow V^{-1}\vec{x}' = DV^{-1}\vec{x}$

Let $\vec{\xi}(t) = V^{-1}\vec{x}(t) \Rightarrow \vec{\xi}' = D\vec{\xi}$

Proper vs. Improper Nodes

For a proper node, all lines coming in are straight lines. Improper nodes have curved lines.

Examples of Phase Portraits

For 2^{nd} order autonomous linear, homogeneous systems $\vec{x}' = A\vec{x}$ with a critical point $(0,0)$.

Assume $A\vec{v}_1 = \lambda_1\vec{v}_1$, $A\vec{v}_2 = \lambda_2\vec{v}_2$ with

$V = [\vec{v}_1 \quad \vec{v}_2]$, $\qquad \vec{v}_1, \vec{v}_2$, lin. indep.

(1) Equal Eigenvalues Example

Decoupled tanks, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\lambda_1 = \lambda_2 = -1, \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\frac{dx}{dt} = -x \Rightarrow x = c_1e^{-t} \to 0$

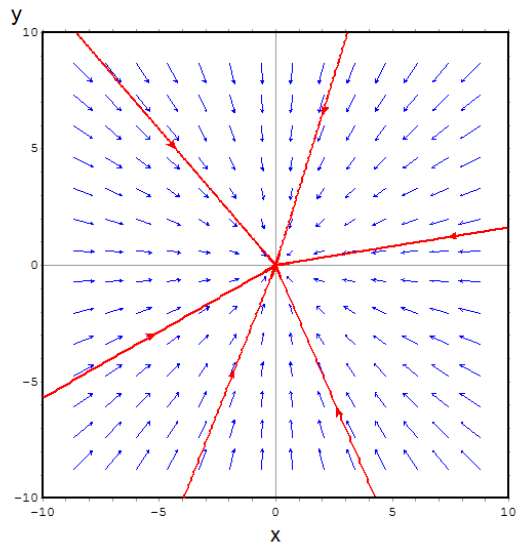
$\frac{dy}{dt} = -y \Rightarrow y = c_2e^{-t} \to 0$

Eliminate t

$y = \frac{c_2}{c_1}\vec{x}_1, \qquad c_1 \neq 0$

Solutions are lines of the form $y = cx$

$(0,0)$ is a "proper node" , asymptotically stable.



(2) Distinct Real eigenvalues with same signs

Let $\vec{\xi}(t) = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \ \vec{\xi}' = D\vec{\xi}$

$D = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}$

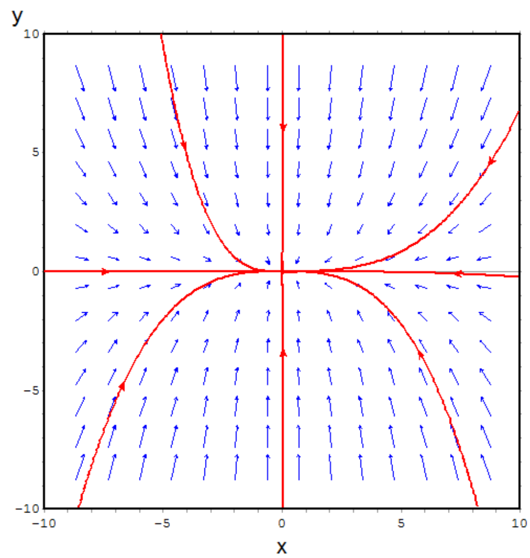
$\frac{d\xi}{dt} = -2\xi \Rightarrow \xi = c_1e^{-2t}$

$\frac{d\eta}{dt} = -6\eta \Rightarrow \eta = c_2e^{-6t}$

$\Rightarrow \eta = \frac{c_2}{c_1}\xi^3 = c\xi^3, \qquad c_1 \neq 0$

Solutions are cubic curves going into 0

$(0,0)$ is "improper node", asymptotically stable.



Recall

Coupled tanks with $f_1 3,\ f_2 = 1$
$\vec{x}' = A\vec{x}, \qquad A = \begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}$
$\lambda_1 = -2, \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \lambda_2 = -6, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
$\vec{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
$x = c_1 e^{-2t} + c_2 e^{-6t}$
$y = 2c_1 e^{-2t} - 2c_2 e^{-6t}$

Using the linear isomorphism get skewed version of previous result.
$(0,0)$ is "improper node", asymptotically stable

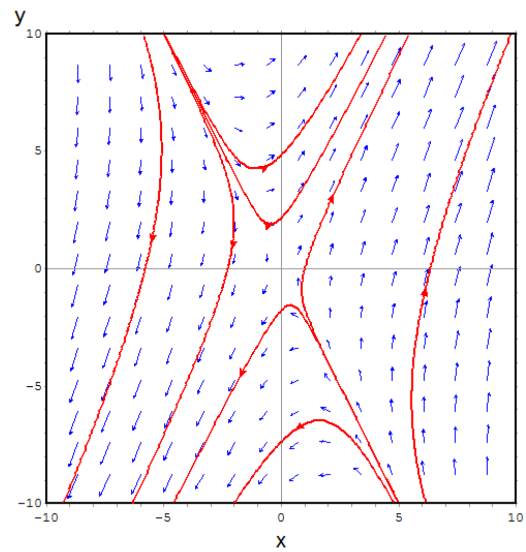
(3) Real Eigenvalues with Opposite Sign

$\vec{\xi}' = D\vec{\xi}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$
$\dfrac{d\xi}{dt} = 3\xi \Rightarrow \xi = c_1 e^{3t} \to \infty$
$\dfrac{d\eta}{dt} = -\eta \Rightarrow \eta = c_2 e^{-t} \to 0$
This produces a "saddle point". Along a single axis, lines are convergent.
Other axis is divergent. All lines approach divergent axis.

Example

$\vec{x}' = A\vec{x}, \qquad A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}, \qquad \lambda_1 = 3, \qquad \vec{v}_1 = [1 \quad 2]$
$\lambda_2 = -1, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = V\vec{\xi} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
Get skewed saddle point.



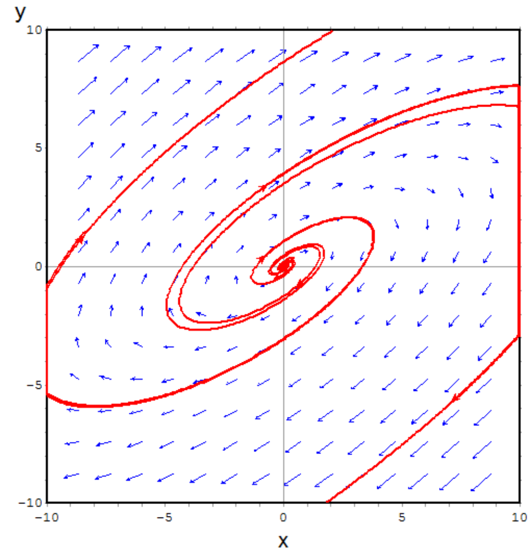
(4) Complex Conjugate eigenvalues

$\lambda = \alpha \pm i\beta, \qquad \vec{v} = \vec{u} \pm i\vec{w}$
Recall: solution matrix
$X(t) = [\vec{x}_1(t) \quad \vec{x}_2(t)] = e^{\alpha t} V R(t)$ where $V = [\vec{u} \quad \vec{v}]$, $R(t) = \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix}$

General solution: write
$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
$\vec{x}(t) = X(t)\vec{c} = e^{\alpha t} V R(t) \vec{c}$
$R(t)\vec{c} = \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c \begin{bmatrix} \cos(\beta t - \delta) \\ -\sin(\beta t - \delta) \end{bmatrix}, \qquad c = \sqrt{c_1^2 + c_2^2}$
Let $\vec{\xi}(t) = c e^{\alpha t} \begin{bmatrix} \cos(\beta t - \delta) \\ -\sin(\beta t - \delta) \end{bmatrix}$
In polar coordinates: Let $\theta = -t$
$\xi = c e^{-\alpha\theta} \cos(\beta\theta + \delta)$
$\eta = c e^{-\alpha\theta} \sin(\beta\theta + \delta)$
We have "spiral point", which is stable if $\alpha < 0$ and unstable if $\alpha > 0$
If $\alpha = 0$ then solutions circle about spiral point.

Example

$A = \begin{bmatrix} -2 & 6 \\ -3 & 4 \end{bmatrix}, \qquad \alpha = 1, \qquad \beta = 3 \Rightarrow$ Unstable
$V = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$



Example: Undamped Harmonic Oscillator

$m\frac{dv}{dt} = -ky, \qquad \frac{dy}{dt} = v, \qquad \omega = \sqrt{\frac{k}{m}}$

$\vec{x}' = A\vec{x}, \qquad \vec{x} = \begin{bmatrix} y \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 + \omega^2 = 0$

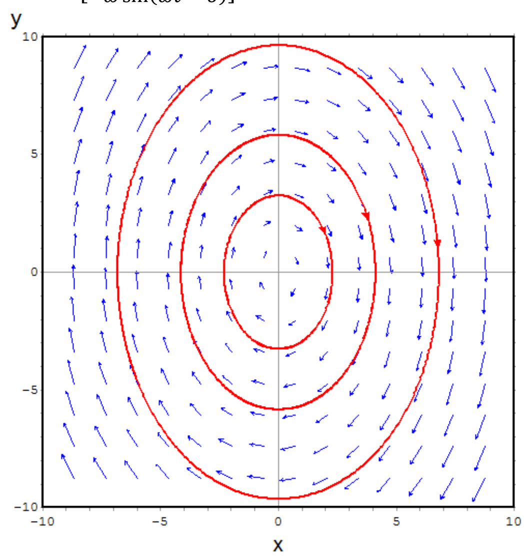
$\lambda_{1,2} = \pm i\omega, \qquad (\alpha = 0)$

(0,0) is "center" and (neutrally) stable.

$(A - \lambda_1 I)\vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ i\omega \end{bmatrix}$

$\Rightarrow \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} 0 \\ \omega \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$

$\vec{x}(t) = c\begin{bmatrix} \cos(\omega t - \delta) \\ -\omega\sin(\omega t - \delta) \end{bmatrix}$



Recall energy conservation:

$\frac{dv}{dy} = \frac{\frac{dv}{dt}}{\frac{dy}{dt}} = -\frac{\omega^2 y}{v}$

$mvdv = -kydy \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = C$

Non-Homogeneous Systems

November-26-12 2:07 PM

Solving Non-Homogeneous Systems with Constant Coefficients

$\vec{x}' = A\vec{x} + \vec{f}(t), \qquad \vec{x}(t_0) = \vec{x}_0$

General Solution

$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$
where $\vec{x}_1, \ \vec{x}_2$ are solutions to homogeneous system
$X(t)\vec{c} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

IC: $\vec{x}(t_0) = X(t_0)\vec{c} + \vec{x}_p(t_0) = \vec{x}_o$
$\Rightarrow \vec{c} = X^{-1}(t_0)\left(\vec{x}_0 - \vec{x}_p(t_0)\right)$

Methods for Finding Particular Solution

1) Use $\mathcal{L}$
2) Undetermined coefficients for "simple" $\vec{f}(t)$
3) Variation of constants

Method of Variation of Constants for linear nonhomogeneous systems

In general, $A = A(t)$
$\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t) \ (**), \qquad \vec{x}(t_0) = \vec{x}_0$

Assume we have a fundamental matrix
$X(t)$ for homogeneous system, $\vec{x}' = A\vec{x} \ (*)$
$X(t) = [\vec{x}_1(t) \quad \vec{x}_2(t)\,], \qquad \vec{x}_1' = A\vec{x}_1, \qquad \vec{x}_2' = A\vec{x}_2$

Recall
General solution for $\vec{x}(t) = X(t)\vec{c}, \ \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
To handle non-homogeneous system (**), assume
$\vec{x}(t) = X(t)\vec{u}(t), \qquad \vec{u}(t) = ?$
$\vec{x}'(t) = X'(t)\vec{u}(t) + X(t)\vec{u}'(t) = A(t)X(t)\vec{u}(t) + \vec{f}(t)$
$\Rightarrow \left(X'(t) - A(t)X(t)\right)\vec{u}(t) + X(t)\vec{u}'(t) = \vec{f}(t)$

1. $X(t)$ satisfies $\vec{X}'(t) = A(t)X(t)$
   Proof: $\vec{X}(t) = [\vec{x}_1'(t) \quad \vec{x}_2'(t)] = [A\vec{x}_1 \quad A\vec{x}_2] = AAX$

$\Rightarrow X(t)\vec{u}'(t) = \vec{f}(t)$

1. $X(t)$ is invertible for all $t \in I$

$\vec{u}'(t) = X^{-1}(t)\vec{f}(t)$
$\vec{u}(t) = \int X^{-1}(t)\vec{f}(t)dt + \vec{c}$
$\vec{u}(t) = c + \int_{t_0}^{t} X^{-1}(s)\vec{f}(s)ds$
$\vec{x}(t) = X(t)\vec{c} + X(t)\int_{t_0}^{t} X^{-1}(s)\vec{f}(s)ds$
    $X(t)\vec{c}$ is general solution of homogeneous equation
    $X(t)\int_{t_0}^{t} X^{-1}(s)\vec{f}(s)ds = \vec{x}_p(t) = \text{particular solution}$

Solve IVP: $t_0 \in I$
$\vec{x}(t_0) = \vec{x}_0 = X(t_0)\vec{c} + 0 \Rightarrow \vec{c} = X^{-1}(t_0)\vec{x}_0$

$$\boxed{\vec{x}(t) = X(t)X^{-1}(t_0)\vec{x}_0 + X(t)\int_{t_0}^{t} X^{-1}(s)\vec{f}(s)ds}$$

Example: Solution with Undetermined Coefficients

Use method 2 for coupled tanks with
$f_1 = 3, \qquad f_2 = -1$
$\vec{x}(t) = \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}, \qquad A = \begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}$
$\vec{f}(t) = \begin{bmatrix} 3c_{\text{in}} \\ 0 \end{bmatrix}$
Assume $\vec{x}_p(t) = \vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \overrightarrow{\text{const}}$
$0 = \vec{x}_p'(t) = A\vec{x}_p(t) + \vec{f}(t\,) \Rightarrow \vec{x}_p = -A^{-1}\vec{f} = c_{\text{in}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solve the IVP, $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\vec{x}(t) = \begin{bmatrix} e^{-2t} & e^{-6t} \\ 2e^{-2t} & -2e^{-6t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} c_{\text{in}} \\ c_{\text{in}} \end{bmatrix}, \qquad \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$V = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$
$V\vec{c} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_{\text{in}} \\ c_{\text{in}} \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4}c_{\text{in}} \\ -\frac{1}{4}c_{\text{in}} \end{bmatrix}$
$\vec{x}(t) = c_{\text{in}}\begin{bmatrix} -\frac{3}{4}e^{-2t} - \frac{1}{4}e^{-6t} + 1 \\ -\frac{3}{2}e^{-2t} + \frac{1}{2}e^{-6t} + 1 \end{bmatrix} = \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}$

Evolution Matrix

November-30-12 1:40 PM

Evolution Matrix

Evolution matrix $\Phi_{t_0}(t)$ generated by A at $t_0 \in I$ is the fundamental matrix $X(t)$ whose i^{th} column is the unique solution to the IVP $\vec{x}' = A\vec{x},\ \vec{x}(t_0) = b_i$, where $\vec{b}_i$ is the i^{th} vector in the standard basis.

Standard 2D basis

$$\{\vec{b}_1, \vec{b}_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

Recall

$$\begin{cases} \vec{x}_1(t) = X(t)X^{-1}(t_0)\vec{b}_1 \\ \vec{x}_2(t) = X(t)X^{-1}\vec{b}_2 \end{cases}$$

$$\Rightarrow \Phi_{t_0}(t) = [\vec{x}_1(t) \quad \vec{x}_2(t)] = X(t)X^{-1}(t_0)[\vec{b}_1 \quad \vec{b}_2]$$

$$\boxed{\Phi_{t_0}(t) = X(t)X^{-1}(t_0)}$$

where $X(t)$ is any fundamental matrix for $\vec{x}' = A\vec{x}$

Note

$$X(t)X^{-1}(s) = X(t)X^{-1}(t_0)X(t_0)X^{-1}(s) = \Phi_{t_0}(t)\Phi_{t_0}^{-1}(s)$$

Therefore solution to system can be rewritten

$$\vec{x}(t) = \Phi_{t_0}(t)\vec{x}_0 + \Phi_{t_0}(t)\int_{t_0}^{t} \Phi_{t_0}^{-1}(s)\vec{f}(s)ds$$

Properties of Evolution Matrix for Autonomous Systems

1) For autonomous systems, we have **"Time-shift immunity"**
   If $\vec{\xi}(t)$ is solution to $\vec{\xi}' = A\vec{\xi}$, then $\vec{x}(t) = \vec{\xi}(t - t_0)$ is solution to $\vec{x}' = A\vec{x}$
   $\Phi_{t_0}(t) = \Phi_0(t - t_0) \equiv \Phi(t - t_0)$ for any $t_0 \in I$
2) $\Phi(0) = I$ (Also applies to non-autonomous)
3) $\Phi'(t) = A\Phi(t)$ (Also applies to non-autonomous)
4) $\Phi(t_1 + t_2) = \Phi(t_2)\Phi(t_1)$
5) $\Phi^{-1}(t) = \Phi(-t)$
6) $\Phi(t) = e^{At} = I + tA + \frac{t^2}{2}A^2 + \cdots = I\sum_{j=0}^{\infty}\frac{t^j}{j!}A^j$

$$\vec{x}(t) = e^{A(t-t_0)}\vec{x}_0 + \int_{t_0}^{t} e^{A(t-s)}f(\vec{s})ds$$

★ $$\vec{x}(t) = \Phi(t - t_0)\vec{x}_0 + \int_{t_0}^{t} \Phi(t-s)\vec{f}(s)ds, \qquad \text{where } \Phi(t) = X(t)X^{-1}(0)$$

Using $\mathcal{L}$ to find $\Phi(t)$

$\Phi'(t) = A\Phi(t), \qquad \Phi(0) = I$

$\mathcal{L}\{\Phi'(t)\} = s\mathcal{L}\{\Phi(t)\} - \Phi(0) = A\mathcal{L}\{\Phi(t)\}$ $(*)$

Solve $(*)$ for $\mathcal{L}\{\Phi(t)\}$

$(sI - A)\mathcal{L}\{\Phi(t)\} = I$

$\Rightarrow \mathcal{L}\{\Phi(t)\} = (sA - I)^{-1}$

$\Phi(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

$\mathcal{L}^{-1}$ is computed element-wise.

Justification of $\Phi(t_1 + t_2) = \Phi(t_2)\Phi(t_1)$

Point pairs $f(0) = \vec{a},\ f(t_1) = \vec{b},\ f(t_2) = \vec{c}$

$\vec{b} = \Phi(t_1)\vec{a}$

$\vec{c} = \Phi(t_2)\vec{b} = \Phi(t_2)\Phi(t_1)\vec{a}$

Example

Coupled tanks, $f_1 = 3,\ f_2 = 1$

$$A = \begin{bmatrix}-4 & 1\\4 & -4\end{bmatrix}, \qquad \vec{f}(t) = \begin{bmatrix}3c_{\text{in}}\\0\end{bmatrix}$$

$$\lambda_1 = -2, \qquad \vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \qquad \lambda_2 = -6, \qquad \vec{v} = \begin{bmatrix}1\\-2\end{bmatrix}$$

$$X(t) = [e^{\lambda_1}\vec{v}_1 \quad e^{\lambda_2}\vec{v}_2] = \begin{bmatrix}e^{-2t} & e^{-6t}\\2e^{-2t} & -2e^{-6t}\end{bmatrix}$$

$$X(0) = \begin{bmatrix}1 & 1\\2 & -2\end{bmatrix} = [\vec{v}_1 \quad \vec{v}_2] = V$$

$$X^{-1}(0) = \begin{bmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{2} & -\frac{1}{4}\end{bmatrix} = V^{-1}$$

$$\Phi(t) = X(t)X^{-1}(0) = \begin{bmatrix}\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-6t} & \frac{1}{4}e^{-2t} - \frac{1}{4}e^{-6t}\\ e^{-2t} - e^{-6t} & \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-6t}\end{bmatrix}$$

Solving the IVP: $\vec{x}_0 = \begin{bmatrix}m_1(0)\\m_2(0)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\vec{x}(t) = \vec{0} + \int_0^t \begin{bmatrix}\frac{3}{2}c_{\text{in}}e^{-2(t-s)} + \frac{3}{2}c_{\text{in}}e^{-6(t-s)}\\ 3c_{\text{in}}e^{-2(t-s)} - 3c_{\text{in}}e^{-2(t-s)}\end{bmatrix}ds$$

$$= \frac{3}{2}c_{\text{in}}\begin{bmatrix}1\\2\end{bmatrix}\int_0^t e^{-2(t-s)}ds + \frac{3}{2}c_{\text{in}}\begin{bmatrix}1\\-2\end{bmatrix}\int_0^t e^{-6(t-s)}ds$$

$$\int_0^t e^{-2(t-s)}ds = \frac{1}{2}(1 - e^{-2t})$$

$$\int_0^t e^{-6(t-s)}ds = \frac{1}{6}(1 - e^{-6t})$$

$$x(t) = c_{\text{in}}\begin{bmatrix}1 - \frac{3}{4}e^{-2t} - \frac{1}{4}e^{-6t}\\ 1 - \frac{3}{2}e^{-2t} + \frac{1}{2}e^{-6t}\end{bmatrix} = \begin{bmatrix}m_1(t)\\m_2(t)\end{bmatrix}$$

"Proof" of $e^{At} = \Phi(t)$

Recall, $A = VDV^{-1}$ where $D = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$

$$e^{At} = I\sum_{j=0}^{\infty}\frac{t^j}{j!}VD^jV = V\left(\sum_{j=0}^{\infty}\frac{t^j}{j!}\begin{bmatrix}\lambda_1^j & 0\\0 & \lambda_2^j\end{bmatrix}\right)V^{-1} = V\begin{bmatrix}\sum_{j=0}^{\infty}\frac{(\lambda_1 t)^j}{j!} & 0\\ 0 & \sum_{j=0}^{\infty}\frac{(\lambda_2 t)^j}{j!}\end{bmatrix}V^{-1}$$

$$= V\begin{bmatrix}e^{\lambda_1 t} & 0\\0 & e^{\lambda_2 t}\end{bmatrix}V^{-1} = \begin{bmatrix}1 & 1\\2 & -2\end{bmatrix}\begin{bmatrix}e^{\lambda_1 t} & 0\\0 & e^{\lambda_2 t}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{2} & -\frac{1}{4}\end{bmatrix}$$

Example

Tanks with $f_1 = 1,\ f_2 = 0$

$$A = \begin{bmatrix}-1 & 0\\1 & -1\end{bmatrix}, \qquad \lambda_1 = \lambda_2 = -1$$

$$\vec{x}_1(t) = \begin{bmatrix}0\\e^{-t}\end{bmatrix}, \qquad \vec{x}_2(t) = \begin{bmatrix}e^{-t}\\te^{-t}\end{bmatrix}$$

$$X(t) = \begin{bmatrix}0 & e^{-t}\\e^{-t} & te^{-t}\end{bmatrix}, \qquad X(0) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = X\hat{}-1(0)$$

$$\Phi(t) = X(t)X^{-1}(0) = \begin{bmatrix}0 & e^{-t}\\e^{-t} & te^{-t}\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}e^{-t} & 0\\te^{-t} & e^{-t}\end{bmatrix} \equiv e^{At}$$

Proof $\Phi(t) = e^{At}$

$$e^{At} = e^{\lambda t I}e^{(A-\lambda I)t} = e^{-t}\left(I + t(A+I) + \frac{t^2}{2!}(A+I)^2 + \cdots\right), \qquad (\lambda_1 = \lambda_2 = -1)$$

$$(A+I)^2 = 0$$

$$e^{At} = e^{-t}\left(I + t\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) = e^{-t}\begin{bmatrix}1 & 0\\t & 1\end{bmatrix}$$

Example of Using $\mathcal{L}$ to solve $\Phi(t)$

$$A = \begin{bmatrix}-4 & 1\\4 & -4\end{bmatrix}, \qquad sI - A = \begin{bmatrix}s+4 & -1\\-4 & s+4\end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+4)^2 - 4}\begin{bmatrix}s+4 & 1\\4 & s+4\end{bmatrix} = \begin{bmatrix}\frac{s+4}{(s+2)(s-6)} & \frac{1}{(s+2)(s+6)}\\ \frac{4}{(s+2)(s+6)} & \frac{s+4}{(s+2)(s+6)}\end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} \mathcal{L}\left\{\frac{s+4}{(s+2)(s-6)}\right\} & \mathcal{L}\left\{\frac{1}{(s+2)(s+6)}\right\} \\ \mathcal{L}\left\{\frac{4}{(s+2)(s+6)}\right\} & \mathcal{L}\left\{\frac{s+4}{(s+2)(s+6)}\right\} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-6t} & \frac{1}{4}e^{-2t} - \frac{1}{4}e^{-6t} \\ e^{-2t} - e^{-6t} & \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-6t} \end{bmatrix}$$

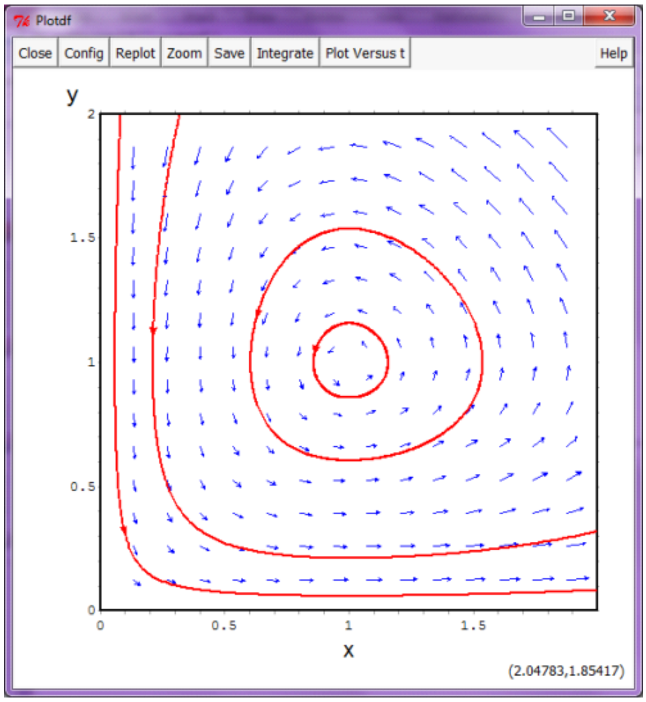
Example

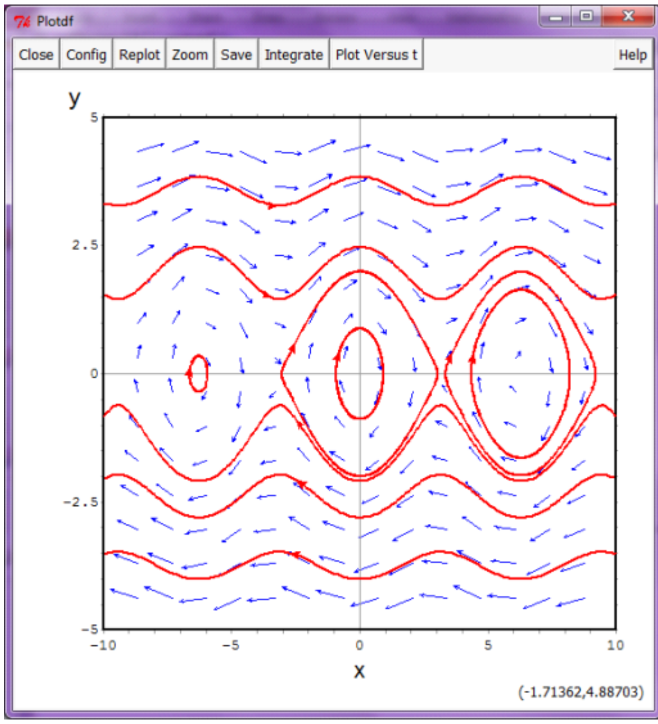
$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

$$\Phi(t) = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)^2} & \frac{1}{s+1} \end{bmatrix}$$

Qualitative Analysis of Nonlinear Systems

December-03-12 1:56 PM





Example

Lotka-Volterra equations

for predator(y) - prey(x) model

$\frac{dx}{dt} = f(x, y), \frac{dy}{dt} = g(x, y)$

$f = ax - bxy$

$g = cxy - dy$

1) Critical points : $f = 0,\ g = 0$
   a. $(0, 0)$
   b. $\left(\frac{d}{c}, \frac{a}{b}\right)$

   Note: If $(x(0), y(0))$ are both positive, then so are $x(t), y(t)$

   $\frac{1}{x} \cdot \frac{dx}{dt} = a - by$

   $\frac{d \ln x}{dt} = a - by \Rightarrow x(t) = ce^{\int (a-by)dt}$

Linearization

Linearize about points (a) and (b)

$a)\ \frac{dx}{dt} \approx ax, \quad \frac{dy}{dt} \approx -dy$

$A = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$

$\lambda_1 = a, \quad \lambda_2 = -d, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}(t) = c_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-dt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

b) Shift origin to $\left(\frac{d}{c}, \frac{a}{b}\right)$

$A = \begin{bmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{bmatrix} \Bigg|_{\left(\frac{d}{c}, \frac{a}{b}\right)} = \begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}$

$\lambda_1 = i\sqrt{ad}, \quad \vec{u} + i\vec{w}$

$\lambda_2 = -i\sqrt{ad}, \quad \vec{u} - i\vec{w}$

$\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{w} = \frac{b}{c}\sqrt{\frac{d}{a}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Example

$\frac{d^2\theta}{dt^2} = -\omega^2 \sin\theta, \quad \omega = \sqrt{\frac{g}{L}}$

Define $v = \frac{d\theta}{dt}$

$\frac{d\theta}{dt} = v \to v = 0$

$\frac{dv}{dt} = -\omega^2 \sin\theta \to \theta_n = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

Linearize about θ_n

$sin\theta = \sin\theta_n + \cos\theta_n (\theta - \theta_n)$

$A_n = \begin{bmatrix} 0 & 1 \\ \omega^2(-1)^{n+1} & 0 \end{bmatrix}$

n odd $\lambda_1 = \omega,\ \vec{v}_1 = \begin{bmatrix} 1 \\ \omega \end{bmatrix},\ \lambda_2 = -\omega,\ \vec{v}_2 = \begin{bmatrix} 1 \\ -\omega \end{bmatrix}$