Review of Vectors on $\mathbb{R}^n$

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1.1 Notations

$\vec{x} = (x^{(1)}, \dots, x^{(m)}) \in \mathbb{R}^n$

$x^{(i)}$ = i-th component of the vector $\vec{x}$

Operations with vectors

Addition: $\vec{x}$ as above, $\vec{y} = (y^{(1)}, \dots, y^{(m)})$

$$\vec{x} + \vec{y} := (x^{(1)} + y^{(1)}, \dots, x^{(m)} + y^{(m)})$$

Scalar multiplication:

$$\alpha\vec{x} := (\alpha x^{(1)}, \dots, \alpha x^{(m)}), \qquad for\ \alpha \in \mathbb{R}, \vec{x} \in \mathbb{R}^n$$

Standard inner product (dot product):

$$\langle \vec{x}, \vec{y} \rangle = x^{(1)}y^{(1)} + \dots + x^{(m)}y^{(m)} = \sum_{i=1}^{m} x^{(i)}y^{(i)} \in \mathbb{R}$$

Norm ("length") of a vector in $\mathbb{R}^n$:

$$\|\vec{x}\| := \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=1}^{m} (x^{(i)})^2}$$

Observe that $\|\vec{x}\| \geq 0$, with equality holding iff $\vec{x} = \vec{0} = (0, \dots, 0)$

1.2 Remark

Basic properties of standard inner product

Bilinearity :

$\vec{x}, \vec{x}_1, \vec{x}_2\ \vec{y}, \vec{y}_1, \vec{y}_2 \in \mathbb{R}^n, \qquad \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

$< \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2, \vec{y} > = \alpha_1, < \vec{x}_1, \vec{y} > + \alpha_2 < \vec{x}_1, \vec{y} >$

$< \vec{x}, \beta_1\vec{y}_1 + \beta_2\vec{y}_2 > = \beta_1 < \vec{x}, \vec{y}_1 > + \beta_2 < \vec{x}, \vec{y}_2 >$

Symmetry:

$< \vec{x}, \vec{y} > = < \vec{y}, \vec{x} > \forall\, \vec{x}, \vec{y} \in \mathbb{R}^n$

Positivity

$< \vec{x}, \vec{x} > \geq 0\ \forall x \in \mathbb{R}^n$ with equality iff $x = \vec{0}$

1.3 Proposition

Cauchy-Schwarz inequality (C-S)

$|< \vec{x}, \vec{y} >| \leq \|\vec{x}\| \times \|\vec{y}\|, \qquad \forall \vec{x}, \vec{y} \in \mathbb{R}^n$

1.5 Corollary (Triangle Inequality) (T)

$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|, \forall \vec{x}, \vec{y} \in \mathbb{R}^n$

1.6 Remark (Homogeneity) (H)

$\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|, \forall \alpha \in \mathbb{R}, \vec{x} \in \mathbb{R}^n$

1.7 Distance

For $\vec{x} = (x^{(1)}, \dots, x^{(m)})$ and $y = (y^{(1)}, \dots, y^{(m)})$ in $\mathbb{R}^m$ define the Euclidian distance between $\vec{x}$ and $\vec{y}$ to be

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \sqrt{\sum_{i=1}^{n} (x^{(i)} - y^{(i)})^2}$$

1.8 Corollary (TT)

$d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}), \forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$

1.9 Ball

For $\vec{a} \in \mathbb{R}^n$ and $r > 0$ denote

$B(\vec{a};\ r) := \{\vec{x} \in \mathbb{R}^n | d(\vec{a}, \vec{x}) < r\}$ - Open Ball

$\bar{B}(\vec{a};\ r) := \{\vec{x} \in \mathbb{R}^n | d(\vec{a}, \vec{x}) \leq r\}$ - Closed Ball

1.11 Notation

For $\vec{x} = (x^{(1)}, \dots, x^{(n)}) \in \mathbb{R}^n$

1-Norm of $\vec{x}$

$$\|\vec{x}\|_1 := \sum_{i=1}^{n} |x^{(i)}|$$

∞-Norm of $\vec{x}$

$$\|\vec{x}\|_\infty := \max(|x^{(1)}|, \dots, |x^{(n)}|)$$

Proof of Cauchy-Schwarz inequality

If $\vec{y} = 0$ then get $0 = 0$

Will assume $\vec{y} \neq \vec{0}$ hence that $\|\vec{y}\| > 0$

Define $f : \mathbb{R} \to \mathbb{R}\ by$

$f(t) = < \vec{x} - t\vec{y}, \vec{x} - t\vec{y} >, \forall t \in \mathbb{R}$

Observe that $f(t) \geq 0, \forall t \in \mathbb{R}$ (By positivity of inner product)

On the other hand, use the bilinearity property to get:

$f(t) = < \vec{x}, \vec{x} > - < t\vec{y}, \vec{x} > - < \vec{x}, t\vec{y} > + < t\vec{y}, t\vec{y} >$

$\quad = \|\vec{x}\|^2 - 2t < \vec{x}, \vec{y} > + t^2\|\vec{y}\|^2$

$\quad = a + bt + ct^2$

So f is a quadratic function such that $f(t) \geq 0\ \forall t \in \mathbb{R}$

For such f, the discriminant $\Delta = b^2 - 4ac$ must satisfy $\Delta \leq 0$

But what is Δ?

$\Delta = b^2 - 4ac = 4(< \vec{x}, \vec{y} >)^2 - 4 \times \|\vec{x}\|^2 \times \|\vec{y}\|^2$

So

$\Delta \leq 0 \Rightarrow (< \vec{x}, \vec{y} >)^2 \leq \|\vec{x}\|^2 \times \|\vec{y}\|^2$

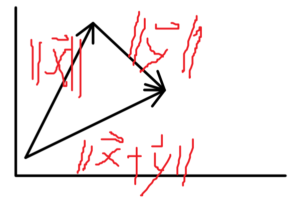
$\Rightarrow |< \vec{x}, \vec{y} >| \leq \|\vec{x}\| \times \|\vec{y}\|$

QED

1.4 Exercise

Determine the cases when C-S holds with equality.

Comment about Triangle Inequality in $\mathbb{R}^2$



Proof of 1.5 Corollary

$\|\vec{x} + \vec{y}\|^2 = < \vec{x} + \vec{y}, \vec{x} + \vec{y} >$

$= < \vec{x}, \vec{x} > + < \vec{x}, \vec{y} > + < \vec{y}, \vec{x} > + < \vec{y}, \vec{y} >$

$= \|\vec{x}\|^2 + 2 < \vec{x}, \vec{y} > + \|\vec{y}\|^2$

$(C - S) \leq \|\vec{x}\|^2 + 2\|\vec{x}\|^2\|\vec{y}\|^2 + \|\vec{y}\|^2 = (\|\vec{x}\| + \|\vec{y}\|)^2$

$\|\vec{x} + \vec{y}\|^2 \leq \|\vec{x}\| + \|\vec{y}\|$

QED

Proof of 1.6 Remark

$\|\alpha\vec{x}\| = \sqrt{< \alpha\vec{x}, \alpha\vec{x} >} = \sqrt{\alpha^2 < \vec{x}, \vec{x} >} = |\alpha|\|\vec{x}\|$

Immediate consequence of (H): every vector $\vec{x} \neq 0$ in $\mathbb{R}^n$ can be written uniquely in the form $\vec{x} = r \times \vec{u}$ where $r > 0$ and $u \in \mathbb{R}^n$ has $\|u\| = 1$ (u is a unit vector)

Proof of 1.8 Corollary

$d(\vec{x}, \vec{z}) = \|\vec{x} - \vec{z}\| = \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| = d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

1.10 Exercise

Let $\vec{x} = (x^{(1)}, \dots, x^{(n)})$ be in $\mathbb{R}^n$. Prove that:

a) $|x^{(i)}| \leq \|\vec{x}\|, \forall\ 1 \leq i < n$

b) $\|\vec{x}\| \leq \sum_{i=1}^{n} |x^{(i)}|$

Solution - by immediate algebra

Sequences in $\mathbb{R}^n$

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2.1 Sequences in $\mathbb{R}^n$

$(\vec{x}_k)_{k=1}^{\infty} = \vec{x}_1, \vec{x}_2, \dots, \vec{x}_k, \dots$
$\vec{x}_k \in \mathbb{R}^n, \vec{a} \in \mathbb{R}^n$

Say that $(\vec{x}_k)_{k=1}^{\infty}$ converges to $\vec{a}$ when the following happens:

$\forall \epsilon > 0, \exists k_0 \in \mathbb{N}$ *such that* $\|\vec{x}_k - \vec{a}\| < \epsilon \ \forall k \geq k_0$

Note:
Can also say
$d(\vec{x}_k, \vec{a}) < \epsilon$, *or* $\vec{x}_k \in B(\vec{a}, \epsilon)$, *instead of* $\|\vec{x}_k - \vec{a}\| < \epsilon$

2.2 Cauchy Sequences in $\mathbb{R}^n$

$(\vec{x}_k)_{k=1}^{\infty}$ *sequence in* $\mathbb{R}^n$
Say that $(\vec{x}_k)_{k=1}^{\infty}$ is a Cauchy sequence when the following happens:
$\forall \varepsilon > 0 \ \exists k_0 \in \mathbb{N}$ *such that* $\|\vec{x}_p - \vec{x}_q\| < \varepsilon \ \forall p, q \geq k_0$

2.3 Component Sequences

$(\vec{x}_k)_{k=1}^{\infty}$ sequence in $\mathbb{R}^n$
Write explicitly
$\vec{x}_k = (x_k^{(1)}, x_k^{(2)}, \dots, x_k^{(n)})$

We get sequences in $\mathbb{R}$
$\left(x_k^{(i)}\right)_{k=1}^{\infty}$ *for* $1 \leq i \leq n$
They are called the component sequences of $(\vec{x}_k)_{k=1}^{\infty}$
Conversely, with n sequences in $\mathbb{R}$ you can assemble them to make a sequence in $\mathbb{R}^n$

2.4 Proposition

$(\vec{x}_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$, $\vec{a} \in \mathbb{R}$. Then
$\vec{x}_k \to \vec{a}$ *in* $\mathbb{R}^n$
$\Leftrightarrow$
$x_k^{(l)} \to a^{(i)}$ *in* $\mathbb{R} \ \forall \ 1 \leq i \leq n$

2.5 Proposition

$(\vec{x}_k)_{k=1}^{\infty}$*sequence in* $\mathbb{R}^n$. Then
$(\vec{x}_k)$ is Cauchy in $\mathbb{R}^n$
$\Leftrightarrow$
$\left(x_k^{(i)}\right)_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$

2.6 Cauchy Theorem in $\mathbb{R}^n$

Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$.
Then $(\vec{x}_k)_{k=1}^{\infty}$ is convergent (to some limit $\vec{a} \in \mathbb{R}^n$) iff it is a Cauchy sequence.

2.7 Bounded Sequences in $\mathbb{R}^n$

Say that a sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ is bounded when $\exists \ r > 0$ such that $\|\vec{x}_k\| \leq r, \forall k \in \mathbb{N}$.

Note:
Can write $\|\vec{x}_k\| = \|\vec{x}_k - \vec{0}\| = d(\vec{x}_k, \vec{0})$
So $\|\vec{x}_k\| \leq r \Leftrightarrow d(\vec{x}, \vec{0}) \leq r \Leftrightarrow \vec{x} \in \bar{B}(\vec{0}; r)$

2.8 Proposition

Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$. Then
$(\vec{x}_k)_{k=1}^{\infty}$ is bounded in $\mathbb{R}^n$
$\Leftrightarrow$
Each of the component sequences $\left(x_k^{(1)}\right)_{k=1}^{\infty}, \dots, \left(x_k^{(n)}\right)_{k=1}^{\infty}$ is bounded in $\mathbb{R}$

2.9 Bolzano-Weierstrass Theorem in $\mathbb{R}^n$

Let $(\vec{x}_k)_{(k=1)}^{\infty}$ be a bounded sequence in $\mathbb{R}^n$. Then we can find indices $1 \leq k(1) < k(2) < \dots < k(p) < \dots$
Such that the subsequence $\left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$ is convergent.

That is, every bounded sequence has a convergent sub-

Will do $\mathbb{R}^n$ versions of two important theorems from MATH 147: Cauchy, and Bolzano-Weierstrass

Remark about Def 2.1
For $(\vec{x}_k)_{k=1}^{\infty}$ *in* $\mathbb{R}^n, \vec{a} \in \mathbb{R}^n$ have
$x_k \to_{k\to\infty} \vec{a} \Leftrightarrow \|\vec{x}_k - \vec{a}\| \to_{k\to\infty} 0$

Proof of proposition 2.4

$\Rightarrow$
Know $\vec{x}_k \to \vec{a}$ *in* $\mathbb{R}$
Want to know that $x_k^{(i)} \to a^{(i)} \ \forall 1 \leq i \leq n$
Fix i. Observe that for all $k \geq 1$
$0 \leq \left|x_k^{(i)} - a^{(i)}\right| = \left|(\vec{x}_k - \vec{a})^{(i)}\right| \leq \|\vec{x}_k - \vec{a}\| \to 0$
By squeeze, $\left|x_k^{(i)} - a^{(i)}\right| \to 0 \Rightarrow x_k^{(i)} \to a^{(i)}$

$\Leftarrow$
Know $x_k^{(i)} \to a^{(i)}$ *in* $\mathbb{R} \ \forall \ 1 \leq i \leq n$. So have
$\left|x_k^{(i)} - a^{(i)}\right| \to 0, 1 \leq i \leq n$

$$\sum_{i=1}^{n} \left|x_k^{(i)} - a^{(i)}\right| \to 0$$

By exercise 1.10(b)

$$0 \leq \|\vec{x}_k - \vec{a}\| \leq \sum_{i=1}^{n} \left|x_k^{(i)} - a^{(i)}\right| \to 0$$

Hence $\|\vec{x}_k \to \vec{a}\| \to 0$ by squeeze and hence $\vec{x}_k \to \vec{a}$

Proof of 2.6 (Cauchy Theorem)

$(\vec{x}_k)_{(k=1)}^{\infty}$ convergent in $\mathbb{R}^n$
$\Leftrightarrow$
Each of $\left(x_k^{(i)}\right)_{k=1}^{\infty}$ is convergent in $\mathbb{R}$
$\Leftrightarrow$
Each of $\left(x_k^{(i)}\right)_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$
$\Leftrightarrow$
$(\vec{x}_k)_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}^n$
$\mathcal{QED}$

2.8 Proof

Left as exercise

Proof of Lemma 2.11

$(\vec{y}_k)_{k=1}^{\infty}$ convergent in $\mathbb{R}^n \Rightarrow \left(x_k^{(i)}\right)_{k=1}^{\infty}$ converges $\forall \ 1 \leq i \leq n$
$(t_k)_{k=1}^{\infty}$ is convergent $\Rightarrow \left(x_k^{n+1}\right)_{k=1}^{\infty}$ is convergent.
So have $\left(x_k^{(i)}\right)_{k=1}^{\infty}$ is convergent for every $1 \leq i \leq n+1$
Using reverse direction for Proposition 2.4 to conclude
$(\vec{x}_k)_{k=1}^{\infty}$ is convergent in $\mathbb{R}^{n+1}$

Proof of Theorem 2.9 (Bolzano-Weierstrass)

By induction on n.
Base case n=1. This is the B-W theorem from Math 147
Induction. Assume the statement is true for n.
Let $(\vec{x}_k)_{k=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^{n+1}$. For every k write $\vec{x}_k = (\vec{y}_k, t_k)$ with $\vec{y}_k \in \mathbb{R}^n$ and $t_k \in R$

Claim 1
$(\vec{y}_k)_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^n$
$(t_k)_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$
This follows from discussion about components of bounded sequences.

Claim 2
Can find an infinite set of indices $Q \subseteq \mathbb{N}$ such that the subsequence $(\vec{y}_k)_{k \in Q}$ is convergent in $\mathbb{R}^n$
Why? The induction hypothesis which says that B-W holds in $\mathbb{R}^n$

Claim 3
Let Q be as in Claim 2. Can find infinite subset $P \subseteq Q$ such that $(t_k)_{k \in P}$ is convergent in $\mathbb{R}$.
We invoke the B-W theorem from Math 147 to the sequence $(t_k)_{k \in Q}$

Claim 4

indices $1 \le k(1) < k(2) < \cdots < k(p) < \cdots$
Such that the subsequence $\left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$ is convergent.

That is, every bounded sequence has a convergent sub-sequences.

2.10 Remarks

1.
For $n = 1$ this is the Bolzano-Weierstrass from MATH 147.
Here we want to prove that the same results holds in $\mathbb{R}^n$ for every n. We will do this by induction on n.

2.
Notation: Subsequences and sub-subsequences of a sequence.
Given a sequence $(\vec{x}_k)_{(k=1)}^{\infty}\ in\ \mathbb{R}^n$. Subsequences of $(\vec{x}_k)_{k=1}^{\infty}$ are of the form $\left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$.
Giving a subsequence is equivalent to giving an infinite subset $P = \{k(1), k(2), \dots, k(p), \dots\} \subseteq \mathbb{N}$
Instead of $\left(\vec{x}_{k(p)}\right)_{(p=1)}^{\infty}$ it is convenient to write $(\vec{x}_k)_{k\in P}$

With this notation, taking a sub-subsequence amounts to dropping from $(\vec{x}_k)_{k\in P}$ to $(\vec{x}_k)_{k\in Q}$ where $Q \subseteq P$ is an infinite set.

3.
Note that if $\vec{x}_k \to \vec{a}$ in $\mathbb{R}^n$ then for any subsequence we will have $\vec{x}_{k(p)} \to_{p\to\infty} \vec{a}$

2.11 Lemma: Inductive Convergence

$(\vec{x}_k)_{k=1}^{\infty}$ sequence $in\ \mathbb{R}^{n+1}$
For every k can write $\vec{x}_k = (\vec{y}_k, t_k)$ with $\vec{y}_k \in \mathbb{R}^n, t_k \in \mathbb{R}$

If $(\vec{y}_k)_{k=1}^{\infty}$ converges in $\mathbb{R}^n$ and if $(t_k)_{k=1}^{\infty}$ converges in $\mathbb{R}$ then $(\vec{x}_k)_{k=1}^{\infty}$ converges in $\mathbb{R}^{n+1}$

We invoke the B-W theorem from Math 147 to the sequence $(t_k)_{k\in Q}$

Claim 4

Let $P \subseteq \mathbb{N}$ be the set of indices from Claim 3. Then the subsequence of $(\vec{x}_k)_{k\in P}$ is convergent in $\mathbb{R}^{n+1}$
Why? We have $\vec{x}_k = (\vec{y}_k, t_k), \forall k \in P$
Have $\vec{y}_k \to \vec{b} \in \mathbb{R}^n, t_k \to s \in \mathbb{R} \Rightarrow \vec{x}_k \to (\vec{b}, s) \in \mathbb{R}^{n+1}$
■

Open and Closed subsets of $\mathbb{R}^n$

September-21-11 11:30 AM

A. Open and Closed

3.1 Definitions

Let A be a subset of $\mathbb{R}^n$

1. A vector $\vec{a} \in$ A is said to be an **interior point** of A when $\exists$ r > 0 such that $B(\vec{a}; r) \subseteq A$

   The set of all interior points of A is called the **interior of A** denoted as $int(A)$
2. A vector $\vec{b} \in \mathbb{R}^n$ is said to be **adherent** to A when it has the property that $B(\vec{b}; r) \cap A \neq \emptyset,\ \forall r > 0$

   The set of all adherent points of A is called the **closure** of A, denoted by $cl(A)$

3.2 Proposition

$A \subseteq R^n, \vec{b} \in \mathbb{R}^n$. Then
$\vec{b} \in cl(A)$
$\Leftrightarrow$
$\exists\ sequence\ (\vec{x}_k)_{k=1}^{\infty}$ in A such that $\vec{x}_k \to \vec{b}$

3.3 Remark and Definition

For every $A \subseteq R^n$ have $int(A) \subseteq A \subseteq cl(A)$

The set-difference $cl(A) \setminus int(A)$ called the **boundary** of A, denoted as $bd(A)$

3.4 Definition

A set $A \subseteq \mathbb{R}^n$ said to be **open** when it satisfied $A = int(A)$
A set $A \subseteq \mathbb{R}^n$ said to be **closed** when it satisfies $A = cl(A)$

Warning

Most subsets $A \subseteq R^n$ are neither open nor closed. So A not open does not imply that A is closed.

3.6 Definition

Say that $A \subseteq \mathbb{R}^n$ has the **"no-escape"** property when the following happens:
Whenever $(\vec{x}_k)_{k=1}^{\infty}$ is a sequence in A such that $\vec{x}_k \to \vec{b} \in \mathbb{R}^n$ then $\vec{b}$ must also belong to A.

3.7 Proposition

For $A \subseteq \mathbb{R}^n$ have $(A\ is\ closed) \Leftrightarrow (A\ has\ the'no-escape'property)$

Proof: Exercise.

3.8 Remark

1. For every $A \subseteq R^n$ have that $int(A)$ is open. Moreover $int(A)$ is the largest possible open set which sites inside A.
2. For every $A \subseteq \mathbb{R}^n$ we have that $cl(A)$ is closed, and in fact it is the smallest possible closed set which contains A.

Proof: in homework

Proof of Proposition 3.2

"⇒" Know $\vec{b} \in$ cl(A).
Then for every $k \in \mathbb{N}$ have $B\left(\vec{b}; \frac{1}{k}\right) \cap A \neq 0$, hence pick $\vec{x}_k \in B\left(\vec{b}, \frac{1}{k}\right) \cap A$. This way we get a sequence in A such that $\|\vec{x}_k - \vec{b}\| < \frac{1}{k}, \forall k \geq 1$
Have $\|\vec{x}_k - \vec{b}\| \to_{k\to\infty} 0$ by squeeze, hence $\vec{x}_k \to \vec{b}$

" ⇐ " Know $\exists(\vec{x}_k)_{k=1}^{\infty}$ in A such that $\vec{x}_k \to \vec{b}$
Let $r > 0$. Since $\vec{x}_k \to \vec{b}$ can find $k_0 \in \mathbb{N}$ such that $\|\vec{x}_k - \vec{b}\| < r, \forall k \geq k_0$
In particular have $\|\vec{x}_{k_0} - \vec{b}\| < r \Rightarrow \vec{x}_{k_0} \in B(\vec{b}; r) \cap A$
So $B(\vec{b}; r) \cap A \neq \emptyset$, and done. QED

3.3 Remark

$int(A) \subseteq A$, by definition of $int(A)$
$A \subseteq cl(A)$
For every $\vec{a} \in$ A can find sequence $(\vec{x}_k)_{(k=1)}^{\infty}$ in A such that $\vec{x}_k \to \vec{a}$. Just take $\vec{x}_k = \vec{a}, \forall k \geq 1$

3.4 Example

Say $n = 2$, let A $= \{(s,t): s,t \in \mathbb{R}, t > 0\} \cup \{(s,0): s \in \mathbb{R}, s \geq 0\}$

Then $int(A) = \{(s,t): s,t \in \mathbb{R}, t > 0\}$
For $\vec{x} = (s,t)$ with $t > 0$, can find $r > 0$ such that $B(\vec{x}; r) \subseteq A$. E.g. take $r = \frac{t}{2}$
But $\vec{x} = (x, 0)$ is not interior to A - there is no $r > 0$ such that $B(\vec{y}, r) \subseteq A$

$cl(A) = \{(s,t): x,t \in \mathbb{R}, t \geq 0\}$
$bd(A) = int(A) \setminus cl(A) = \{(s,0): x \in \mathbb{R}\}$

Compact subsets of $\mathbb{R}^n$

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B Compact Sets

3.9 Definition

A subset $A \subseteq \mathbb{R}^n$ is said to be **bounded** when $\exists r > 0$ such that $\|\vec{x}\| \leq r, \forall \vec{x} \in A$

Note

"$\|\vec{x}\| \leq r, \forall \vec{x} \in A$" is equivalent to saying that $A \subseteq \bar{B}(\vec{0}; r)$. Could also use an open ball; pick $r' > r$ then have $\|\vec{x}\| < r', \forall \vec{x} \in A$ hence $A \subseteq B(\vec{0}, r')$

3.10 Definition

A subset $A \subseteq \mathbb{R}^n$ is said to be **compact** when it is both closed and bounded.

Note

There are several equivalent descriptions of compactness (Some of them extend to spaces more general than $\mathbb{R}^n$ - see PMath 351)

3.11 Definition

A subset A $\subseteq R^n$ is said to be **sequentially compact** when the following happens:
For every sequence $(\vec{x}_k)_{k=1}^{\infty}$ in A, one can find a convergent subsequence $\left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$ such that the limit $\vec{a} = \lim_{p \to a} \vec{x}_{k(p)}$ still belongs to A

3.12 Theorem

For $A \subseteq \mathbb{R}^n$ have that A is compact iff A is sequentially compact.

C Duality Open $\leftrightarrow$ Closed

Via taking complements

3.13 Duality interior vs. closure

For every $A \subseteq \mathbb{R}^n$ have that
$int(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus cl(A)$
$cl(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus int(A)$

3.14 Corollary (Duality open vs. closed)

For $A \subseteq \mathbb{R}^n$ have (A is closed) $\Leftrightarrow$ ($\mathbb{R}^n \setminus A$ is open)

3.15 Remark

We have one description for what it means that $A \subseteq \mathbb{R}^n$ is open.
$A\ open \Leftrightarrow A = int(A) \Leftrightarrow every\ \vec{a} \in A\ is\ an\ interior\ point\ of\ A$

We have three equivalent descriptions for what it means that $A \subseteq \mathbb{R}^n$ is closed:

1. $A = cl(A)$ (by Definition 3.5.2)
2. A has the "no-escape" property (Proposition 3.7)
3. $\mathbb{R}^n \setminus A$ is an open set (Corollary 3.14)

Proof of Theorem 3.12

"$\Rightarrow$" Know that A is closed and bounded. Let $(\vec{x}_k)_{k=1}^{\infty}$ be a sequence in A.
A is bounded $\Rightarrow (\vec{x}_k)_{k=1}^{\infty}$ is a bounded sequence $\Rightarrow \exists \left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$ convergent.
Denote the $\lim_{p\to\infty} \vec{x}_{k(p)} =: \vec{a} (\in \mathbb{R}^n)$
Since A is closed, it has the no escape property, therefore $\vec{a} \in$ A

" $\Leftarrow$ "
Know A is sequentially compact. Want to prove that A is closed and bounded.
This is problem 7 in homework 2.
QED

Note

Theorem 3.12 is part of a theorem of **Heine-Borel**

Proof of Proposition 3.13

Will do first equality, second can by done by similar argument or the 2nd can be deduced using the first.

So prove the first equality
" $\subseteq$ "
Take $\vec{b} \in$ int($\mathbb{R}^n \setminus$ A). So $\exists r > 0$ s.t. $B(\vec{b}; r) \in \mathbb{R}^n \setminus A$
But then $B(\vec{b}; r) \cap A = \emptyset$ and it follows that $\vec{b}$ is not adherent to A. Hence $\vec{b} \notin$ cl(A). Hence $\vec{b} \in \mathbb{R}^n \setminus cl(A)$

" $\supseteq$ "
Take $\vec{b} \in \mathbb{R}^n \setminus$ cl(A) $\Rightarrow \vec{b} \notin$ cl(A) $\Rightarrow \vec{b}$ is not adherent to A.
From Def 3.1.2 it follows that $\exists r > 0$ such that $B(\vec{b}; r) \cap A = \emptyset$
But if $B(\vec{b}; r) \cap A = \emptyset$, then must have the $B(\vec{b}; r) \subseteq \mathbb{R}^n \setminus A$
Finally from $B(\vec{b}; r) \subseteq \mathbb{R}^n \setminus A$ we conclude that $\vec{b} \in$ int($\mathbb{R}^n \setminus$ A)
QED for first formula

Proof of Corollary 3.14

" $\Rightarrow$ "
A closed $\Rightarrow cl(A) = A \Rightarrow int(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus cl(A) = \mathbb{R}^n \setminus A$
$\Rightarrow \mathbb{R}^n \setminus A$ is open (it is equal to its interior)

" $\Leftarrow$ "
$\mathbb{R}^n \setminus A$ is open $\Rightarrow int(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus A$
$\Rightarrow \mathbb{R}^n \setminus cl(A) = \mathbb{R}^n \setminus A \Rightarrow cl(A) = A$ (by taking complements again)
$\Rightarrow$ A is closed
■

Continuous Functions

September-26-11 11:30 AM

4.1 Definition

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}^m, A \neq \emptyset$

1. Let $\vec{a} \in$ A. Say that A is **continuous** at $\vec{a}$ when the following happens:
   $\forall \varepsilon > 0, \exists \delta > 0\ s.t. \|f(\vec{x}) - f(\vec{a})\| < \varepsilon\ \forall x \in A\ with\ \|\vec{x} - \vec{a}\| < \delta$
2. Let B be a subset of A. Say that f is **continuous on B** when f is continuous at every $\vec{a} \in$ B
   Note
   In particular, may have B=A, get definition for "f is continuous on A"

4.2 Remark

Given $\varepsilon > 0$ have to find $\delta > 0$ such that
$f\big(B(\vec{A}; \delta) \cap A\big) \subseteq B(f(\vec{a}); \varepsilon)$

4.3 Definition

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}^m, \vec{a} \in A$. Say that f **respects sequences** in A which converge to $\vec{a}$ when the following happens:
Whenever $(\vec{x}_k)_{k=1}^{\infty}$ is a sequence in A such that $\vec{x}_k \to_{k\to\infty} \vec{a}$ it follows that $f(\vec{x}_k) \to_{k\to\infty} f(\vec{a})$

4.4 Proposition

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}^m, \vec{a} \in A$ Then
f respects sequences in A which converge to $\vec{a}$
$\Leftrightarrow$
f is continuous at $\vec{a}$

4.5 Definition

$A \subseteq \mathbb{R}^n, f: A \to R^m$. For every $\vec{a} \in A$, write explicitly $f(\vec{a}) = \left(f^{(1)}(\vec{a}), \dots, f^{(m)}(\vec{a})\right)$
Get n functions $f^{(1)}, \dots, f^{(m)} : A \to \mathbb{R}$

For $1 \le j \le m$, the function $f^{(j)}: A \to \mathbb{R}$ is called the **j-th component** of f

4.6 Proposition

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}^m, \vec{a} \in A$
f is continuous at $\vec{a}$
$\Leftrightarrow$
Each of the component functions $f^{(1)}, \dots, f^{(m)}$ is continuous at $\vec{a}$

Proof of Proposition 4.4

" $\Rightarrow$ " Know f respects sequences convergent at $\vec{a}$
Want f satisfies $\varepsilon - \delta$ at $\vec{a}$
So fix an $\varepsilon > 0$. Need to prove that $\exists \delta > 0$ such that
$(*)\ \|f(x) - f(a)\| < \varepsilon\ \forall \vec{x} \in A\ s.t. \|\vec{x} - \vec{a}\| < \delta$
Assume by contradiction that I cannot find a $\delta > 0$ such that (*) holds. So no matter what $\delta > 0$ I try, (*) will fail.

Try $\delta = 1$, and it fails.
Hence $\exists \vec{x}_1 \in A\ s.t. \|\vec{x}_1 - \vec{a}\| < 1$, but nevertheless $\|f(\vec{x}_1) - f(\vec{a})\| \ge 1$
For each $k \in \mathbb{N}$, take $\delta = \frac{1}{k}$, and it fails. $\exists x_k \in A\ s.t. \|\vec{x}_k - \vec{a}\| < \frac{1}{k}$ but nevertheless $\|(f(\vec{x}_k) - f(\vec{a})\| \ge \varepsilon$

Observe that in this way we get a sequence $(\vec{x}_k)_{k=1}^{\infty}$ in A where $\|\vec{x}_k - \vec{a}\| < \frac{1}{k} \forall k \in \mathbb{N} \Rightarrow \|\vec{x}_k - \vec{a}\| \to_{k\to\infty} 0, hence\ \vec{x}_k \to_{k\to\infty} \vec{a}$
And yet $\|f(\vec{x}_k) - f(\vec{a})\| \ge \varepsilon, \forall k \in N\ hence\ \|f(\vec{x}_k) - f(\vec{a})\|! \to 0, f(\vec{x}_k)! \to f(\vec{a})$.
So f does not respect the sequence $\vec{x}_k \to \vec{a}$, contradiction with the hypothesis.

Hence the assumption that there is no delta for which (*) works leads to contradiction. Hence $\exists \delta$. Done with " $\Rightarrow$ "

Proof of " $\Leftarrow$ "
Exercise, on homework 3

Proof of Proposition 4.6

f continuous at $\vec{a}$
$\Leftrightarrow$
f respects sequences in A which converge to $\vec{a}$
$\Leftrightarrow (*)$
Each of $f^{(1)}, \dots, f^{(m)}$ respects sequences in A which converge to $\vec{a}$
$\Leftrightarrow$
Each of $f^{(1)}, .., f^{(m)}$ is continuous at $\vec{a}$

(*)
Take $(\vec{x}_k)_{k=1}^{\infty}$ in A such that $\vec{x}_k \to \vec{a}$
For every $k \in \mathbb{N}$, write $f(\vec{x}_k) = \left(f^{(1)}(\vec{x}_k), \dots, f^{(m)}(\vec{x}_k)\right)$
Know from prop 2.4 that $f(\vec{x}_k) \to f(\vec{a})$ iff $f^{(j)}(\vec{x}_k) \to f^{(j)}(\vec{a})$ $\forall 1 \le j \le m$

Uniform Continuity

September-28-11 11:30 AM

5.1 Remark

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}^m$

Suppose we want to discuss at the same time the continuity of f at several points of A: $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_p \in A$

Have $\varepsilon > 0, \forall 1 \leq k \leq p$ we find $\delta_k > 0 \; s.t. x \in A \; \|\vec{x} - \vec{a}_k\| < \delta_k \Rightarrow \|f(\vec{x}) - f(\vec{a}_k)\| < \varepsilon$

To find a single delta which works for all $\vec{a}_k$ take $\delta \coloneqq \min\{\delta_k : 1 \leq k \leq p\} > 0$ and works $\forall \vec{a}_k$

But what happens if we did this for infinitely many points in A at the same time, or all the points of A.

Here we can't always find a $\delta > 0$ good for all $\vec{a}'s$ at the same time.

5.2 Uniform Continuity

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}^m$.

Say that f is **uniformly continuous** on A when the following happens:

$\forall \varepsilon > 0, \exists \delta > 0 : \; \|f(\vec{x}) - f(\vec{a})\| < \varepsilon \; \forall \vec{x}, \vec{a} \in A : \|\vec{x} - \vec{a}\| < \delta$

5.4 Proposition

Let $A \subseteq \mathbb{R}^n$ be a compact set.

Let $f: A \to \mathbb{R}^m$ be a function. If f is continuous on A then f is uniformly continuous on A

5.5 Definition

$B \subseteq A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}^m, function$

We say that f is **uniformly continuous** on B when the restriction of F to B is uniformly continuous

$\forall \varepsilon > 0, \exists \delta > 0 \; s.t. \left\|f(\vec{x}) - f(\vec{b})\right\| < \varepsilon \; \forall \vec{x}, \vec{b} \in B \; s.t. \left\|\vec{x} - \vec{b}\right\| < \delta$

Please us this definition in Problem 6(a) of Homework 3

5.3 Example

f continuous on A, but not uniformly continuous on A

Let $A = (0,1) \times (0,1) \subseteq \mathbb{R}^2$

$f: A \to \mathbb{R}, \qquad f\big((s,t)\big) = \frac{s}{t}$

Observe that f is continuous at every $\vec{a} = (s,t) \in A$

Indeed, check with sequences. Suppose $\vec{x}_k \to \vec{a}$ where $\vec{x}_k = (s_k, t_k) \in A$

Then $s_k \to s, t_k \to t$

Take ratio of convergent sequence as in Calc 1, get

$\frac{s_k}{t_k} \to \frac{s}{t}$

Hence $f(\vec{x}_k) \to f(\vec{a})$

So have that f is continuous on A

Claim: But fi is not uniformly continuous on A

Opponent gives $\varepsilon = \frac{1}{2}$

Can I find $\delta > 0 \; s.t.$

$|f(\vec{x}) - f(\vec{a})| < \frac{1}{2} \; \forall \vec{x}, \vec{a} \in A \; with \; \|\vec{x} - \vec{a}\|$

Assume $\exists \delta$ which satisfies the above.

Consider the sequence $(\vec{x}_k)_{k=1}^{\infty}$ in A where $x_k = \left(\frac{1}{k}, \frac{1}{k^2}\right) \; \forall k \geq 2$

Note that $\|\vec{x}_k - \vec{x}_{k+1}\| \to 0$

Hence $\exists k_0 \in \mathbb{N}$ s.t. $\|\vec{x}_k - \vec{x}_{k+1}\| < \delta \; \forall k \geq k_0$. In particular $\left\|\vec{x}_{k_0} - \vec{x}_{k_0+1}\right\| < \delta$ so it should follow that $\left|f\left(\vec{x}_{k_0}\right) - f\left(\vec{x}_{k_0+1}\right)\right| < \frac{1}{2}$

But $f\left(\vec{x}_{k_0}\right) = \frac{\frac{1}{k_0}}{\frac{1}{k_0^2}} = k_0$. Similarly $f\left(\vec{x}_{k_0+1}\right) = k_0 + 1$

$\left|f\left(\vec{x}_{k_0}\right) - f\left(\vec{x}_{k_0+1}\right)\right| = |k_0 - k_0 - 1| = 1 < \frac{1}{2}$

Contradiction, coming from the assumption that δ exists.

Proof of Proposition 5.4

Given $\varepsilon > 0$, Want to find $\delta > 0$ s.t.

$\vec{x}, \vec{a} \in A, \|\vec{x} - \vec{a}\| < \varepsilon \Rightarrow \|f(\vec{x}) - f(\vec{a})\| < \varepsilon$

Assume by contradiction that no such δ exists.

Pick $k \in \mathbb{N}$, use $\delta = \frac{1}{k}$. We can find $\vec{a}_k, \vec{x}_k$ in A such that $\|\vec{x}_k - \vec{a}_k\| < \frac{1}{k}$ but nevertheless $\|f(\vec{x}_k) - f(\vec{a}_k)\| \geq \varepsilon$. In this way we find two sequences in A, $(\vec{x}_k)_{k=1}^{\infty}$ and $(\vec{a}_k)_{k=1}^{\infty}$ is compact and hence sequentially compact. So can find $1 \leq k(1) < k(2) < \cdots < k(p) < \cdots$ such that $\left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$ converges to a limit $\vec{x}_0 \in A$

Claim: For the same $1 \leq k(1) \leq k(2) \leq \cdots \leq k(p) \leq \cdots$ we have that $\lim_{p \to \infty} \vec{a}_{k(p)} = \vec{x}_0$

For every $p \in \mathbb{N}$ write $\left\|\vec{a}_{k(p)} - \vec{x}_0\right\| \leq \left\|\vec{a}_{k(p)} - \vec{x}_{k(p)}\right\| + \left\|\vec{x}_{k(p)} - \vec{x}_0\right\| \leq \frac{1}{k(p)} + \left\|\vec{x}_{k(p)} - \vec{x}_0\right\| \to 0 + 0 = 0$

So by squeeze, $\left\|\vec{a}_{k(p)} - \vec{x}_0\right\| \to 0$. Done claim

Now, f is continuous at $\vec{x}_0$ so it respects $\vec{x}_{k(p)} \to \vec{x}_0$ and $\vec{a}_{k(p)} \to \vec{x}_0$. So $f\left(\vec{x}_{k(p)}\right) \to f(\vec{x}_0)$ and $f\left(\vec{a}_{k(p)}\right) \to f(\vec{x})$

$\left\|f\left(\vec{x}_{k(p)}\right) - f\left(\vec{a}_{k(p)}\right)\right\| \leq \left\|f\left(\vec{x}_{k(p)}\right) - f(\vec{x}_0)\right\| + \left\|f(\vec{x}_0) - f\left(\vec{a}_{k(p)}\right)\right\| \to 0 + 0 = 0$

Contradiction with construction of $\vec{x}_k, \vec{a}_k$ which said $\left\|f\left(\vec{x}_{k(p)}\right) - f\left(\vec{a}_{k(p)}\right)\right\| \geq \varepsilon \; \forall p \in \mathbb{N}$

So assumption that I cannot find a δ leads to contradiction. It remains that we can find δ.

QED

Extreme Value Theorem

September-30-11 12:05 PM

Supremum / Infemum

This is about global minimum and maximum of a continuous function on a compact set. Will use the concepts $\inf(A)$ and $\sup(A)$ for a bounded nonempty subset $A \subseteq \mathbb{R}$.
$\inf(A)$ = smallest possible limit of a sequence in A
$\sup(B)$ = largest possible limit of a sequence in A

Have that $\inf(A)$ is the greatest lower bound (GLB) for A
  i) $\inf(A) \leq a, \forall a \in A$
  ii) If $\alpha \in \mathbb{R}$ has the property that $\alpha \leq a, \forall a \in A$, then it follows that $\inf(A) \geq \alpha$

$\sup(A)$ is the lowest upper bound (LUB) for A

Note:

For a general bounded set A, $\inf A$ and $\sup A$ may or may not belong to A

6.1 Remark

$K \subseteq \mathbb{R}$ a nonempty compact set.
Then K is bounded, hence can talk about $\alpha = \inf K$ *and* $\beta = \sup K$. We are certain that $\alpha, \beta \in K$
(Why? Because K is closed so it has "no-escape" property for sequences.

6.2 Definition

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}^m$

1. The **image** of f is the set $f(A) = \{\vec{y} \in \mathbb{R}^m | \exists \vec{x} \in A\ s.t.\ f(\vec{x}) = \vec{y}\}$
2. We say that f is **bounded in A** if f(A) is a bounded subset of $\mathbb{R}^m$. Equivalently, this means that $\exists r > 0$ s.t. $\|f(\vec{x})\| \leq r\ \forall \vec{x} \in A$

6.3 Remark and Notation (special case m=1)

$A \subseteq \mathbb{R}^m, f: A \to \mathbb{R}$. Then
f is bounded $\Leftrightarrow \exists r > 0\ s.t. |f(\vec{x})| \leq r\ \forall \vec{x} \in A$
Here $f(A)$ is a bounded subset of $\mathbb{R}$

So we can talk about inf and sup of the set $F(A) \subseteq \mathbb{R}^n$. We abbreviate them as follows:

$$\inf_A(f) := \inf\{f(\vec{x}) | \vec{x} \in A\}$$
$$\sup_A(f) := \sup\{f(\vec{x}) | \vec{x} \in A\}$$

Also, use the notation for the **oscillation of f on A**

$$\mathrm{osc}_A(f) := \sup_A(f) - \inf_A(f)$$

6.4 Definition

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}$ a bounded function.
An element $\vec{a} \in A$ is said to be:

- A **global minimum** for f on A when $f(\vec{a}) = \inf_A(f)$
- A **global maximum** for f on A when $f(\vec{a}) = \sup_A(f)$

Note

A bounded function f on A may or may not have a global min/max and if it does, then it may have one or several.

6.6 Theorem (EVT)

$A \subseteq \mathbb{R}^n$ compact, $f: A \to \mathbb{R}$ continuous. Then f is bounded, and has at least one point of global max and at least one point of global min.

We will derive EVT from the following fact (important on its own)

6.7 Proposition

$A \subseteq \mathbb{R}^n$ compact, $f: A \to \mathbb{R}^m$ continuous. Then the image set $f(A) \subseteq \mathbb{R}^m$ is a compact set of $\mathbb{R}^m$.

6.5 Example

$A = (0,1) \times (0,1) \subseteq \mathbb{R}^2$
$f: A \to \mathbb{R}$ defined by $f\big((s,t)\big) = |s - t|\ \forall 0 < s, t < 1$
$f(A) = [0,1)$ hence $\inf_A(f) = 0, \sup_A(f) = 1$
f has many points of global min: all points (s, s) with $0 < s < 1$
But f has no points of global max. There is no point $\vec{a} \in$ A such that $f(\vec{a}) = 1$

Proof of Proposition 6.7

Denote $f(A) = B \subseteq \mathbb{R}^m$
We will verify that B is sequentially compact (know this this is equivalent to compact - Theorem 3.12)
So let us fix a sequence $(\vec{y}_k)_{k=1}^{\infty}$ in B. Have to prove that $(\vec{y}_k)_{k=1}^{\infty}$ has a convergent subsequence with limit still in B.
For every $k \in \mathbb{N}$ have $\vec{y}_k \in B = f(A)$, hence can find $\vec{x}_k \in A$ s.t. $f(\vec{x}_k) = \vec{y}_k$

A is compact by hypothesis, hence it is sequentially compact. So we can find $1 \leq k(1) < k(2) < \cdots < k(p) < \cdots$ s.t. $\vec{x}_{k(p)} \to \vec{a} \in A$
Function f is continuous on A, hence respects convergent sequences in A, so have $f\big(\vec{x}_{k(p)}\big) \to f(\vec{a}) \Rightarrow \vec{y}_{k_p} \to f(\vec{a}) = \vec{b} \in$ B
So we have found a convergent subsequence $\big(\vec{y}_{k(p)}\big)_{p=1}^{\infty}$ of $(\vec{y}_k)_{k=1}^{\infty}$ which converges to a value of B. QED

Proof of Proposition 6.6 (EVT)

Have $A \subseteq \mathbb{R}^m$ compact, $f: A \to \mathbb{R}$ continuous
Want: f is bounded, and $\exists \vec{a}_1, \vec{a}_2 \in A\ s.t.\ f(\vec{a}_1) \leq f(\vec{x}) \leq f(\vec{a}_2)\ \forall \vec{x} \in A$

Denote $f(A) = K \subseteq \mathbb{R}$
Then K is compact by proposition 6.7
So we can talk about $\alpha = \inf(K)$, $\beta = \sup(K)$ and moreover $\alpha, \beta \in K$ (By Remark 6.1)
Since $\alpha, \beta \in K = f(A)$ we can find $\vec{a}_1, \vec{a}_2 \in A\ s.t.\ f(\vec{a}_1) = \alpha, f(\vec{a}_2) = \beta$
But then for every $\vec{x} \in$ A we can write $f(\vec{x}) \in K$
$\alpha \leq f(\vec{x}) \leq \beta \Rightarrow f(\vec{a}_1) \leq f(\vec{x}) \leq f(\vec{a}_2)\ \forall \vec{x} \in A$
QED

Integration Intro

October-05-11 11:32 AM

Goal

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}$ Want to associate to f a real number, called the integral of f on A denoted

$$\int_A f(\vec{x})d\vec{x}$$

What kind of $A \subseteq \mathbb{R}^n$? A will be a bounded subset of $\mathbb{R}^n$
What kind of f? f will be in any case a bounded function. But need more conditions.
Case of f continuous, but will also allow some discontinuities.

Historical Note

- Idea that a continuous function has an integral - Cauchy (~1820)
- Concept of integrable function - Riemann (~1850)

Rectangles and their divisions

October-05-11 11:41 AM

We prefer half-open rectangles

7.1 Definition

We call a **half-open rectangle** in $\mathbb{R}^n$ a set of the form
$P = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]$ where $a_i < b_i\ \forall 1 \le i \le n$, and are in $\mathbb{R}$
$P = \{\vec{x} \in \mathbb{R}^n | a_i < x^{(i)} \le b_i\ \forall 1 \le i \le n\}$

For $P = \prod_{i=1}^{n} (a_i, b_i]$ we denote $vol(P) = \prod_{i=1}^{n} (b_i - a_i)$
$diam(P) = \sup\{\|\vec{x} - \vec{y}\| | \vec{x}, \vec{y} \in P\} = \|\vec{b} - \vec{a}\|$
where $\vec{a} = (a_1, a_2, \ldots, a_n)$,$\vec{b} = (b_1, b_2, \ldots, b_n)$

7.2 Notation and Remark

We denote by $\mathcal{P}_n$ the collection of all half-open rectangles in $\mathbb{R}^n$
Note: $\mathcal{P}_n$ is a set of sets

$P \in \mathcal{P}_n$ means P is a half-open rectangle

Note that
$P, Q \in \mathcal{P}_n, P \cap Q \ne \emptyset \Rightarrow P \cap Q \in \mathcal{P}_n$

Exercise: Verify this by algebra.

7.3 Definition

Let $P \in \mathcal{P}_n$. By a **division of P** we understand a set $\Delta = \{P_1, P_2, \ldots, P_r\}$ of half-open rectangles such that
$\bigcup_{i=1}^{r} P_i = P$ and $P_i \cap P_j = \emptyset\ \forall i \ne j$

Notation

$\|\Delta\| = \max(diam(P_i), 1 \le i \le r)$

7.4 Remark

Special case of division: **grid divisions**.
$P = \prod_{i=1}^{n} (a_i, b_i] \in \mathcal{P}_n$
A grid division of P is obtained by decomposing each $(a_i, b_i]$ and then taking the Cartesian products
$(a_i, b_i] = \bigcup_{j=1}^{r_i} J_j^{(i)} = J_1^{(i)}, J_2^{(i)}, \ldots, J_{r_i}^{(i)}, \qquad J_j^{(i)} = \left(x_j^{(i)}, y_j^{(i)}\right] \in \mathbb{R}$
Then P is divided into $r = r_1 r_2 \ldots r_m$ rectangles of the form
$J_{i_1}^{(1)} \times J_{i_2}^{(2)} \times \cdots \times J_{i_n}^{(n)}$
with $1 \le i_1 \le r_1, 1 \le i_m \le r_m$

7.5 Definition

$P \in \mathcal{P}_m$ and let $\Delta = \{P_1, \ldots, P_r\}, \Gamma = \{Q_1, \ldots, Q_s\}$ be divisions of O
Say that Γ refines Δ (denote $\Gamma \prec \Delta$)
When for every $1 \le j \le s$ there exists $1 \le i \le r$such that $Q_j \subseteq P_i$

7.6 Remark

If $\Gamma \prec \Delta$ then can write $\Gamma = \{Q_{11}, Q_{1s_1}, \ldots, Q_{r1}, Q_{r2}, \ldots, Q_{rs_r}\}$
Where $\{Q_{i1}, \ldots, Q_{is_1}\}$ is a division of P_i

7.7 Remark

Let $\Delta = \{P_1, \ldots, P_r\}$ be any division of P. One can find a grid-division Γ such that $\Gamma \prec \Delta$
Proof: Exercise
Geometric idea: extend lines of division for each sub-rectangle.

7.8 Proposition

Let $\mathcal{P} \in \mathcal{P}_n$ and let Δ', Δ'' be two divisions of $\mathcal{P}$. Then one can find a division Γ of $\mathcal{P}$ such that $\Gamma \prec \Delta'$ and $\Gamma \prec \Delta''$.
Say that Γ is a common refinement for Δ' and Δ''

7.9 Remark

$P \in \mathcal{P}_n, \Delta = \{P_1, \ldots, P_r\}$ is a division of P then
$\sum_{i=1}^{r} vol(P_i) = vol(P)$

Proof of Proposition 7.8

Write $\Delta' = \{P_1', P_2,', \ldots, P_r'\}$
$\Delta'' = \{P_1'', P_2'', \ldots, P_s''\}$
Put $\Gamma = \{P_i' \cap P_j'' | 1 \le i \le r, 1 \le j \le s, where\ P_i' \cap P_i'' \ne \emptyset\}$
So Γ consists of some q number of half-open rectangles, where $q \le r \times s$
Have that Γ is a division of $\mathcal{P}$.
Verification is by immediate Boolean algebra. Exercise.

We observe that $\Gamma \prec \Delta'$. Indeed every rectangle $\mathcal{P}_i' \cap \mathcal{P}_j''$ of Γ is included in a rectangle of Δ', namely $\mathcal{P}_i' \cap P_j'' \subseteq P_i'$. Same argument with $P_i' \cap P_j''$ gives $\Gamma \prec \Delta''$ QED

Proof of Remark 7.9

What do we so if Δ is not a grid division?
If $\Delta = \{P_1, \ldots, P_r\}$ is not a grid division then refine it to a grid division $\Gamma = \{Q_1, \ldots, Q_s\}$ then reduce
$\sum_{d=1}^{s} vol(Q_i) = vol(P)$ to $\sum_{i=1}^{r} vol(P_i)$ by suitably grouping terms

Definition of Integral

October-07-11 11:53 AM

Riemann integral → ~1850
We will use Darboux sums → ~1870

8.1 Definition

$\mathcal{P} \in \mathcal{P}_n$ Let $f: \mathcal{P} \to \mathbb{R}$ be a bounded function. Let $\Delta = \{P_1, \dots, P_r\}$ be a division of $\mathcal{P}$
Then the **upper Darboux sum** for f and Δ is

$$U(f,\Delta) = \sum_{i=1}^{r} vol(P_i) \times \sup_{P_i}(f)$$

And the **lower Darboux sum** for f and Δ is

$$L(f,\Delta) = \sum_{i=1}^{r} vol(P_i) \times \inf_{P_i}(f)$$

8.2 Remark

$\mathcal{P}, f, \Delta$ as defined above.

$$U(f,\Delta) - L(f,\Delta) = \sum_{i=1}^{r} vol(P_i) \left(\sup_{P_i} f - \inf_{P_i} f\right)$$

$$= \sum_{i=1}^{r} vol(P_i) \times osc_{P_i}(f) \geq 0$$

8.3 Lemma

$P \in \mathcal{P}_n, f: P \to \mathbb{R}$ bounded function.
Suppose Δ, Γ are divisions of P such that $\Gamma \prec \Delta$
Then $U(f,\Gamma) \leq U(f,\Delta)$ and $L(f,\Gamma) \geq L(f,\Delta)$
$\Rightarrow U(f,\Gamma) - L(f,\Gamma) \leq U(f,\Delta) - L(f,\Delta)$

8.4 Proposition

$P \in \mathcal{P}_n, f: P \to \mathbb{R}$ bounded function. Let Δ', Δ''be two divisions. Then $L(f,\Delta') \leq U(f,\Delta'')$

8.5 Remark

$P \in \mathcal{P}_n, f: P \to \mathbb{R}$ bounded. Consider the following set of real numbers:
$S = \{s \in \mathbb{R} | \exists division\, \Delta\, of\, L(f,\Delta) = s\}$
$T = \{t \in \mathbb{R} | \exists division\, \Delta\, of\, P\, with\, U(f,\Delta) = t\}$
Then Prop 8.4 says that $s \leq t\ \forall s \in S, \forall t \in T$
Make some observations from here:

a) S is bounded above (since every $t \in T$ is an upper bound for S)
   Hence can talk about $\sup(S)$
   Observe that $\sup(S) \leq t, \forall t \in T$ (since t is some upper bound for S, while $\sup(S)$ is the smallest upper bound for S
b) T is bounded below (e.g. $\sup(S)$) is a lower bound for T. Hence can consider $\inf(T)$, and will have $\inf(T) \geq \sup(S)$

Have $\sup S \leq \inf T$
When can this hold with equality?
Some equivalent conditions for this:

1. $\sup S = \inf T$
2. $\forall \varepsilon > 0\ \exists s \in S\ and\ t \in T\ s.t.\ s - t < \varepsilon$
3. $\exists$ sequences $(s_k)_{k=1}^{\infty}$ in S and $(t_k)_{k=1}^{\infty}$ in T such that $t_k - s_k \to 0$

Exercise

Now recall that we had $S = \{s \in S | \exists\, division\, \Delta\, of\, P\, with\, L(f,\Delta) = s\}$
Hence $\sup(S) = \sup\{L(f,\Delta) | \Delta\, division\, of\, P\}$
Likewise
$\inf(T) = \inf\{U(f,\Delta) | \Delta\, division\, of\, P\}$

8.6 Definition

$P \in \mathcal{P}_n, f: P \to \mathbb{R}$ bounded function

- Two number $\sup\{L(f,\Delta) | \Delta\, division\, of\, P\}$ is called the **lower integral** of f, denoted
  $$\underline{\int_P} f \ \ or\ \ \underline{\int_P} f(\vec{x}) d\vec{x}$$
- The number $\inf\{U(f,\Delta) | \Delta\, division\, of\, P\}$ is called the **upper integral** of f, denoted
  $$\overline{\int_P} f \ or\ \overline{\int_P} f(\vec{x}) d\vec{x}$$

8.7 Proposition

$P \in \mathcal{P}_n, f: P \to \mathbb{R}\ bounded$
Then $l\int_P f \leq u \int_P f$ Moreover, the following are equivalent:

1) $l\int_P f = u\int_P f$

Proof of Lemma 8.3

Will show the inequality for U. L is similar.

Write $\Delta = \{P_1, \dots, P_r\}, \Gamma = \{Q_{1,1}, \dots, Q_{1,s_1}, \dots, Q_{r,1}, \dots, Q_{r,s_r}\}$ where $Q_{i,1} \cup \dots \cup Q_{i,s_i} = P_i\ \forall i$
For every $1 \leq i \leq r$ and $1 \leq j \leq s_i$ have that $\sup_{Q_{i,j}}(f) \leq \sup_{P_i}(\mathrm{f})$ This is just because $Q_{i,j} \subseteq P_i$

Then write

$$U(f,\Gamma) = \sum_{i=1}^{r} \left(\sum_{j=1}^{s_i} vol(Q_{i,j}) \cdot \sup_{Q_{i,j}}(f) \right) \leq \sum_{i=1}^{r} \left(\sum_{j=1}^{s_i} vol(Q_{i,j}) \sup_{P_i}(f) \right)$$

$$= \sum_{i=1}^{r} \left(\sum_{j=1}^{s_i} vol(Q_{i,j}) \right) \sup_{P_i}(f) = \sum_{i=1}^{r} vol(P_i) \cdot \sup_{P_i}(f) = U(f,\Delta)$$

QED

Proof of Proposition 8.4

Can find division Γ of P such that $\Gamma \prec \Delta'$ and $\Gamma \prec \Delta''$ (from Lecture 7, prop 7.8)
Then $L(f,\Delta') \leq L(f,\Gamma) \leq U(f,\Gamma) \leq U(f,\Delta'')$
Lemma 8.3, Remark 8.2, Lemma 8.3
QED

Proof of Proposition 8.7

The inequality $l\int_P f \leq u\int_P f$ is just the inequality $\sup S \leq \inf T$ from remark 8.5
The equivalent conditions 1, 2, 3, are suitable re-writings of the "(inf=sup)" equivalences in remark 8.5
However, condition 2 from (inf=sup) says less. It says $\exists s \in S, t \in T$ with $t - s < \varepsilon$
That is, $\exists \Delta', \Delta''$ divisions of P such that $U(f,\Delta'') - L(f,\Delta') < \varepsilon$
But then let Δ be a division of P such that $\Delta \prec \Delta', \Delta \prec \Delta''$. Then have $U(f,\Delta) \leq U(f,\Delta'')$ and $L(f,\Delta \geq L(f,\Delta'') \Rightarrow U(f,\Delta) - L(f,\Delta) \leq U(f,\Delta'') - L(f,\Delta') < \varepsilon$
This is how 2 is fixed. Same for 3.

Proof of Proposition 8.9

Denote $I \coloneqq \int_P f$
$Have\ I = l\int_P f\ = \sup\{L(f,\Delta) | \Delta\, division\, of\, P\}$
Hence $I \geq L(f,\Delta_k), \forall k \geq 1$
Likewise
$I = u\int_P f = \inf\{U(f,\Delta) | \Delta\, division\, of\, P\}$
$\Rightarrow I \leq U(f,\Delta k), \forall k \geq 1$

So have $L(f,\Delta_k) \leq I \leq U(f,\Delta_k), \forall k \geq 1$
Then $|I - L(f,\Delta_k)| = I - L(f,\Delta_k) \leq U(f,\Delta_k) - L(f,\Delta_k) \to 0$
So $|I - L(f,\Delta_k)| \to 0$ hence $L(f,\Delta_k) \to I$

Also $U(f,\Delta_k) = L(f,\Delta_k) + \big(U(f,\Delta_k) - L(f,\Delta_k)\big) \to I + 0 = 0$
QED

2) For every $\varepsilon > 0$ there exists a division Δ of P with $U(f,\Delta) - L(f,\Delta) < \varepsilon$
3) There exists a sequence of divisions $(\Delta_k)_{k=1}^{\infty}$ of P such that $U(f,\Delta_k) - L(f,\Delta_k) \to 0$

8.8 Definition

$P \in \mathcal{P}_n, f: P \to \mathbb{R}$ bounded.
If $l \int_P f = u \int_P f$ then we say that f is **integrable on P** and we define its integral to be the common value of $l \int_P f$, $u \int_P f$. Notation:

$$\int_P f$$

8.9 Proposition

$P \in \mathcal{P}_n, f: P \to \mathbb{R}$ bounded, **integrable.** Suppose $(\Delta_k)_{k=1}^{\infty}$ is a sequence of divisions of P such that $U(f,\Delta_k) - L(f,\Delta_k) \to 0$
Then we have

$$\lim_{k\to\infty} U(f,\Delta_k) = \int_P f = \lim_{k\to\infty} L(f,\Delta_k)$$

Linear Combinations of Integrable Functions

October-14-11 11:54 AM

9.1 Remark

$A \subseteq \mathbb{R}^n, \quad f, g: A \to \mathbb{R}, bounded$
Consider the sum $h = f + g$
$h: A \to \mathbb{R}, \quad h(\vec{x}) = f(\vec{x}) + g(\vec{x}), \forall \vec{x} \in A$
Have that $\sup_A h \leq \sup_A f + \sup_A g$ and $\inf_A h \geq \inf_A f + \inf_A g$

9.2 Lemma

$P \in \mathcal{P}_n, \quad f, g: P \to \mathbb{R}$ bounded
Consider the sum $f + g$. Then for every division Δ of P we have
$U(f + g, \Delta) \leq U(f, \Delta) + U(g, \Delta)$ and $L(f + g, \Delta) \geq L(f, \Delta) + L(g, \Delta)$

9.3 Proposition

$P \in \mathcal{P}_n, \quad f, g: P \to \mathbb{R}$ bounded, integrable.
Then $f + g$ is also bounded and integrable, and has $\int_P f + g = \int_P f + \int_P g$

9.4 Remark

$P \in \mathbb{P}, f: P \to \mathbb{R}$ bounded, integrable.
Then αf is bounded and integrable and has $\int_p \alpha f = \alpha \int_P f$

9.5 Theorem

$P \in \mathcal{P}_n$. Let $Int_b(P, \mathbb{R}) = \{f: p \to \mathbb{R} | f$ bounded and integrable$\}$
Then $Int_b(P, \mathbb{R})$ is closed under linear combinations and the map
$Int_b(P, \mathbb{R}) \to \mathbb{R}: f \mapsto \int_P f$ is linear

Question

What about $f \cdot g$, for $f, g \in Int_b(P, \mathbb{R})$

9.6 Lemma

If $f \in Int_b(P, \mathbb{R})$ then $f^2 \in Int_b(P, \mathbb{R})$
Where $f^2: P \to \mathbb{R}$ is defined by $(f^2)(\vec{x}) = \left(f(\vec{x})\right)^2 \ \forall \vec{x} \in P$

9.7 Proposition

$f, g \in Int_b(P, \mathbb{R}) \Rightarrow f \cdot g \in Int_b(P, \mathbb{R})$
Where $f \cdot g: P \to \mathbb{R}$ defined by $(f \cdot g)(\vec{x}) = f(\vec{x}) \cdot g(\vec{x}), \vec{x} \in P$

Remark 9.1

Addition

$\alpha := \sup_A(f) = \sup\{ f(\vec{x}) | \vec{x} \in A\}$
$\beta := \sup_A(g) = \sup\{g(\vec{x}) | \vec{x} \in A\}$
For every $x \in A$ have $h(\vec{x}) = f(\vec{x}) + g(\vec{x}) \geq \alpha + \beta$
So $\alpha + \beta$ is an upper bound for $\{h(\vec{x}) | \vec{x} \in A\}$. Hence we have
$\sup_A(h) \leq \alpha + \beta = \sup_A(f) + \sup_A(g)$
Same for inf

Proof of Lemma 9.2

Write $\Delta = \{P_1, \dots, P_r\}$. Then

$$U(f + g, \Delta) = \sum_{i=1}^{r} vol(P_i) \cdot \sup_{P_i}(f + g) \leq \sum_{i=1}^{r} vol(P_i) \left(\sup_{P_i}(f) + \sup_{P_i}(g)\right)$$

$$= \left(\sum_{i=1}^{r} vol(P_i) \sup_{P_i}(f)\right) + \left(\sum_{i=1}^{r} vol(P_i) \sup_{P_i}(g)\right) = U(f, \Delta) + U(g, \Delta)$$

Inequality for $L(f + g, \Delta)$ done in the same way.

Proof of Proposition 9.3

Use the integrability criterion for f and for g.
Get sequences $(\Delta'_k)_{k=1}^{\infty}$ and $(\Delta''_k)_{k=1}^{\infty}$ of divisions of P such that
$U(f, \Delta'_k) - L(f, \Delta'_k) \to 0$ and $U(g, \Delta''_k) - L(g, \Delta''_k) \to 0$
For every $k \geq 1$ let Δ_k be a division of P such that $\Delta_k \prec \Delta'_k, \Delta_k \prec \Delta''_k$
Then also have
$U(f, \Delta_k) - L(f, \Delta_k) \to 0$ and $U(g, \Delta_k) - L(g, \Delta_k) \to 0$
For every $k \geq 1$ have
$U(f + g, \Delta_k) \leq U(f, \Delta_k) + U(g, \Delta_k)$ and $L(f + g, \Delta_k) \geq L(f, \Delta_k) + L(g, \Delta_k)$
$U(f + g, \Delta_k) - L(f + g, \Delta_k) \leq U(f, \Delta_k) - L(f, \Delta_k) + U(g, \Delta_k) - L(g, \Delta_k)$
$\to 0 + 0 = 0$
So by squeeze, $U(f + g, \Delta_k) - L(f + g, \Delta_k) \to 0$ so $f + g$ is integrable.

Moreover, Prop 8.9 says that
$\int_P f + g = \lim_{k \to \infty} U(f + g, \Delta_k) = \lim_{k \to \infty} L(f + g, \Delta_k)$
$U(f + g, \Delta_k) \leq U(f, \Delta_k) + U(g, \Delta_k)$
$L(f + g, \Delta_k) \geq L(f, \Delta_k) + L(g, \Delta_k)$

But then just make $k \to \infty$
$\int_P f + \int_P g \leq \int_P f + g \leq \int_P f + \int_P g$
So get
$\int_P f + g = \int_P f + \int_P g$ as claimed QED

9.4 Remark

$P \in \mathcal{P}, f: P \to \mathbb{R}$ bounded. Let $\alpha \in \mathbb{R}$, consider new function αf
$(\alpha f: P \to \mathbb{R}$ defined by $(\alpha f)(\vec{x}) = \alpha f(\vec{x}) \ \forall \vec{x} \in P)$
αf is bounded (immediate)

Have 3 cases: $\alpha > 0, \alpha = 0, \alpha < 0$

Case 1:
For every division Δ of P get $U(\alpha f, \Delta) = \alpha U(f, \Delta)$ and $L(\alpha f, \Delta) = \alpha L(f, \Delta)$
Take infimum of U's and Supremum of L's.
$u \int_P \alpha f = \alpha \left(u \int_P f \right), l \int_P \alpha f = \alpha \left(l \int_P f \right)$
In particular, if f is integrable then αf is integrable as well with
$\int_P \alpha f = \alpha \int_P f$

Case 2:
Have $\alpha f = 0$. αf is integrable with $\int \alpha f = 0$

Case 3:
For every division Δ of P have
$U(\alpha f, \Delta) = \alpha L(f, \Delta)$
$L(\alpha f, \Delta) = \alpha L(f, \Delta)$
Problem 4. a) in homework 5. Have there case $\alpha = -1$. General $\alpha < 0$ is treated in the same way.

This implies further that
$u \int_P \alpha f = \alpha \left(l \int_P f \right)$ and $l \int_P \alpha f = \alpha \left(u \int_P f \right)$
If f is integrable, still get conclusion that αf is integrable with
$\int_p \alpha f = \alpha \int f$

Proof of Theorem 9.5

Statement amounts to 2 things:

1) If $f, g \in Int_b(P, \mathbb{R})$ then $f+g \in Int_b(P, \mathbb{R})$ and $\int_P f+g = \int_P f + \int_P g$
   This is Proposition 9.3
2) If $f \in Int_b(P, \mathbb{R})$ and $\alpha \in \mathbb{R}$ then $\alpha f \in Int_b(P, \mathbb{R})$ and $\int_P \alpha f = \alpha \int_P f$
   This is Proposition 9.4

Proof of Lemma 9.6

f^2 bounded - immediate
Take $r > 0$ s.t. $|f(\vec{x})| \leq r, \forall \vec{x} \in P$
Then $|f^2(\vec{x})| \leq r^2, \forall \vec{x} \in P$
But why is f^2 integrable?

Recall that if $\Delta = \{P_1, \dots, P_r\}$ is a division of P then
$$U(f, \Delta) - L(f, \Delta) = \sum_{i=1}^{r} vol(P_i) \cdot osc_{P_i}(f)$$

Claim 1

Let $r > 0$ be such that $|f(\vec{x})| \leq r, \forall \vec{x} \in P$
Then for every $\emptyset \neq A \subseteq P$ we have $osc_A(f^2) \leq 2r \cdot osc_A(f)$

Verification of Claim 1

Denote $\omega \coloneqq osc_A(f)$
Have $\omega \coloneqq \sup_{\vec{x}, \vec{y} \in A} |f(\vec{x}) - f(\vec{y})|$
In particular, have that $|f(\vec{x}) - f(\vec{y})| \leq \omega \forall \vec{x}, \vec{y} \in A$
But then for $\vec{x}, \vec{y} \in A$ write
$|(f^2)(\vec{x}) - (f^2)(\vec{y})| = \left|\left(f(\vec{x})\right)^2 - \left(f(\vec{y})\right)^2\right| = \left|\left(f(\vec{x}) - f(\vec{y})\right)\left(f(\vec{x}) + f(\vec{y})\right)\right|$

Proof of Proposition 9.7

$(f+g)^2 = f^2 + 2fg + g^2$
$$\Rightarrow f \cdot g = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$$
$f+g \in Int_b(P, \mathbb{R})$ by 9.3
$(f+g)^2, f^2, g^2 \in Int_b(P, \mathbb{R})$ by 9.6
$\left((f+g)^2 - f^2 - g^2\right) \in Int_b(P, \mathbb{R})$ by 9.3
$f \cdot g = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right) \in Int_b(P, \mathbb{R})$ by 9.4
QED

Integrals Respect Inequalities

October-21-11 11:18 AM

10.1 Remark

$P \in \mathcal{P}_n, f: P \to \mathbb{R}$ bounded.
Suppose that $\alpha, \beta \in \mathbb{R}$ such that $\alpha \le f(\vec{x}) \le \beta \; \forall \vec{x} \in P$
Then for every division $\Delta = \{P_1, \dots, P_r\}$ of P we get

$$U(f,\Delta) = \sum_{i=1}^{r} vol(P_i) \cdot \sup_{P_i}(f) \le \beta vol(P) \text{ and}$$

$$L(f,\Delta) = \sum_{i=1}^{r} vol(P_i) \cdot \inf_{P_i}(f) \ge \alpha \, vol(P).$$

Then $\alpha \, vol(P) \le l \int_P f \le u \int_P f \le \beta vol(P)$. In particular, if f is integrable $\alpha \cdot vol(P) \le \int_P f \le \beta vol(P)$.
This is like a "mean value theorem"

$$\alpha \le \frac{1}{vol(P)} \int_P f \le b$$

10.2 Proposition

$P \in \mathcal{P}_n$, let $f, g: P \to \mathbb{R}$ be bounded, integrable functions such that $f(\vec{x}) \le g(\vec{x}) \; \forall \vec{x} \in P \; (f \le g)$. Then $\int_P f \le \int_P g$

10.3 Proposition

$P \in \mathcal{P}_n, f: P \to \mathbb{R}$ bounded, integrable. Consider $|f|: P \to \mathbb{R}$ defined by $|f|(\vec{x}) = |f(\vec{x})|$.
Then $|f|$ is bounded and integrable and $\left|\int_P f\right| \le \int_P |f|$

Implications of Remark 10.1

If $f(\vec{x}) = c \; \forall \vec{x}$, let $\alpha = \beta = c$. Then f is integrable with $\int_P f = c \cdot vol(P)$
If f is non-negative let $\alpha = 0$, then Assuming f is integrable $\alpha \cdot vol(P) \le \int_P f \Rightarrow 0 \le \int_P f$

Proof of Propositions 10.2

Let $h(\vec{x}) = g(\vec{x}) - f(\vec{x})$. h is bounded and integrable and $\int_P h = \int_P g - \int_P f$
Since $g \ge f$, h is non-negative, so $\int_P h \ge 0$. Hence $\int_P g \ge \int_P f$.

Proof of Proposition 10.3

Verification of bounded f integrable will be on homework.
Similar to proof of 9.6
$-|f|(\vec{x}) = -|f(\vec{x})| \le |f(\vec{x})| = |f|(\vec{x}) \; \forall \vec{x} \in P$ So $-|f| \le f \le |f|$ by prop 10.2

$$\int_P -|f| \le \int_P f \le \int_P |f| \Rightarrow -\int_P -|f| \le \int_P f \le \int_P |f| \Rightarrow \left|\int_P f\right| \le \int_P |f|$$

Integrals over more general domains in $\mathbb{R}$

October-21-11 11:31 AM

11.2 Lemma

$P, Q \in \mathcal{P}_n$ such that $Q \subseteq P$

Let $g: Q \to \mathbb{R}$ and let $f: P \to \mathbb{R}$ be defined by $f(\vec{x}) = \begin{cases} g(\vec{x}), & \vec{x} \in Q \\ 0, & \vec{x} \notin Q \end{cases}$

Then we have that
g is bounded and is integrable on Q
$\Leftrightarrow$
f is bounded and integrable on P
Moreover, if these conditions hold then have $\int_Q g = \int_P f$

11.3 Definition and Proposition

Let $A \subseteq \mathbb{R}^n$ be a (nonempty and) bounded set, and let $f: A \to \mathbb{R}$ be a bounded function. Pick a half-open rectangle $P \in \mathcal{P}_n$ such that $P \supseteq A$ and extend f to a function:
$\tilde{f}: P \to \mathbb{R}$ defined by

$$\tilde{f}(\vec{x}) = \begin{cases} f(\vec{x}) \ if \ \vec{x} \in A \\ 0 \ if \ \vec{x} \in \frac{P}{A} \end{cases}$$

Then it makes sense to declare:
f is integrable on A
$\Leftrightarrow$
$\tilde{f}$ is integrable on P
Moreover, if f is integrable them it makes sense to declare

$$\int_A f := \int_P \tilde{f}$$

11.4 Notation

$A \subseteq \mathbb{R}^n$ is bounded
Denote $Int_b(A, \mathbb{R}) = \{f: A \to \mathbb{R} \mid f \text{ is bounded and integrable}\}$

11.5 Theorem

$A \subseteq \mathbb{R}$, bounded. Then the set of functions $Int_b(A, \mathbb{R})$ is closed under linear combinations, and have

$$\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g \ \forall f, g \in Int_b(A, \mathbb{R}), \forall \alpha, \beta \in \mathbb{R}$$

11.6 Remark

Other properties of the integral also go through in the same way.

- $f, g \in Int_b(A, \mathbb{R}) \Rightarrow f \cdot g \in Int_b(A, \mathbb{R})$
- $f \in Int_b(A, \mathbb{R}) \Rightarrow |f| \in \mathrm{Int}_b(A, \mathbb{R})$ and $\left|\int_A f\right| \leq \int_A |f|$

11.7 Remark

$A \subseteq \mathbb{R}^n$ bounded, let $f: A \to \mathbb{R}$ be defined by $f(\vec{x}) = 1 \ \forall \vec{x} \in A$
Can we be sure that $f \in Int_b(A, \mathbb{R})$?

Say e.g. $n = 2$ and $A = \{(s,t) \in \mathbb{R}^2 \mid 0 < s, t < 1 \ \ s, t \in \mathbb{Q}\}$
Then $A \subseteq P = (0,1] \times (0,1]$
f extends to $\tilde{f}: P \to \mathbb{R}$ where

$$\tilde{f}((s,t)) = \begin{cases} 1, & (s,t) \in A \\ 0, & (s,t) \notin A \end{cases}$$

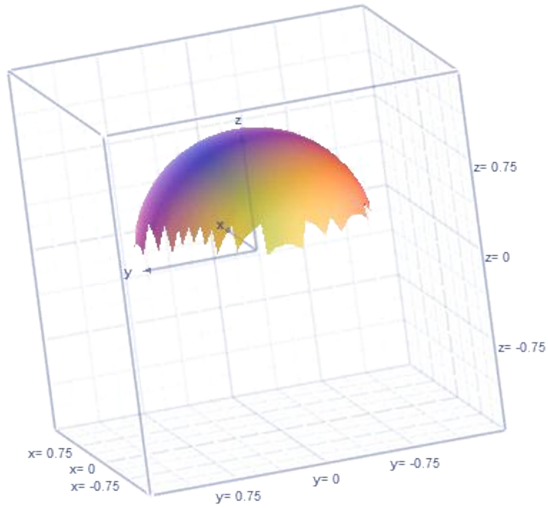
Not integrable.

What was the problem?
One way to look at it: $bd(A)$ was way too large
$bd(A) = cl(A)/int(A) = [0,1] \times [0,1]$

Will prove that things improve if we assume that $bd(A)$ is "small"

11.1 Example

$n = 2$. Look at the function f defined by the formula:
$f((s,t)) = \sqrt{1 - (s^2 + t^2)}$
$f: D \to \mathbb{R}, D = \{(s,t) \in \mathbb{R}^2 : s^2 + t^2 \leq 1\} = \bar{B}((0,0), 1)$



Range is half a sphere, Domain is not a rectangle, what do we do?

Proof of Lemma 11.2

Exercise
Direct verifications by using criterion with sequences of divisions. (Prop 8.7 and Prop 8.9)

Proof of Proposition 11.3

Why does the definition make sense?

Must verify that the definition is independent of the choice of P. So suppose that someone else picks $P_1 \in \mathcal{P}_n$ extends f to $\widetilde{f_1}: P_1 \to \mathbb{R}$ by

$$\widetilde{f_1}(x) = \begin{cases} f(\vec{x}), if \ \vec{x} \in A \\ 0 \ if \ \vec{x} \in \frac{P_1}{A} \end{cases}$$

Must verify that
$\tilde{f}$ integrable on P $\Leftrightarrow$ $\widetilde{f_1}$ integrable on P_1
Moreover, if these conditions hold then want $\int_P \tilde{f} = \int_{P_1} \widetilde{f_1}$

Denote $Q = P \cap P_1$. have $Q \in \mathcal{P}_n$ and $Q \supseteq A$
Let $f: Q \to \mathbb{R}$ be defined by

$$g(\vec{x}) = \begin{cases} f(\vec{x}), & \vec{x} \in A \\ 0, & \vec{x} \in Q/A \end{cases}$$

Observe: $Q \subseteq P$ and $\tilde{f}$ extends g with 0.
$Q \subseteq P_1$ and $\widetilde{f_1}$ extends g with 0.
Apply lemma 11.2 twice
$\tilde{f}$ integrable on P $\Leftrightarrow$
g integrable on Q $\Leftrightarrow$
$\widetilde{f_1}$ integrable on P_1
If these considerations hold then Lemma 11.2 also says

$$\int_P \tilde{f} = \int_Q g = \int_{P_1} \widetilde{f_1}$$

Proof of Theorem 11.5

Take $P \in \mathcal{P}_n$ such that $P \supseteq A$
Extend $f, g \in Int_b(A, \mathbb{R})$ to $\tilde{f}, \tilde{g} \in Int_b(P, \mathbb{R})$, then we use Theorem 9.5 for $\tilde{f}, \tilde{g}$.
QED

Integrability for Continuous Functions Modulo Null Sets

October-24-11 11:28 AM

12.1 Definition

$C \subseteq \mathbb{R}^n$ is a **null set** when the following happens:
$\forall \varepsilon > 0, \exists$ a finite family $Q_1, \dots, Q_s \in \mathcal{P}_n$ such that
$\bigcup_{i=1}^{s} Q_i \supseteq C$ and $\sum_{i=1}^{s} vol(Q_i) < \varepsilon$

12.3 Remark

In definition 12.1 there were two requirements

1) $\bigcup_{i=1}^{s} Q_i \supseteq C$
2) $\sum_{i=1}^{s} vol(Q_i) < \varepsilon$

But did not ask for

3) $Q_i \cap Q_j = \emptyset \; \forall i \neq j$

But observe that if C is a null set, then we can always arrange $Q_1, \dots, Q_s$ to also satisfy (3). This is done by refining $Q_1, \dots, Q_s$ as necessary.

11.4 Remark

These are some obvious properties satisfied by null sets

- If $C \subseteq \mathbb{R}^n$ is a null set and if $D \subseteq C$ then D is a null set as well
- If $C_1, C_2 \subseteq \mathbb{R}^n$ are null sets then $C_1 \cup C_2$ is also a null set.

Lemma ('Two Ways of Being Small')

$P \in \mathcal{P}_n, \tilde{f}: P \to \mathbb{R}$ bounded function.
Suppose that $\forall \varepsilon > 0$ we can find a division
$\Delta = \{Q_1, \dots, Q_u, R_1, \dots, R_v\}$
Such that (Way 1) + (Way 2) hold.
(Way 1):

$$\sum_{j=1}^{u} vol(Q_j) < \varepsilon$$

(Way 2):

$$osc_{R_k}(\tilde{f}) < \varepsilon \; \forall 1 \leq k \leq v$$

Then $\tilde{f}$ is integrable on P

12.5 Theorem

$A \subseteq \mathbb{R}^n$ bounded (nonempty) set such that $bd(A)$ is a null set.
Let $f: A \to \mathbb{R}$ be a bounded function.
Suppose we found $B, G \subseteq A$ (B-bad, G-good) such that

i) $B \cup G = A, B \cap G = \emptyset$
ii) f is continuous at every $\vec{x} \in$ G
iii) B is a null set

Then f is integrable on A

Note

For exam might need to know individual parts or the outline of the whole proof of the above theorem.

12.6 Corollary (Special case $B = \emptyset$)

$A \subseteq \mathbb{R}^n$ bounded with $bd(A)$ is a null set. $f: A \to \mathbb{R}$ is a bounded continuous function. Then f is integrable.

12.2 Example of Null Set

$C = \{(t,t) \in \mathbb{R}^2 : 0 \leq t \leq 1\}$
Claim: C is a null subset of $\mathbb{R}^2$
Verification of Claim: Given $\varepsilon > 0$
Pick $k \in \mathbb{N}$ s.t. $\frac{1}{k} < \frac{\varepsilon}{2}$. For $0 \leq i \leq k$ let $Q_i = \left(\frac{i-1}{k}, \frac{i}{k}\right] \times \left(\frac{i-1}{k}, \frac{i}{k}\right] \in \mathcal{P}_2$

Then $\bigcup_{i=0}^{k} Q_i \supseteq C, \quad \sum_{i=0}^{k} vol(Q_i) = \sum_{i=0}^{k} \frac{1}{k^2} = \frac{k+1}{k^2} \leq \frac{2k}{k^2} = \frac{2}{k} < \varepsilon$

Comment

This example generalizes naturally to cases when C is the graph of a p-Lipschitz function
$h: D \to \mathbb{R}^n$
$D \subseteq \mathbb{R}^m$ with $m < n$

Proof of Lemma

Use integrability criterion from Prop 8.7, in the form with ε. Given $\varepsilon > 0$ have to find a division Δ of P such that $U(\tilde{f}, \Delta) - L(\tilde{f}, \Delta) < \varepsilon$

We apply the hypothesis for a suitable $\varepsilon' > 0$.

Let $\varepsilon' = \dfrac{\varepsilon}{1 + \mathrm{vol(P)} + \mathrm{osc_P}(\tilde{f})}$

Hypothesis gives us $\Delta = \{Q_1, \dots, Q_u, R_1, \dots, R_v\}$ such that
$\sum_{i=1}^{u} vol(Q_j) < \varepsilon'$ and $osc_{R_k}(\tilde{f}) < \varepsilon' \forall 1 \leq k \leq v$
Calculate

$$U(\tilde{f}, \Delta) - L(\tilde{f}, \Delta) = \sum_{j=1}^{n} vol(Q_j) \cdot osc_{Q_j}(\tilde{f}) + \sum_{k=1}^{v} vol(R_k) \cdot osc_{P_k}(\tilde{f})$$

$$< \sum_{j=1}^{n} vol(Q_j) \cdot osc_P(\tilde{f}) + \sum_{k=1}^{v} vol(R_k) \cdot \varepsilon' < osc_P(\tilde{f})\varepsilon' + \varepsilon' vol(P) = \varepsilon' \left(osc_P(\tilde{f}) + vol(P)\right)$$

$$= \frac{\varepsilon \left(osc_P(\tilde{f}) + vol(P)\right)}{1 + vol(P) + osc_P(\tilde{f})} < \varepsilon$$

Proof of Theorem 12.5

(Using A, f, B, G as in the theorem definition)
Enclose A in a rectangle $P \in \mathcal{P}_n$ and extend f to a function $\tilde{f}: P \to \mathbb{R}$ by

$$\tilde{f}(\vec{x}) = \begin{cases} f(\vec{x}), & \vec{x} \in A \\ 0, & \vec{x} \notin A \end{cases}$$

WLOG (by enlarging P as necessary) may assume that $cl(A) \subseteq int(P)$
Consider the set $C = B \cup bd(A) \subseteq cl(A) \subseteq int(P)$
Observe C is a null set.

Claim 1:

$\tilde{f}$ is continuous at every $\vec{x} \in P \backslash C$

Verification of Claim 1:

Fix $\vec{x} \in \mathrm{P \backslash C}$. Observe that $\vec{x} \in \left(\mathrm{P \backslash cl(A)}\right) \cup \mathrm{int(A)}$ (everywhere except the boundary)

Case I: $\vec{x} \in \mathrm{P \backslash cl(A) = int(P \backslash A)}$
In this case, can find $r > 0$ such that $B(\vec{x}; r) \cap A = \emptyset$. Hence $\tilde{f} \equiv 0$ on $B(\vec{x}; r) \cap P$ and it follows that $\tilde{f}$ is continuous at $\vec{x}$

Case II: $\vec{x} \in \mathrm{int(A)}$. In this case can find $r > 0$ such that $B(\vec{x}; r) \subseteq A$. For this $r > 0$ we have that $\tilde{f}(\vec{y}) = f(\vec{y}), \forall \vec{y} \in B(\vec{x}; r)$. But observe that $\vec{x} \in$ G since $\vec{x} \notin$ C $\Rightarrow \vec{x} \notin$ B
So $\vec{x} \in G \Rightarrow f$ is continuous at $\vec{x} \Rightarrow \tilde{f}$ is continuous at $\vec{x}$
Done with claim.

Claim 2:

For every $\varepsilon > 0$ we can find some $Q_1', \dots, Q_s' \in \mathcal{P}_n$ such that $C \subseteq Q_1' \cup \dots \cup Q_s' \subseteq P$ with
$\sum_{j=1}^{s} vol(Q_j') < \varepsilon$ (Since C is a null set)
and such that $\tilde{f}$ is uniformly continuous on $P \backslash (Q_1' \cup \dots \cup Q_s{}') \subseteq P \backslash C$

Verification of Claim 2

C is a null set, $C \subseteq int(P) \Rightarrow$ can find $Q_1, \dots, Q_s \in \mathcal{P}_n$ with $C \subseteq Q_1 \cup \dots \cup Q_s \subseteq int(P)$
and such that $\sum_{j=1}^{s} vol(Q_j) < \frac{\varepsilon}{2}$
For $1 \leq j \leq s$ pick $Q_j' \in \mathcal{P}_n, Q_j' \subseteq int(P)$ such that $Q_j \subseteq int(Q_j')$ and $vol(Q_j') < 2 \cdot vol(Q_j)$
Then $C \subseteq Q_1 \cup \dots \cup Q_s \subseteq int(Q_1') \cup \dots \cup int(Q_s') \subseteq int(Q_1' \cup \dots Q_s')$ and

$$\sum_{j=1}^{s} vol(Q_i') < 2\sum_{j=1}^{s} vol(Q_j) < 2\cdot\frac{\varepsilon}{2} = \varepsilon$$

Consider the compact set $K = cl(P)\setminus int(Q_1' \cup \dots \cup Q_s')$. K is compact since $cl(P)$ is compact and removing an open set.

$\tilde{f}$ is continuous at every point of K (where for $\vec{y} \in cl(P)\setminus$P we put $f(\vec{y}) = 0$) By claim 1.

Since it is continuous at point in a compact set K, f is uniformly continuous on K

Therefore, f is continuous on $P\setminus C = P\setminus int(Q_1' \cup \dots \cup Q_s') \subseteq K$

Claim 3

Given $\varepsilon > 0$ can find a division $\Delta = \{Q_1'', \dots, Q_u'', R_1, \dots, R_v\}$ of P such that $\sum_{j=1}^{u} vol(Q_j'') < \varepsilon$ and such that $osc_{R_k}(\tilde{f}) < \varepsilon\ \forall 1 \le r \le v$

Verification of Claim 3

Take $Q_1', \dots, Q_s'$ as in Claim 2. Make them become disjoint by performing intersections and by eliminating redundant pieces.

In this way, $Q_1', \dots, Q_s' \to Q_1'', \dots, Q_u''$ and $\sum_{j=1}^{s} vol(Q_i'') < \varepsilon$

On the other hand, $\tilde{f}$ is uniformly continuous on $P\setminus(Q_1'', \dots, Q_u'')$

Hence $\exists\delta > 0$ s.t. $\vec{x}, \vec{y} \in P\setminus(Q_1'', \dots, Q_u''),\ \|\vec{x} - \vec{y}\| < \delta \Rightarrow |\tilde{f}(\vec{x}) - \tilde{f}(\vec{y})| < \varepsilon$

Complete $Q_1'', \dots, Q_u''$ to a division $\{Q_1'', \dots, Q_u'', R_1, \dots, R_v\}$ such that $diam(R_k) < \delta\ \forall 1 \le k \le v$

Then $osc_{R_k}(\tilde{f}) < \varepsilon\ \forall 1 \le k \le v$

By the above 'Two-ways of being small' Lemma, $\tilde{f}$ is integrable on P

Therefore, $\tilde{f}$ is integrable on $A \subseteq P$

How to Calculate Integrals I

November-02-11 11:30 AM

13.1 Remark

$A \subseteq R^n$ *bounded*, $f: A \to \mathbb{R}$ *bounded function*
From L12 have good criterion (Theorem 12.5) for f to be integrable.

13.2 Remark and Notation

Say $n = p + q$, *with* $p, q \in \mathbb{N}$

- For $A \subseteq \mathbb{R}^p, B \subseteq \mathbb{R}^q$ define Cartesian product
  $A \times B = \{(\vec{a}, \vec{b}) : \vec{a} \in A, \vec{b} \in B\} \subseteq \mathbb{R}^n$

- Every $P \in \mathcal{P}_n$ can be written as
  $P = M \times N$ with $M \in \mathcal{P}_p, N \in \mathcal{P}_q$
  $P = (a_1, b_1] \times \cdots \times (a_p, b_p] \times (a_{p+1}, b_{p+1}] \times \cdots \times (a_n, b_n]$

- Let $P = M \times N$ be as above
  Let $f: P \to \mathbb{R}$ be a function
  For every $\vec{v} \in$ M define **partial function** $f_{\vec{v}}: N \to \mathbb{R}$
  by $f_{\vec{v}}(\vec{w}) \coloneqq f(\vec{v}, \vec{w}), \quad \vec{w} \in W$

Notation

Notation used sometimes for $f_{\vec{v}}$ is $f(\vec{v}, \cdot)$

13.3 Theorem (Fubini)

$P = M \times N$ and $f: P \to \mathbb{R}$ as above.
Suppose that

i) $f \in Int_b(P, \mathbb{R})$
ii) For every $\vec{v} \in$ M, the partial function $f_{\vec{v}}: N \to \mathbb{R}$ belongs to $Int_b(N, \mathbb{R})$

Define a function $F: M \to \mathbb{R}$ by

$$F(\vec{v}) = \int_N f_{\vec{v}}, \qquad \vec{v} \in M$$

Then $F \in Int_b(M, \mathbb{R})$ and $\boxed{\int_M F = \int_P f}$

13.4 Remark

Write $\vec{x} \in$ P as $\vec{x} = (\vec{v}, \vec{w})$ with $\vec{v} \in V$ and $\vec{w} \in$ W
$\int_P f$ is also written $\int_P f(\vec{x}) d\vec{x}$ or as $\int_P f(\vec{v}, \vec{w}) d\vec{v} d\vec{w}$
$= \int_P f(\vec{v}, \vec{w}) d(\vec{v}, \vec{w})$

Left hand side of boxed formula is

$$\int_M F(\vec{v}) dv = \int_M \left(\int_N f_{\vec{v}}(\vec{w}) d\vec{w} \right) d\vec{v} = \int_M \left(\int_N f(\vec{v}, \vec{w}) d\vec{w} \right) d\vec{v}$$

So can say that

$$\int_{M \times N} f(\vec{v}, \vec{w}) d\vec{v} d\vec{w} = \int_M \left(\int_N f(\vec{v}, \vec{w}) d\vec{w} \right) d\vec{v}$$

Result: Reduces dimensionality of integrals to be calculated.

13.6 Remark

By symmetry, Fubini also applies to iterated integrals with components considered in another order.

$$\int_P f(\vec{v}, \vec{w}) d(\vec{v}, \vec{w}) = \int_N \left(\int_M f(\vec{v}, \vec{w}) d\vec{v} \right) d\vec{w}$$

Holding if:

i) $f \in Int_b(P, \mathbb{R})$
ii) $f_{\vec{w}} \in Int_b(M, \mathbb{R}), \qquad \forall \vec{w} \in N$ *where* $f_{\vec{w}} = f(\cdot, w)$

Or could, by example have
$P = (a_1, b_1] \times (a_2, b_2] \times (a_3, b_3] \subseteq \mathbb{R}^3$

$$\int_P f(x, y, z) d(x, y, z) = \int_{(a_1, b_1] \times (a_3, b_3]} \left(\int_{a_2}^{b_2} f(x, y, z) dy \right) d(x, z)$$

With two suitable conditions i), ii)

Example

$n = 2$
$A = \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 \leq 1\}$, closed unit disk
$bd(A) = \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 = 1\}$ a null set in $\mathbb{R}^2$
Due to theorem 12.5 every continuous function $f: A \to \mathbb{R}$ is integrable.
But how to calculate $\int_A f$

Concrete example to follow:
$f: A \to \mathbb{R}, \qquad f((s, t)) = \sqrt{1 - (s^2 + t^2)}$

We calculate $\int_A f$ by a method called "theorem of Fubini"
Enclose $A \subseteq P = (-2,2] \times (-2,2]$
$\tilde{f}: P \to \mathbb{R}$ by putting $\tilde{f}(\vec{x}) = 0 \ \forall \vec{x} \in P \backslash A$
By definition have $\int_A f = \int_P \tilde{f}$ and we calcuate $\int_P \tilde{f}$ with Fubini
For $v \in M = (-2, -2]$ look at the partial function $\tilde{f_v}: N \to \mathbb{R}$
Have $\tilde{f_v} = 0$ for $v \in (-2, -1] \cup [1,2]$
For $v \in (-1,1)$ we get

$$\tilde{f_v}: (-2,2] \to \mathbb{R}, \qquad \tilde{f_v}(w) = \tilde{f}(v, w) = \begin{cases} \sqrt{1 - (v^2 + w^2)}, & |w| \leq \sqrt{1 - v^2} \\ 0, & otherwise \end{cases}$$

Note that $\tilde{f_v}$ is continuous hence integrable. So hypothesis (ii) of Fubini holds. Also have hypothesis (i) since $f \in Int_b(P, \mathbb{R})$

So apply Fubini. Define $F: (-2,2] \to \mathbb{R}$ by

$$F(v) = \int_{-2}^{2} \tilde{f_v}(w) dw = \begin{cases} \dfrac{\pi(1 - v^2)}{2}, & v \in (-1,1) \\ 0, & v \in (-2, -1] \cup [1,2] \end{cases}$$

Finally,

$$\int_A f = \int_P \tilde{f} = \int_{-2}^{2} F(v) dv = \int_{-1}^{1} \frac{\pi(1 - v^2)}{2} dv = \frac{2\pi}{3}$$

How to Calculate Integrals II

November-04-11 11:48 AM

(A) Integrals and Volumes

14.1 Definition

$A \subseteq \mathbb{R}^n, f: A \to \mathbb{R}$ such that $f(\vec{x}) \geq 0, \forall \vec{x} \in A$
Graph of f is $\Gamma = \{(\vec{x}, z) \in \mathbb{R}^{n+1} | \vec{x} \in A, x \in \mathbb{R}, z = f(\vec{x})\}$
The set $S = \{(\vec{x}, z) \in \mathbb{R}^{n+1} | \vec{x} \in A, z \in \mathbb{R}, 0 \leq x \leq f(\vec{x})\}$ is called the **subgraph** of f.

14.3 Proposition

$A \subseteq \mathbb{R}^n$ bounded set, $f \in Int_b(A, \mathbb{R})$ such that $f(\vec{x}) \geq 0, \forall \vec{x} \in A$. Let $S \subseteq \mathbb{R}^{n+1}$ be the subgraph of f. Then S has volume (in $\mathbb{R}^{n+1}$ and $vol(S) = \int_A f$.

Comment

The proposition equates

$$\int_A f = \int_S 1$$

LHS is n dimensional, RHS is n+1 dimensional.
Proof by following Darboux sums. Darboux sums for f can be interpreted as volumes in $\mathbb{R}^{n+1}$, which "approximate" $vol(S)$

14.6 Remark

In calculations it is sometimes convenient to replaces values of functions on a null set.

Underlying fact:

$P \in \mathcal{P}_n,\ f, g: P \to \mathbb{R}$ bounded functions.
Suppose $\exists N \subseteq P$ null set such that $f(\vec{x}) = g(\vec{x})\ \forall \vec{x} \in P \backslash N$. If $f \in Int_b(P, \mathbb{R})$ then $g \in Int(P, \mathbb{R})$ and $\int_P g = \int_P f$

Proof of fact

Done by analysis of divisions of P

(B) Polar Coordinates

14.8 Definition

For $0 \leq r_1 < r_2$ the set $A = \left\{(s,t) \in \mathbb{R}^2 \middle| r_1 < \sqrt{s^2+t^2} \leq r_2\right\}$ will be called the **half-open annulus** of radii r_1 and r_2 centered at (0,0)

For such Annulus A, the map $T: (r_1, r_2] \times (0, 2\pi] \to A$
$T\big((r,\theta)\big) = (r\cos\theta, r\sin\theta)$
is called **parameterization** of A by polar coordinates.

On $R = (r_1, r_2] \times (0, 2\pi]$
Vertical segments (constant r) become circles of radius r inside A.
Horizontal segments (constant θ) become chords of angle θ in A.

T is a bijective map between $(r_1, r_2] \times (0, 2\pi]$ and A

14.9 Proposition

A and R as above. Let $f: A \to \mathbb{R}$ be a bounded function. Let $g: R \to \mathbb{R}$ be the composted function $g = f \circ T$
$g(\vec{x}) = f(T(\vec{x})), \quad \vec{x} \in R$
More precisely, $g\big((r,\theta)\big) = f(r\cos\theta, r\sin\theta)$

$$\text{Then } \int_A f\big((s,t)\big) d(s,t) = \int_R g\big((r,\theta)\big) r\, d(r,\theta) \quad [PC]$$

Where does the r in $\int_R g\big((r,\theta)\big) r\, d(r,\theta)$ come from?
r is the **Jacobian** of T at (r,θ)

$$\int_A f = \int_R g \cdot J, \qquad J: R \to \mathbb{R};\ J\big((r,\theta)\big) = r\ \ \forall (r,\theta) \in R$$

J is the Jacobian function for polar coordinates .
The discussion of Jacobian is in terms of partial derivatives (taken for $T: R \to A$)

14.2/4 Example

$n = 2, A = \{(s,t) \in \mathbb{R}^2 |\ s^2 + t^2 \leq 1\}$
$f: A \to \mathbb{R}$ defined by $f\big((s,t)\big) = \sqrt{1 - (s^2+t^2)}$

Subgraph of f is

$$S = \left\{(s,t,z) \in \mathbb{R}^3 \middle| s^2+t^2 \leq 1, 0 \leq z \leq \sqrt{1-(s^2+t^2)}\right\} = \{(s,t,z) \in \mathbb{R}^3 | s^2+t^2+z^2 \leq 1, z \geq 0\}$$

On Wednesday calculated $\int_A f = \frac{2\pi}{3}$. S subgraph $\int$ of f has $vol(S) = \frac{2\pi}{3}$
Moral: Volume of closed unit ball in $\mathbb{R}^3$ is equal to $\frac{4\pi}{3}$

14.5 Remark

Another way to calculate volume of unit ball in $\mathbb{R}^3$. Take the open unit ball. $B = \{(s,t,z) | s^2+t^2+z^2 < 1\} \subseteq \mathbb{R}^3$
Enclose B with $C = (-1,1] \times (-1,1] \times (-1,1]$

$$\text{Have } vol(B) = \int_B 1 = \int_C I_B(\vec{x}) d\vec{x} = \int_{-1}^{1} \left(\int_{(-1,1]\times(-1,1]} I_B(s,t,z) d(s,t) \right) dz$$

Fix z and look at partial function
$(-1,1] \times (-1,1] \to \mathbb{R}, \qquad (s,t) \mapsto I_B(s,t,z)$

$$I_B(s,t,z) = \begin{cases} 1\ if\ (s,t,z) \in B \\ 0\ if\ (s,t,z) \notin B \end{cases} = \begin{cases} 1\ if\ d\big((s,t),(0,0)\big) < \sqrt{1-z^2} \\ \qquad 0\ otherwise \end{cases}$$

$$\text{Get } \int_{(-1,1]\times(-1,1]} I_B(s,t,z) d(s,t) = \pi(1-z^2)$$

$$vol(B) = \int_{-1}^{1} \pi(1-z^2) dz = \pi\left[z - \frac{z^3}{3}\right]_{-1}^{1} = \frac{4\pi}{3}$$

14.6 Illustration of Use

Let $B = \{(s,t,z) \in \mathbb{R}^3 | s^2+t^2+z^2 < 1\}, \bar{B} = \{(s,t,z) \in \mathbb{R}^3 | s^2+t^2+z^2 \leq 1\}$
How do I know $vol\ (B) = vol\ (\bar{B})$?
Have $B, \bar{B} \subseteq P = (-2,2] \times (-2,2] \times (-2,2]$

$$\text{So } vol(B) = \int_P I_B(\vec{x}) d\vec{x}, \qquad vol(\bar{B}) = \int_P I_{\bar{B}}(\vec{x}) d\vec{x}$$

Take $f = I_B, g = I_{\bar{B}}$ in 'fact', have that f, g agree on $P \backslash N$ where $N = \{(s,t,z) \in \mathbb{R}^3 | s^2+t^2+z^2 = 1\}$ (null set)

14.7 Polar Coordinates Example

Look again at $A = \{(s,t) \in \mathbb{R}^n | s^2+t^2 \leq 1\}$
$f: A \to \mathbb{R}$ defined by $f\big((s,t)\big) = \sqrt{1-(s^2+t^2)}$

$$\text{Calculated in 2 ways that } \int_A f = \frac{2\pi}{3}$$

Now a third way. Write A as a union of circles of radii $r \in [0,1]$ centered at (0,0). On circle of radius r have $s^2+t^2 = r^2$ hence $f\big((s,t)\big) = \sqrt{1-r^2}$

$$\text{Could then have } \int_A f\ ? = \int_0^1 2\pi r\sqrt{1-r^2} dr$$

Check

$$\int_0^1 2\pi r\sqrt{1-r^2} dr = \int_0^1 \pi\sqrt{u} du$$

$u = 1 - r^2 \Rightarrow du = -2rdr$

$$\int_0^1 \pi\sqrt{u} du = \left[\pi \frac{2}{3} u^{\frac{3}{2}}\right]_1^0 = \frac{2\pi}{3}$$

But why does this hold? Is it Fubini?

14.10 Example

Make $r_1 = 0, r_2 = 1$
$A = \left\{(s,t) \in \mathbb{R}^2 : 0 < \sqrt{s^2+t^2} \leq 1\right\} = \bar{B}\big((0,0);1\big) \setminus \{(0,0)\}$ This is called a "Punctured Disk"
Let $f: A \to \mathbb{R}, \qquad f\big((s,t)\big) = \sqrt{1-(s^2+t^2)}$
Have $f \in Int_b(A, \mathbb{R})$
$bd(A) = \{(s,t) \in \mathbb{R}^2 : s^2+t^2 = 1\} \cup \{(0,0)\}$ null set
f is bounded and continuous on A so f is integrable on A
Let $R = (0,1] \times (0,2\pi]$, define $g: R \to \mathbb{R}\ by$

$$g\big((r,\theta)\big) = f\big((r\cos\theta, r\sin\theta)\big) = \sqrt{1 - \big((r\cos\theta)^2 + (r\sin\theta)^2\big)} = \sqrt{1-r^2}$$

Integrate for polar coordinates:

$$\int_A f = \int_{(0,1]\times(0,2\theta]} r\sqrt{1-r^2}\, d(r,\theta) = \int_0^1 \left(\int_0^{2\pi} r\sqrt{1-r^2}\, d\theta \right) dr = \int_0^1 2\pi r\sqrt{1-r^2} dr = \frac{2\pi}{3}$$

Directional Derivatives

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15.1 Definition

$A \subseteq \mathbb{R}^n, \vec{a} \in int(A)$, Let $\vec{v}$ be any vector in A
Let $f: A \to \mathbb{R}$ be a function.

If $\lim\limits_{\substack{t\to 0\\ t\neq 0}} \dfrac{f(\vec{a}+t\vec{v})-f(\vec{a})}{t} \in \mathbb{R}$ exists
Then we say that f has **directional derivative** at $\vec{a}$ in direction $\vec{v}$
Notation for the limit:
$(\partial_{\vec{v}} f)(\vec{a}) := \lim\limits_{\substack{t\to 0\\ t\neq 0}} \dfrac{f(\vec{a}+t\vec{v})-f(\vec{a})}{t}$

15.2 Remark

Notations as in Definition 15.1

1. If $\vec{v}=\vec{0}$, then $(\partial_{\vec{0}})(\vec{a})$ is sure to exist, and $(\partial_{\vec{0}})(\vec{a})=0$
2. Now suppose $\vec{v}\neq\vec{0}$, hence $\|\vec{v}\|>0$
   Have $\vec{a}\in \text{int}(A)$, so $\exists r>0$ s.t. $B(\vec{a},r)\subseteq A$
   Then it makes sense to define
   $\varphi: \left(-\dfrac{r}{\|\vec{v}\|}, \dfrac{r}{\|\vec{v}\|}\right) \to \mathbb{R}$ by $\varphi(t)=f(\vec{a}+t\vec{v})$, $\quad -\dfrac{r}{\|\vec{v}\|}<t<\dfrac{r}{\|\vec{v}\|}$
   Indeed, if $|t|<\dfrac{r}{\|\vec{v}\|}$, hence $\|(\vec{a}+t\vec{v})-\vec{a}\|=\|t\vec{v}\|=|t|\|\vec{v}\|<r$
   So $-\dfrac{r}{\|\vec{v}\|}<t<\dfrac{r}{\|\vec{v}\|} \Rightarrow \vec{a}+t\vec{v}\in B(\vec{a};r)\subseteq A,\quad$ and $f(\vec{a}+t\vec{v})$ is defined
   φ is called the **partial function** of f around the point $\vec{a}$ in direction $\vec{v}$

   $\lim\limits_{\substack{t\to 0\\ t\neq 0}} \dfrac{f(\vec{a}+t\vec{v})-f(\vec{a})}{t} = \lim\limits_{\substack{t\to 0\\ t\neq 0}} \dfrac{\varphi(t)-\varphi(0)}{t} = \varphi'(0)$ When the derivative exists

15.3 Definition

$A\subseteq\mathbb{R}^n, \vec{a}\in int(A)$. Fix $1\le i\le n$
Let $\vec{e_i}=(0,\dots,0,1,0,\dots,0)$ be the i^{th} vector of the standard basis of $\mathbb{R}^n$
If $(\partial_{\vec{e_i}} f)(\vec{a})$ exists then this is called the i^{th} **partial derivative** of f at $\vec{a}$ denoted as $(\partial_i f)(\vec{a})$

15.4 Definition

$f: A\to\mathbb{R}, \qquad a\in int(A)$ and suppose that $(\partial_i f)(\vec{a})$ exists for every $1\le i\le n$.
The vector $\big((\partial_1 f)(\vec{a}), (\partial_2 f)(\vec{a}), \dots, (\partial_n f)(\vec{a})\big)\in\mathbb{R}$
is called the **gradient vector** of f at $\vec{a}$, denoted $(\nabla f)(\vec{a})$
∇ = "Nabla" or "Grad" for gradient.

15.5 Proposition

$A\subseteq\mathbb{R}^n, \qquad f: A\to\mathbb{R}, \qquad \vec{a}\in int(A)$
Let $\vec{v}\neq\vec{0}$ be in $\mathbb{R}^n$ and suppose that $(\partial_{\vec{v}} f)(\vec{a})$ exists.
Then for every $\alpha\in\mathbb{R}$ the directional derivative exists as well
[H - Homogeneity] $\boxed{(\partial_{\alpha\vec{v}} f)(\vec{a}) = \alpha(\partial_{\vec{v}} f)(\vec{a})}$

15.6 Remark

$A\subseteq\mathbb{R}^n, \qquad f: A\to\mathbb{R}, \qquad \vec{a}\in int(A)$
Suppose that $(\partial_{\vec{v}} f)(\vec{a})$ exists for all $\vec{v}\in\mathbb{R}^n$. So can define function:
$L: \mathbb{R}^n\to\mathbb{R}, \qquad L(\vec{v}) = (\partial_{\vec{v}} f)(\vec{a})$

Proposition 15.5 says $L(\alpha\vec{v})=\alpha L(\vec{v})\ \ \forall\alpha\in\mathbb{R}, \vec{v}\in\mathbb{R}^n$

Proof of Proposition 15.5

If $\alpha=0$ then [H] amounts to $0=0$ so assume $\alpha\neq 0$. Denote $\alpha\vec{v}=\vec{w}$

Must verify existence of
$\lim\limits_{\substack{t\to 0\\ t\neq 0}} \dfrac{f(\vec{a}+t\vec{w})-f(\vec{a})}{t} = \lim\limits_{\substack{t\to 0\\ t\neq 0}} \dfrac{f(\vec{a}+t\alpha\vec{v})-f(\vec{a})}{t\alpha}\alpha =_1 \lim\limits_{\substack{s\to 0\\ s\neq 0}} \dfrac{f(\vec{a}+s\vec{v})-f(\vec{a})}{s}\alpha$
$=_1$: Put $s=t\alpha$ when $t\to 0, t\neq 0$ get $s\to 0, s\neq 0$
This limit does exist and equals $(\partial_{\vec{v}} f)(\vec{a})\alpha$ ■

Question

Isn't L additive as well? So it would be a linear function
If yes, then for every $\vec{v}=(v^{(1)},\dots,v^{(m)})\in\mathbb{R}$ write $v=\sum\limits_{i=1}^{n} v^{(i)}\vec{e_i}$ and get
$L(\vec{v})=\sum\limits_{i=1}^{n} v^{(i)}L(\vec{e_i}) \Rightarrow (\partial_{\vec{v}} f)(\vec{a}) = \sum\limits_{i=1}^{n} v^{(i)}(\delta_i f)(\vec{a})$

Answer

No :(

Problem 4 in homework 7 gives a function $f: \mathbb{R}^2\to\mathbb{R}$ such that f is continuous and $(\partial_{\vec{v}} f)(\vec{a})$ exist for all $\vec{a}\in\mathbb{R}^2, \vec{v}\in\mathbb{R}^2$
And yet, if we put
$L(\vec{v}) = (\partial_{\vec{v}f})(\vec{0})\ \vec{v}\in\mathbb{R}^2$
Then L is not linear.

What do we do to get the answer "Yes"?
Go to the concept of a C^1 function

C^1 functions

November-11-11 11:58 AM

16.1 Remark

Directional/partial derivatives as functions.
$A \subseteq \mathbb{R}^n$ open set, $f: A \to \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$
If $(\partial_{\vec{v}} f)(\vec{a})$ exists for every $\vec{a} \in A$ then we get a new function
$\partial_{\vec{v}} f: A \to \mathbb{R}$
called the **directional derivative** of f in direction $\vec{v}$

Special case: $\vec{v} = \vec{e_i}$
If $(\partial_i f)(\vec{a})$ exists for every $\vec{a} \in$ A then we get a new function
$\partial_i f: A \to \mathbb{R}$
called the $\boldsymbol{i^{th}}$ **partial derivative** of f.

16.2 Definition

$A \subseteq \mathbb{R}^n$. A function $f: A \to \mathbb{R}$ is said to be a $\boldsymbol{C^1}$**-function** when it has the following properties:

- f is continuous on A
- f has partial derivatives at every $\vec{a} \in A$
- The new functions $\partial_i f: A \to \mathbb{R}$, $1 \le i \le n$ are continuous on A

The collection of all C^1-functions from A to $\mathbb{R}$ is denoted $C^1(A, \mathbb{R})$

Note

One uses the notation
$C^0(A, \mathbb{R}) = \{f: A \to \mathbb{R} \mid f \text{ is continuous on A}\}$
Will also encounter $C^2(A, \mathbb{R}), C^3(A, \mathbb{R}), \dots, C^\infty(A, \mathbb{R})$
$C^n(A, \mathbb{R})$ defined as the set of all continuous functions whose partial derivatives are in $C^{n-1}(A, \mathbb{R})$

16.4 Theorem

$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R})$.
Then for every $\vec{a} \in$ A we have
$$\lim_{\substack{\vec{x} \to \vec{a} \\ \vec{x} \ne \vec{a}}} \frac{|f(\vec{x}) - f(\vec{a}) - \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a}) \rangle|}{\|\vec{x} - \vec{a}\|} = 0 \quad \text{[L-approx]}$$
where $(\nabla f)(\vec{a}) = ((\partial_1 f)(\vec{a}), \dots, (\partial_n f)(\vec{a}))$

To prove this we do

16.5 Lemma

(Mean Value Theorem in direction i, $1 \le i \le n$)

$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R})$, $\vec{a} \in A$.
Let $r > 0$ be such that $B(\vec{a}; r) \subseteq A$.
Let i be an index in $\{1, \dots, n\}$ and let $\vec{x}, \vec{y} \in B(\vec{a}; r)$ be such that they only possibly differ on the component i (So $x^{(j)} = y^{(y)}, \forall 1 \le j \le n, j \ne i$)

Then $\exists \vec{b} \in B(\vec{a}; r)$ such that
$f(\vec{y}) - f(\vec{x}) = (y^{(i)} - x^{(i)})(\partial_i f)(\vec{b})$ $[MVT \text{ direction } i]$

16.6 Definition (Geometry)

$\vec{x}, \vec{y} \in \mathbb{R}^n$
The line segment connecting $\vec{x}$ and $\vec{y}$ is the set
$Co(\vec{x}, \vec{y}) \coloneqq \{(1-t)\vec{x} + t\vec{y} \mid t \in [0, 1]\}$

$\vec{x} = \vec{x} + \vec{0}$
$\vec{y} = \vec{x} + (\vec{y} - \vec{x})$
We do $\vec{x} + t(\vec{y} - \vec{x}), 0 \le t \le 1$ to cover the line segment from $\vec{x}$ to $\vec{y}$

16.7 Proposition

MVT in direction $\vec{v}$

$A \subseteq \mathbb{R}^n$ open, $f: A \to \mathbb{R}, \vec{v} \ne 0 \; in \; \mathbb{R}^n$
Suppose that

- f is continuous on A
- $(\partial_{\vec{v}} f)(\vec{a})$ exists for every $\vec{a} \in A$
- The new function $\partial_{\vec{v}} f: A \to \mathbb{R}$ is continuous on A

Suppose we have $\vec{x}, \vec{y} \in A$ such that $\vec{y} - \vec{x} = \alpha \vec{v}$ for some $\alpha \in \mathbb{R}$ and such that $Co(\vec{x}, \vec{y}) \subseteq A$.
Then $\exists \vec{b} \in Co(\vec{x}, \vec{y}) \; s.t. f(\vec{y}) - f(\vec{x}) = \alpha(\partial_{\vec{v}f})(\vec{b})$

Geometric Interpretation of L-Approx.

Instead of getting a tangent line to the graph of f, we get a tangent hyperplane to the graph of f.
The hyperplane is an n-dimensional subset of $\mathbb{R}^{n+1}$

16.8 Remark (Geometry)

Given $m \in \mathbb{N}, \vec{p} \in \mathbb{R}^m$
How do we write the equation of a hyperplane $H \subseteq \mathbb{R}^m$ that passes through $\vec{p}$

16.3 Remark

For $f \in C^1(A, \mathbb{R})$, will prove a theorem of local linear approximation.

Look at the (known) special case $n = 1$. Make $A = (\alpha, \beta) \subseteq \mathbb{R}, a \in (\alpha, \beta)$
$f: A \to \mathbb{R}$ differentiable at a.
Approximate formula says $f(x) \approx f(a) + f'(a) \cdot (x - a)$ for x close to a.

So have
$\lim_{x \to a}(f(x) - f(a) - f'(a) \times (x - a)) = 0$
But in fact have more!
Have $\lim_{\substack{x \to a \\ x \ne a}} \left| \frac{f(x) - f(a) - f'(a) \times (x - a)}{x - a} \right| = \lim_{\substack{x \to a \\ x \ne a}} \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| = 0$
Call the above (first formula) L-Approx. in 1 variable.

Proof of Lemma 16.5

Case when $x^{(i)} = y^{(i)}$ trivial and get $0 = 0$
So assume that $x^{(i)} \ne y^{(i)}, \; say \; x^{(i)} < y^{(i)}$
Denote $x^{(i)} = \alpha, y^{(i)} = \beta$
Define $\Psi: (\alpha, \beta] \to \mathbb{R}$ by $\Psi(s) = f\left(\left(x^{(1)}, \dots, x^{(i-1)}, s, x^{(i+1)}, \dots, x^{(n)}\right)\right) \forall \alpha \le s \le \beta$
Note that $\Psi(\alpha) = f(\vec{x}), \Psi(\beta) = f(\vec{y})$ and Ψ is continuous on $(\alpha, \beta]$
(Why? Check with sequences using the continuity of f and that $x_k \to_{k \to \infty} x$ in $(\alpha, \beta]$
$\Rightarrow (x^{(1)}, \dots, x^{(i-1)}, x_k, x^{(i+1)}, \dots, x^{(n)}) \to (x^{(1)}, \dots, x^{(i-1)}, s, x^{(i+1)}, \dots, x^{(n)})$

Claim

Take s such that $\alpha < s < \beta$ and put $\vec{b} = (x^{(1)}, \dots, x^{(i-1)}, s, x^{(i+1)}, \dots, x^{(n)}) \in B(\vec{a}; r)$
Then Ψ is differentiable at s, and $\Psi'(s) = (\partial_i f)(\vec{b})$

Verification of Claim

$\Psi(s) = f(\vec{b})$ by definition of Ψ
$\Psi(s + h) = f\left(\left(x^{(1)}, \dots, x^{(i-1)}, s + h, x^{(i+1)}, \dots, x^{(n)}\right)\right) = f(\vec{b} + h\vec{e_i})$
So $\frac{\Psi(s+h) - \Psi(s)}{h} = \frac{f(\vec{b} + h\vec{e_i}) - f(\vec{b})}{h}$
Take limit $h \to 0$ $(h \ne 0)$. Get claim since the expression on the right hand tends to $(\partial_i f)(\vec{b})$ ∎

Due to claim, we can apply MVT from Calculus I to Ψ.
Gives $\exists s, \alpha < s < \beta$, such that $\frac{\Psi(\beta) - \Psi(\alpha)}{\beta - \alpha} = \Psi'(s)$
Convert $\Psi(\alpha) = f(\vec{x}), \Psi(\beta) = f(\vec{y}), \; \alpha = x^{(i)}, \beta = y^{(i)}$
$\Psi'(x) = (\partial_i f)(\vec{b})$ for $\vec{b} = (x^{(1)}, \dots, x^{(i-1)}, s, x^{(i+1)}, \dots, x^{(n)})$
$\frac{f(\vec{y}) - f(\vec{x})}{y^{(i)} - x^{(i)}} = (\partial_i f)(\vec{b})$
and done. QED

Proof of Theorem 16.4

Important Proof

Fix $r > 0$ such that $B(\vec{a}; r) \subseteq A$. So $f(\vec{x})$ makes sense for any $\vec{x}$ such that $\|\vec{x} - \vec{a}\| < r$
Given $\varepsilon > 0$. Want to find $0 < \delta < r$ such that
$$\begin{pmatrix} \|\vec{x} - \vec{a}\| < \delta \\ \vec{x} \ne \vec{a} \end{pmatrix} \Rightarrow \frac{|f(\vec{x}) - f(\vec{a}) - \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a}) \rangle|}{\|\vec{x} - \vec{a}\|} < \varepsilon \qquad [Want]$$

Know

For every $1 \le i \le n$ know that $\partial_i f: A \to \mathbb{R}$ is continuous at $\vec{a}$, hence $\exists 0 < \delta_i < r$ such that
$\|\vec{x} - \vec{a}\| < \delta_i \Rightarrow |(\partial_i f)(\vec{x}) - (\partial_i f)(\vec{a})| < \frac{\varepsilon}{\sqrt{n}}$
Take $\delta = \min(\delta_1, \dots, \delta_n)$. So $0 < \delta < r$ and have
$\|\vec{x} - \vec{a}\| < \delta \Rightarrow |(\partial_i f)(\vec{x}) - (\partial_i f)(\vec{a})| < \frac{\varepsilon}{\sqrt{n}} \; \forall 1 \le i \le n \qquad [Know\ 1]$

Will show that δ works in [Want].
So pick $\vec{x} = (x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in B(\vec{a}; \delta) \setminus \{\vec{a}\} \Rightarrow 0 < \|\vec{x} - \vec{a}\| < \delta$

Define $\vec{x}_0, \vec{x}_1, \dots, \vec{x}_n \in B(\vec{a}; \delta)$ as follows
$\vec{x}_0 = (a^{(1)}, a^{(2)}, \dots, a^{(n)}) = \vec{a}$
$\vec{x}_1 = (x^{(1)}, a^{(2)}, \dots, x^{(n)})$
$\vec{x}_2 = (x^{(1)}, x^{(2)}, a^{(3)}, \dots, a^{(n)})$
$\vec{x}_n = (x^{(1)}, x^{(2)}, \dots, x^{(n)}) = \vec{x}$

Note that $\|\vec{x}_i - \vec{a}\| \le \|\vec{x} - \vec{a}\| < \delta, \; \forall 0 \le i \le n$
Write $f(\vec{x}) - f(\vec{a}) = f(\vec{x}_n) - f(\vec{x}_0) = \sum_{i=1}^{n} f(\vec{x}_n) - f(\vec{x}_{n-1})$
Observe
$\vec{x}_i = (x^{(1)}, \dots, x^{(i-1)}, x^{(i)}, a^{(i+1)}, \dots, a^{(n)})$
$\vec{x}_{i-1} = (x^{(1)}, \dots, x^{(i-1)}, a^{(i)}, a^{(i+1)}, \dots, a^{(n)})$

Given $m \in \mathbb{N}, \vec{p} \in \mathbb{R}^m$
How do we write the equation of a hyperplane $H \subseteq \mathbb{R}^m$ that passes through $\vec{p}$

One Possibility
$H = \{\vec{p} + \alpha_1 \vec{y}_1 + \cdots + \alpha_{n-1}\vec{y}_{n-1} \mid \alpha_1, \dots, \alpha_n \in \mathbb{R}^{m-1}\}$ [Hyp 1]
where $\vec{y}_1, \dots, \vec{y}_{n-1}$ are linearly independent.

Another Possibility
$H = \{\vec{q} \in \mathbb{R}^m \mid (\vec{q} - \vec{p}) \perp \vec{z}\}$ [Hyp 2]
with $\vec{z} \neq \vec{0}$ in $\mathbb{R}^m$ called the **normal vector**

Relation between [Hyp 1] and [Hyp 2]:
$span(\vec{z}) \perp span\{\vec{y}_1, \dots, \vec{y}_{n-1}\}$

16.9 Remark
$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R})$
Consider the graph
$\Gamma = \{(\vec{x}, t) \in \mathbb{R}^{n+1} \mid \vec{x} \in A, t = f(\vec{x})\} \subseteq \mathbb{R}^{n+1}$
Pick $\vec{a} \in$ A, look at $\vec{p} = (\vec{a}, f(\vec{a})) \in \Gamma$

$$\lim_{\substack{\vec{x} \to \vec{a} \\ \vec{x} \neq \vec{a}}} \frac{|f(\vec{x}) - f(\vec{a}) - \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle|}{\|\vec{x} - \vec{a}\|} = 0$$
So $f(\vec{x}) \approx f(\vec{a}) + \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle, for\ \vec{x} \in B(\vec{a}, \delta), small\ \delta$
This is a linear function in *x*.
$\vec{p}' = (\vec{x}, f(\vec{x})) \approx (\vec{x}, f(\vec{a}) + \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle) = \vec{q}$

Claim
$\vec{q} \in H$, where H is a special hyperplane going through $\vec{p}$

Tangent Plane
$\vec{y}_i = (0, 0, \dots, 0, 1, 0, \dots, 0, (\partial_i f)(\vec{a})), \quad 1 \le i \le n$
$$H = \left\{\vec{p} + \sum_{i=1}^{n} \alpha_i \vec{y}_i \mid \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\right\}$$

$\vec{w} = (-(\nabla f)(\vec{a}), 1)$
$H = \{q \in \mathbb{R}^{n+1} \mid (\vec{q} - \vec{p}) \perp \vec{w}\}$

16.10 Proposition
$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R}), \vec{a} \in A$
Then for every $\vec{v} \in \mathbb{R}^n$ the direction derivative $(\partial_{\vec{v}} f)(\vec{a})$ exists and
$\boxed{(\partial_{\vec{v}} f)(\vec{a}) = \langle \vec{v}, (\nabla f)(\vec{a})\rangle}$

Note that this is a linear function of $\vec{v}$

$$L(\vec{v}) = \langle \vec{v}, (\nabla f)(\vec{a})\rangle = \sum_{i=1}^{n} v^{(i)} (\partial_i f)(\vec{a})$$

16.11 Remark
$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R}), \vec{a} \in A$
Suppose $(\nabla f)(\vec{a}) \neq \vec{0}$. Look at various unit vectors $\vec{u} \in \mathbb{R}^n$, $(\|\vec{u}\| = 1)$

Have
$(\partial_{\vec{u}} f)(\vec{a}) = \langle \vec{u}, (\nabla f)(\vec{a})\rangle \le \|\vec{u}\| \cdot \|(\nabla f)(\vec{a})\| = \|(\nabla f)(\vec{a})\|$
Equality holds precisely when $\vec{u} \parallel (\nabla f)(\vec{a})$
$\therefore \vec{u}_0 = \dfrac{(\nabla f)(\vec{a})}{\|(\nabla f)(\vec{a})\|}$, gives
$(\partial_{\vec{u}_0} f)(\vec{a}) = (\nabla f)(\vec{a}) = \max\{(\partial_{\vec{u}} f)(\vec{a}) \mid \vec{u} \in \mathbb{R}^n, \|\vec{u}\| = 1\}$

Informal interpretation:
f is increasing fastest in the direction of the gradient vector.

Observe
$\vec{x}_i = (x^{(1)}, \dots, x^{(i-1)}, x^{(i)}, a^{(i+1)}, \dots, a^{(n)})$
$\vec{x}_{i-1} = (x^{(1)}, \dots, x^{(i-1)}, a^{(i)}, a^{(i+1)}, \dots, a^{(n)})$
Can apply MVT in direction i, and get $\exists \vec{b}_i \in B(\vec{a}; \delta)$ such that
$f(\vec{x}_i) - f(\vec{x}_{i-1}) = (x^{(i)} - a^{(i)})(\partial_i f)(\vec{b}_i)$

So
$$f(\vec{x}) - f(\vec{a}) = \sum_{i=1}^{n} f(\vec{x}_i) - f(\vec{x}_{i-1}) = \sum_{i=1}^{n} (x^{(i)} - a^{(i)})(\partial_i f)(\vec{b}_i) = \langle \vec{x} - \vec{a}, \vec{w}\rangle \quad \text{[Know 2]}$$
Where $\vec{w} = ((\partial_1 f)(\vec{b}_1), (\partial_2 f)(\vec{b}_2), \dots, (\partial_n f)(\vec{b}_n))$

Observe $\|\vec{w} - (\nabla f)(\vec{a})\|^2 = \sum_{i=1}^{n} \left((\partial_i f)(\vec{b}_i) - (\partial_i f)(\vec{a})\right)^2$
$\vec{b}_i \in B(\vec{a}; \delta) \Rightarrow |(\partial_i f)(\vec{b}) - (\partial_i f)(\vec{a})| < \dfrac{\varepsilon}{\sqrt{n}}$ by [Know 1]
$$\sum_{i=1}^{n} \left((\partial_i f)(\vec{b}_i) - (\partial_i f)(\vec{a})\right)^2 < \sum_{i=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2$$

Hence
$\|\vec{w} - (\nabla f)(\vec{a})\| < \varepsilon$ [Know 3]

Now calculate
$|f(\vec{x}) - f(\vec{a}) - \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle|$
$=_1 |\langle \vec{x} - \vec{a}, \vec{w}\rangle - \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle| =_2 |\langle \vec{x} - \vec{a}, \vec{w} - (\nabla f)(\vec{a})\rangle|$
$\le_3 \|\vec{x} - \vec{a}\| \cdot \|\vec{w} - (\nabla f)(\vec{a})\| < \|\vec{x} - \vec{a}\| \cdot \varepsilon$

1: Know 2
2: Bilinearity of inner product
3: By Cauchy-Schwartz
4: Know 3

In summary, get
$$\frac{|f(\vec{x}) - f(\vec{a}) - \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle|}{\|\vec{x} - \vec{a}\|} < \varepsilon$$
QED.

Remark 16.9
Pick $\vec{a} \in$ A, look at $\vec{p} = (\vec{a}, f(\vec{a})) \in \Gamma$

Recall (L-Approx)
$$\lim_{\substack{\vec{x} \to \vec{a} \\ \vec{x} \neq \vec{a}}} \frac{|f(\vec{x}) - f(\vec{a}) - \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle|}{\|\vec{x} - \vec{a}\|} = 0$$
So $f(\vec{x}) \approx f(\vec{a}) + \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle, for\ \vec{x} \in B(\vec{a}, \delta), small\ \delta$
This is a linear function in *x*.
$\vec{p}' = (\vec{x}, f(\vec{x})) \approx (\vec{x}, f(\vec{a}) + \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle)$

Claim
$\vec{q} \in H$, where H is a special hyperplane going through $\vec{p}$

Calculate
$\vec{p} - \vec{q} = (\vec{x}, f(\vec{a}) + \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle) - (\vec{a}, f(\vec{a})) = (\vec{x} - \vec{a}, \langle \vec{x} - \vec{a}, (\nabla f)(\vec{a})\rangle)$
Denote $\vec{v} \coloneqq \vec{x} - \vec{a} = (v^{(1)}, v^{(2)}, \dots, v^{(n)})$
Then $\vec{q} - \vec{p} = \left(v^{(1)}, \dots, v^{(n)}, \sum_{i=1}^{n} v^{(i)} \cdot (\partial_i f)(\vec{a})\right)$
$= v^{(1)}(1, 0, \dots, 0, (\partial_1 f)(\vec{a})) + v^{(2)}(0, 1, 0, \dots, 0, (\partial_2 f)(\vec{a})) + \cdots$
$+ v^{(n)}(0, 0, \dots, 1, (\partial_n f)(\vec{a}))$
So get $\vec{q} = \vec{p} + \sum_{i=1}^{n} v^{(i)} \vec{y}_i$ where $\vec{y}_i = (0, 0, \dots, 0, 1, 0, \dots, 0, (\partial_i f)(\vec{a})), \quad 1 \le i \le n$
So $\vec{q} \in$ H where
$$H = \left\{\vec{p} + \sum_{i=1}^{n} \alpha_i \vec{y}_i \mid \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\right\}$$

What about the normal vector to Γ at $\vec{p}$?
Need $\vec{w} \in \mathbb{R}^{n+1}$ such that $\vec{w} \perp \vec{y}, \ \forall 1 \le i \le n$
Look for $\vec{w}$ in the form $(w^{(1)}, w^{(2)}, \dots, w^{(n)}, 1)$
So $0 = \langle \vec{y}_i, \vec{w}\rangle = w^{(i)} + (\partial_i f)(\vec{a}) \Rightarrow w^{(i)} = -(\partial_i f)(\vec{a})$

Conclusion
$\vec{w} = (-(\nabla f)(\vec{a}), 1)$

Proof of Proposition 16.10
Will assume $\vec{v} \neq \vec{0}$ (for $\vec{v} = \vec{0}$ we know that $(\partial_{\vec{v}} f)(\vec{a})$ exists and is equal to 0)

Recall (L-Approx.) for f at $\vec{a} \in$ A

$$\lim_{\substack{\vec{x}\to\vec{a}\\ \vec{x}\neq\vec{a}}} \frac{|f(\vec{x}) - f(\vec{a}) - \langle \vec{x}-\vec{a}, (\nabla f)(\vec{a})\rangle|}{\|\vec{x}-\vec{a}\|} = 0, \qquad \text{set } \vec{x} = \vec{a} + t\vec{v} \text{ where } t \to 0, t \neq 0$$

Get $\vec{x} - \vec{a} = t\vec{v}$, hence $\langle \vec{x}-\vec{a}, (\nabla f)(\vec{a})\rangle = \langle t\vec{v}, (\nabla f)(\vec{a})\rangle = t\langle \vec{v}, (\nabla f)(\vec{a})\rangle$

also, $\|\vec{x}-\vec{a}\| = \|t\vec{v}\| = |t|\|\vec{v}\|$

So (L-Approx.) becomes

$$\lim_{\substack{t\to 0\\ t\neq 0}} \frac{|f(\vec{a}+t\vec{v}) - f(\vec{a}) - t\langle \vec{v}, (\nabla f)(\vec{a})\rangle|}{|t|\cdot\|\vec{v}\|} = 0, \text{multiply by } \|\vec{v}\|$$

$$\lim_{\substack{t\to 0\\ t\neq 0}} \frac{|f(\vec{a}+t\vec{v}) - f(\vec{a}) - t\langle \vec{v}, (\nabla f)(\vec{a})\rangle|}{|t|} = 0 \cdot \|\vec{v}\| = 0$$

$$\lim_{\substack{t\to 0\\ t\neq 0}} \left| \frac{f(\vec{a}+t\vec{v}) - f(\vec{a})}{t} - \langle \vec{v}, (\nabla f)(\vec{a})\rangle \right| = 0$$

It follows that $\lim\limits_{\substack{t\to 0\\ t\neq 0}} \dfrac{f(\vec{a}+t\vec{v}) - f(\vec{a})}{t}$ exists and is equal to $\langle \vec{v}, (\nabla f)(\vec{a})\rangle$

QED

$C^1(A, \mathbb{R}^m)$ and the Chain Rule

November-21-11 11:59 AM

17.1 Definition

$A \subseteq \mathbb{R}^n$ open, $f: A \to \mathbb{R}^m$ $(m \in \mathbb{N})$
For every $\vec{x} \in A$ write $f(\vec{x}) = \left(f^{(1)}(\vec{x}), f^{(2)}(\vec{x}), \dots, f^{(m)}(\vec{x})\right)$
And in this way we get functions $f^{(i)}: A \to \mathbb{R}, 1 \le i \le m$ called the **components** of f.
Compare to L4 about continuity, Def. 4.5, Prop. 4.6

If $f^{(i)} \in C^1(A, \mathbb{R}), \forall 1 \le i \le m$ then we say that $f \in C^1(A, \mathbb{R}^m)$

17.2 Definition

$A \subseteq \mathbb{R}^n$ open, $f = \left(f^{(1)}, f^{(2)}, \dots, f^{(m)}\right) \in C^1(A, \mathbb{R}^m)$
For every $\vec{a} \in A$ the matrix

$$(Jf)(\vec{a}) = \begin{bmatrix} (\nabla f^{(1)})(\vec{a}) \\ (\nabla f^{(2)})(\vec{a}) \\ \vdots \\ (\nabla f^{(m)})(\vec{a}) \end{bmatrix} = \begin{bmatrix} (\partial_1 f^{(1)})(\vec{a}) & \dots & (\partial_n f^{(1)})(\vec{a}) \\ \vdots & \ddots & \vdots \\ (\partial_1 f^{(m)})(\vec{a}) & \dots & (\partial_n f^{(m)})(\vec{a}) \end{bmatrix}$$

is called the **Jacobian matrix** of f at $\vec{a}$.

Note

$(Jf)(\vec{a}) \in M_{m\times n}(\mathbb{R})$
$(Jf)(\vec{a})_{(i,j)} = \left(\partial_j f^{(i)}\right)(\vec{a})$
$(Jf)(\vec{a})_i = \left(\nabla f^{(i)}\right)(\vec{a})$

17.3 Remark

1: $m = 1$

Have $f \in C^1(A, \mathbb{R})$, so $(Jf)(\vec{a}) \in M_{1\times n}(\mathbb{R})$
$(Jf)(\vec{a})$ is $(\nabla f)(\vec{a})$, treated as a row-matrix

2: $n = 1$ $(m \in \mathbb{N})$

Take $A = I =$ open interval in $\mathbb{R}$, $f: I \to \mathbb{R}^m$
Have $f = \left(f^{(1)}, f^{(2)}, \dots, f^{(m)}\right)$ with $f^{(i)}: I \to \mathbb{R}$
Have $f \in C^1(I, \mathbb{R}^m) \Leftrightarrow \left(f^{(i)} \in C^1(I, \mathbb{R})\ \forall 1 \le i \le m\right)$
Means that $\left(f^{(i)}\right)'$ exists and is continuous on I

Such f is called a **path** in $\mathbb{R}^m$

For every $a \in I$, the derivative $f'(a) = \left(\left(f^{(1)}\right)'(a), \dots, \left(f^{(m)}\right)'(a)\right) \in \mathbb{R}^m$
is called the **velocity vector** of f at a.

Have $(Jf)(a) = \begin{bmatrix} \left(f^{(1)}\right)(a) \\ \vdots \\ \left(f^{(m)}\right)(a) \end{bmatrix} \in M_{m\times 1}(\mathbb{R})$

So $(Jf)(a)$ is the velocity vector $f'(a)$, treated as a column matrix.

17.4 Remark

Can do algebraic operations with C^1 functions

1. $A \subseteq \mathbb{R}^n$ open, $f, g \in C^1(A, \mathbb{R})$ Then $f + g, f \cdot g \in C^1(A, \mathbb{R})$
   with formulas for partial derivatives as in calculus 1
2. $A \subseteq \mathbb{R}^n$ open, $f, g \in C^1(A, \mathbb{R}^m)$, $\alpha, \beta \in \mathbb{R}$
   Form new function:
   $h: \alpha f + \beta g$, $h: A \to \mathbb{R}^m$
   $h(\vec{x}) = \alpha f(\vec{x}) + \beta g(\vec{x}) \in \mathbb{R}^m$, $\forall \vec{x} \in A$
   For $1 \le i \le m$ have $h^{(i)} + \alpha f^{(i)} + \beta g^{(i)} \in C^1(A, \mathbb{R}) \Rightarrow h \in C^1(A, \mathbb{R}^m)$

   Moreover, for $\vec{a} \in A$ and $1 \le i \le m, 1 \le j \le n$ have
   $(Jh)(\vec{a})_{(i,j)} = \left(\partial_j h^{(i)}\right)(\vec{a}) = \alpha\left(\partial_j f^{(i)}\right)(\vec{a}) + \beta\left(\partial_j g^{(i)}\right)(\vec{a})$
   $= \alpha(Jf)(\vec{a})_{(i,j)} + \beta(Jg)(\vec{a})_{(i,j)}$
   $\therefore (Jh)(\vec{a}) = \alpha(Jf)(\vec{a}) + \beta(Jg)(\vec{a})$

Linearity of Jacobian

$\boxed{\left(J(\alpha f + \beta g)\right)(\vec{a}) = \alpha(Jf)(\vec{a}) + \beta(Jg)(\vec{a})}$ $(L - J)$, Linearity of Jacobian

Moral

$C^1(A, \mathbb{R}^m)$ is a vector space of functions, and Jf is linear.

17.5 Theorem (Chain Rule)

$m, n, p \in \mathbb{N}$, $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ open sets
$f \in C^1(A, \mathbb{R}^m)$ such that $f(\vec{a}) \in B\ \forall \vec{a} \in A$, $g = C^1(B, \mathbb{R}^p)$

Consider the composed function
$h: A \to \mathbb{R}^p$, $h = g \circ f$
Then $h \in C^1(A, \mathbb{R}^p)$ and for every $\vec{a} \in A$ have
$\boxed{(Jh)(\vec{a}) = (Jg)\left(f(\vec{a})\right) \times (Jf)(\vec{a})}$ $(M - J)$, Multiplicativity of Jacobian

Aside

Examples for Remark 17.4

$f \cdot g$ have
$\partial_j(fg) = (\partial_j f)g + g(\partial_j g)$, $1 \le j \le n$
Leibnitz rule, applied to partial functions for fg in the j^{th} direction.

More general than $f + g$ can do linear combinations $\alpha f + \beta g, \alpha, \beta \in \mathbb{R}$
Have $\alpha f + \beta g \in C^1(A, \mathbb{R})$ and
$\partial_j(\alpha f + \beta g) = \alpha(\partial_j f) + \beta(\partial_j f)$, $1 \le j \le n$
Linearity of derivative from Calc 1 applied to partial functions in direction j

17.9 Proof of 2, by assuming 3

Have $f \in C^1(A, \mathbb{R}^m)$ with $f(\vec{x}) \in B$, $\forall \vec{x} \in A \subseteq \mathbb{R}^n$
Have $u \in C^1(B, \mathbb{R}), v = u \circ f: A \to \mathbb{R}$
Fix $\vec{a} \in A, j \in \{1, \dots, n\}$. Want to verify that

$$\lim_{\substack{t\to 0 \\ t\neq 0}} \frac{v(\vec{a} + te_j) - v(\vec{a})}{t} \text{ exists and is equal to } \sum_{i=1}^{m} (\partial_i u)(\vec{b}) \cdot \left(\partial_j f^{(i)}\right)(\vec{a})$$

Pick $r > 0$ such that $B(\vec{a}; r) \subseteq A$
Define $\varphi: (-r, r) \to \mathbb{R}$, $\varphi(t) = v\left(\vec{a} + t\vec{e_j}\right)$

Have $\dfrac{v\left(\vec{a} + t\vec{e_j}\right) - v(\vec{a})}{t} = \dfrac{\varphi(t) - \varphi(0)}{t}$

So need that $\varphi'(0)$ exists and is given by the right formula.
Consider the path $\gamma: (-r, r) \to B \subseteq \mathbb{R}^m$, $\gamma(t) = f\left(\vec{a} + t\vec{e_j}\right)$, $-r < t < r$
$\varphi(t) = v\left(\vec{a} + te_j\right) = u\left(f\left(\vec{a} + t\vec{e_j}\right)\right) = u\left(\gamma(t)\right)$
So $\varphi(t) = u\left(\gamma(t)\right), -r < t < r$
Formula from 3 applies, gives

$$\varphi'(0) = \sum_{i=1}^{m} (\partial_i u)\left(\gamma(0)\right) \cdot \left(\gamma^{(t)}\right)(0) = \sum_{i=1}^{m} (\partial_i u)(\vec{b}) \cdot \left(\partial_j f^{(i)}\right)(\vec{a})$$

QED

Left to prove special case of Chain rule for $n = p = 1$

Proof of Lemma 17.10

$$\frac{\left\|\gamma(t_0 + s) - \left(\vec{b} + s\vec{v}\right)\right\|}{|s|} = \left\|\frac{1}{s}\left(\gamma(t + s) - \vec{b} - s\vec{v}\right)\right\|$$

$$= \left\|\frac{1}{s}\left(\gamma(t_0 + s) - \gamma(t_0)\right) - \vec{v}\right\|$$

What is component i of this vector? It is:

$$\frac{1}{s}\left(\gamma^{(i)}(t_0 + s) - \gamma^{(i)}(t_0)\right) - \left(\gamma^{(i)}\right)'(t_0)$$

So have

$$\frac{\left\|\gamma(t_0 + s) - \left(\vec{b} + s\vec{v}\right)\right\|}{|s|} = \left\|\frac{1}{s}\left(\gamma(t_0 + s) - \gamma(t_0)\right) - \vec{v}\right\|$$

$$\le \sum_{i=1}^{n} \left|\frac{\gamma^{(i)}(t_0 + s) - \gamma^{(i)}(t_0)}{s} - \left(\gamma^{(i)}\right)'(t_0)\right| \to 0 \text{ by definition of } \left(\gamma^{(i)}\right)'(t_0)$$

■

Proof of Proposition 17.11

Fix $t_0 \in I$ for which we verify the claim. Denote $\vec{b} \coloneqq \gamma(t_0)$, $\vec{v} \coloneqq \gamma'(t_0)$
Must prove that h is differentiable at t_0 with
$h'(t_0) = \left\langle(\nabla g)(\vec{b}), \vec{v}\right\rangle = (\partial_{\vec{v}} g)(\vec{b})$

So what we want is $\displaystyle\lim_{\substack{s\to 0 \\ s\neq 0}} \frac{h(t_0 + s) - h(t_0)}{s} = (\partial_{\vec{v}} g)(\vec{b})$ [*Want*]

Calculate

$$\frac{h(t_0 + s) - h(t_0)}{s} = \frac{g\left(\gamma(t_0 + s)\right) - g\left(\gamma(t_0)\right)}{s} = \frac{g\left(\gamma(t_0 + s)\right) - g(\vec{b})}{s}$$

$$= \frac{g\left(\gamma(t_0 + s)\right) - g\left(\vec{b} + s\vec{v}\right)}{s} + \frac{g\left(\vec{b} + s\vec{v}\right) - g(\vec{b})}{s}$$

We know that $\displaystyle\lim_{\substack{s\to 0 \\ s\neq 0}} \frac{g\left(\vec{b} + s\vec{v}\right) - g(\vec{b})}{s} = (\partial_{\vec{v}} g)(\vec{b})$

So [Want] will follow if we prove

$$\lim_{\substack{s\to 0 \\ s\neq 0}} \left|\frac{g\left(\gamma(t_0 + s)\right) - g\left(\vec{b} + s\vec{v}\right)}{s}\right| = 0 \quad [Want']$$

To prove [Want'] we will use a Lipschitz condition for g.
Fix $r > 0$ such that $B(\vec{b}; r) \subseteq B$. Use problem 4 in homework 8 for the compact convex set $K = \bar{B}\left(\vec{b}; \frac{r}{2}\right)$ to get $c > 0$ such that
$|g(\vec{x}) - g(\vec{y})| \le c\|\vec{x} - \vec{y}\|$, $\forall \vec{x}, \vec{y} \in K$ [*Lip*]

γ is continuous at t_0 hence can find $l > 0$ such that $(t_0 - l, t_0 + l) \subseteq I$ and such that $t \in (t_0 - l, t_0 + l) \Rightarrow \|\gamma(t) - \gamma(t_0)\| < \frac{r}{2}$
So for $|s| < l$, have $\left\|\gamma(t_0 + s) - \vec{b}\right\| < \frac{r}{2} \Rightarrow \gamma(t_0 + s) \in K$
For $|s| < \frac{1}{1 + \|\vec{v}\|} \times \frac{r}{2}$ we also have that

Then $h \in C^1(A, \mathbb{R}^p)$ and for every $\vec{a} \in A$ have

$\boxed{(Jh)(\vec{a}) = (Jg)\big(f(\vec{a})\big) \times (Jf)(\vec{a})}$ $(M-J)$, Multiplicativity of Jacobian

Aside

The chain rule from calc 1 is the special case of this where $m = n = p = 1$

17.6 Remark

Equation (M-J) is usually written in terms of entries:
For $1 \le k \le p$, $\quad 1 \le j \le n$, have

$$(Jh)(\vec{a})_{(i,j)} = \sum_{i=1}^{m} (Jg)(\vec{b})_{(k,i)} \times (Jf)(\vec{a})_{(i,j)}$$

Write $f = \left(f^{(1)}, \dots, f^{(m)}\right),\ g = \left(g^{(1)}, \dots, g^{(p)}\right),\ h = \left(h^{(1)}, \dots, h^{(p)}\right)$

$$\left(\partial_j h^{(k)}\right)(\vec{a}) = \sum_{i=1}^{m} \left(\partial_i g^{(k)}\right)(\vec{b}) \times \left(\partial_j f^{(i)}\right)(\vec{a})$$

Denote $u := g^{(k)}$, $\quad v := h^{(k)}$, $\quad$ What is the relation between u and v?

$h(\vec{x}) = g\big(f(\vec{x})\big) = \left(g^{(1)}\big(f(\vec{x})\big), \dots, g^{(p)}\big(f(\vec{x})\big)\right)$

Take component $k \Rightarrow h^{(k)}(\vec{x}) = g^{(k)}\big(f(\vec{x})\big) \Rightarrow v(\vec{x}) = u\big(f(\vec{x})\big)$

The modified (M-J) says

$$\boxed{\begin{array}{l} \displaystyle (\partial_j v)(\vec{a}) = \sum_{i=1}^{m} (\partial_i u)(\vec{b}) \times \left(\partial_j f^{(i)}\right)(\vec{a}) \\ \text{for } \vec{b} = f(\vec{a}) \text{ and } v(\vec{x}) = u\big(f(\vec{x})\big), \quad \vec{x} \in A \end{array}} \quad (C-R) \text{ Chain Rule, p=1}$$

Notation

To make it more suggestive, people write

$$(\partial_i v)(\vec{a}) \equiv \frac{\partial v}{\partial x^{(i)}}(\vec{a}), \qquad \frac{\partial u}{\partial y^{(i)}}(\vec{b}) \equiv (\partial_i u)(\vec{b})$$

$$\frac{\partial v}{\partial x^{(j)}}(\vec{a}) = \sum_{i=1}^{m} \frac{\partial u}{\partial y^{(i)}}(\vec{b}) \times \frac{\partial f^{(i)}}{\partial x^{(j)}}(\vec{a})$$

Summarized

$$\frac{\partial v}{\partial x^{(j)}} = \sum_{i=1}^{m} \frac{\partial u}{\partial y^{(i)}} \cdot \frac{\partial y^{(i)}}{\partial x^{(j)}}$$

Imprecise in two ways: $\frac{\partial y^{(i)}}{\partial x^{(i)}}$ should be $\frac{\partial f^{(i)}}{\partial x^{(i)}}$, and does not specify to what points the derivatives should be applied.

17.7 Remark

Special case when $n = p = 1$.
Take $I \subseteq \mathbb{R}$ open interval
$\gamma: I \to \mathbb{R}^m$ a C^1-path
Let $B \subseteq \mathbb{R}^m$ open such that $\gamma(t) \in B, \forall t \in I$.
Let g be in $C^1(B, \mathbb{R})$
Consider composed function $h = g \circ \gamma \in C^1(I, \mathbb{R})$

$$h'(t) = \sum_{i=1}^{m} \partial_i g\big(\gamma(t)\big) \times \left(\gamma^{(i)}\right)'(t) \qquad (C-R) \text{ Chain rule } p = n = 1$$

$$h'(t) = \big\langle (\nabla g)\big(\gamma(t)\big), \gamma'(t) \big\rangle$$

17.8 Remark

Had 3 formulas for the chain rule:

1. $(M-J)$ In Theorem 17.5
2. $(C-R)$ for $p = 1$ in Remark 17.6
3. $(C-R)$ for $n = p = 1$ in Remark 17.7

Clearly $1 \Rightarrow 2 \Rightarrow 3$ because 2 and 3 are special cases.
Conversely, $2 \Rightarrow 1$. Saw this in Remark 17.6 - just have to fix a value $k \in \{1, \dots, p\}$ with $u = g^{(k)},\ v = h^{(k)}$

Observe that $3 \Rightarrow 2$ (Proof 17.9)

17.10 Lemma

$I \subseteq \mathbb{R}$ open interval, $\gamma: I \to \mathbb{R}^m$ a C^1-path
Fix $t_0 \in I$, denote $\vec{b} := \gamma(t_0),\ \gamma'(t_0) := \vec{v}$

Then $\displaystyle \lim_{\substack{s \to 0 \\ s \ne 0}} \frac{\left\| \gamma(t_0 + s) - \left(\vec{b} + s\vec{v}\right) \right\|}{|s|} = 0$

This is an approximation lemma: $\gamma(t_0 + s) \approx \gamma(t_0) + s\gamma'(t_0)$

17.11 Proposition ("CR for $n = p = 1$")

$I \subseteq \mathbb{R}$ open interval, $\gamma: I \to \mathbb{R}^m$ a C^1 path.
$B \subseteq \mathbb{R}^m$ open such that $\gamma(t) \in B,\ \forall t \in I$
Let g be a function on $C^1(B, \mathbb{R})$ and let $h = g \circ \gamma$ so $h: I \to \mathbb{R},\ h(t) = g\big(\gamma(t)\big), t \in I$
Then $h \in C^1(I, \mathbb{R})$ and

$$h'(t) = \sum_{i=1}^{m} (\partial_i g)\big(\gamma(t)\big) \cdot \left(\gamma^{(i)}\right)'(t) = \big\langle (\nabla g)\big(\gamma(t)\big), \gamma'(t) \big\rangle$$

such that $t \in (t_0 - l, t_0 + l) \Rightarrow \|\gamma(t) - \gamma(t_0)\| < \frac{r}{2}$

So for $|s| < l$, have $\left\|\gamma(t_0 + s) - \vec{b}\right\| < \frac{r}{2} \Rightarrow \gamma(t_0 + s) \in K$

For $|s| < \frac{1}{1 + \|\vec{v}\|} \times \frac{r}{2}$ we also have that

$$\left\|\left(\vec{b} + s\vec{v}\right) - \vec{b}\right\| = |s|\|\vec{v}\| < \frac{\|\vec{v}\|}{1 + \|\vec{v}\|} \times \frac{r}{2} < \frac{r}{2} \Rightarrow \vec{b} + s\vec{v} \in \text{K}$$

So for $|s| < \min\left(l, \frac{r}{2(1 + \|\vec{v}\|)}\right)$ [Lip] will apply to $\vec{x} = \gamma(t_0 + s),\ \vec{y} = \vec{b} + s\vec{v}$

$$\frac{\left|g\big(\gamma(t_0 + s)\big) - g\left(\vec{b} + s\vec{v}\right)\right|}{|s|} \le \frac{c\left\|\gamma(t_0 + s) - \left(\vec{b} + s\vec{v}\right)\right\|}{|s|}$$

But lemma 17.10 says that

$\dfrac{\left|\gamma(t_0 + s) - \left(\vec{b} + s\vec{v}\right)\right|}{|s|} \to 0$ So by squeeze we get

$\dfrac{\left\|g\big(\gamma(t_0 + s)\big) - g\left(\vec{b} + s\vec{v}\right)\right|}{|s|} \to_{s \to 0} 0$ Which is [Want']

The fact that $h': I \to \mathbb{R}$ is continuous comes from immediately from the formula

$$h'(t) = \sum_{i=1}^{m} (\partial_i g)\big(\gamma(t)\big) \cdot \left(\gamma^{(i)}\right)'(t)$$

because $(\partial_i g), \gamma(t), \left(\gamma^{(i)}\right)'$ are all continuous.

QED

Special case when $m = n$

November-30-11 11:31 AM

If $m = n$ then the Jacobian matrix is a **square matrix.** Can talk about **determinant** and about **invertibility**.

Recall

For $M \in M_{n\times n}(\mathbb{R})$ have M invertible $\Leftrightarrow \exists X \in M_{n\times n}(\mathbb{R})$ such that $MX = I_n = XM$

Various other descriptions M invertible $\Leftrightarrow \ker N = \emptyset \Leftrightarrow \det M \neq 0$

18.1 Remark

For every $n \geq 1$, the formula for $n \times n$ determinant is a polynomial expression in the entries of the matrix. That is, $\exists$ polynomial P_n of n^2 indeterminates such that

$M = [t_{ij}]_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{R}) \Rightarrow det(M) = P_n(t_{11}, t_{12}, \dots, t_{nn})$

Therefore, P_n is a continuous function on $\mathbb{R}^{n^2}$

18.2 Lemma

Small Perturbation of Invertible Matrices

Let $M = [\alpha_{ij}]_{1\leq i,j\leq n}$ be an invertible matrix. $\exists \lambda > 0$ with the following property:

If $N = [\beta_{ij}]_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{R})$ is such that $|\alpha_{ij} - \beta_{ij}| < \lambda,\ \forall 1 \leq i,j \leq n$ then N is invertible as well.

18.3 Proposition

$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R}^n)$, $\vec{a} \in A$ such that $(Jf)(\vec{a})$ is invertible.

Then $\exists r > 0$ s.t. $B(\vec{a}; r) \subseteq A$ and s.t. f is one-to-one and injective on $B(\vec{a}; r)$.

18.4 Definition

$U, V \in \mathbb{R}^n$ open sets

A C^1-**diffeomorphism** between U and V is a bijection $f: U \to V$ such that both f and its inverse $g: V \to U$ are C^1-functions.

18.5 Theorem

$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R}^n)$, $\vec{a} \in A$ s.t. $(Jf)(\vec{a})$ is an invertible $n \times n$ matrix. Denote $f(\vec{a}) = \vec{b}$.

Then $\exists U, V \subseteq \mathbb{R}^n$ open sets such that

i) $\vec{a} \in U \subseteq A, \quad \vec{b} \in V$
ii) f maps U onto V bijectively
iii) The function $g: V \to U$ which inverts f is a C^1-function and has $(Jg)(\vec{b}) = \left((Jf)(\vec{a})\right)^{-1}$

In short, we get a C^1-diffeomorphism produced by f on an open neighbourhood of $\vec{a}$

18.6 Remark

Discussion around the steps in proof of Theorem 18.5

a) One can find $r > 0$ such that $U = B(\vec{a}; r) \subseteq A$ and such that f is one-to-one on U. So we can put $V := f(U) = \{f(\vec{x}) | \vec{x} \in U\}$ and have that f gives a bijection from U to V with an inverse $g: V \to U$.
b) It can be proved that by reducing r if necessary, one can arrange that V is open, and such that g is C^1-function.
c) For $g: V \to U$ as in b, one proves that $(Jg)(\vec{b}) = \left((Jf)(\vec{a})\right)^{-1}$

Determinant Example

$\det\left(\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\right) = P_2(t_{11}, t_{12}, t_{21}, t_{22}) = t_{11}t_{22} - t_{12}t_{21}$

Proof of Lemma 18.2

Denote $|\det(M)| = \varepsilon > 0$

So $|P_n(\alpha_{11}, \dots, \alpha_{nn})| = |\det(M)| = \varepsilon$ where P_n is as in Remark 18.1

Write continuity of P_n at $(\alpha_{11}, \dots, \alpha_{nn}) \in \mathbb{R}^{n^2}$ for $\frac{\varepsilon}{2}$, $\exists \delta > 0\ s.t.$

$\|(\beta_{11}, \dots, \beta_{nn}) - (\alpha_{11}, \dots, \alpha_{nn})\| < \delta \Rightarrow |P_n(\beta_1, , \dots, \beta_{nn}) - P_n(\alpha_{11}, \dots, \alpha_{nn})| < \frac{\varepsilon}{2}$

Set $\lambda = \frac{\delta}{n}$. Will show that this λ satisfies the Lemma.

Pick a matrix $N = [\beta_{ij}]_{1\leq i,j\leq n}$ such that $|\alpha_{ij} - \beta_{ij}| < \lambda\ \forall 1 \leq i,j \leq n$

Will show that N is invertible.

Observe first that

$$\|(\beta_{11}, \dots, \beta_{nn}) - (\alpha_{11}, \dots, \alpha_{nn})\| = \sqrt{\sum_{i,j=1}^{n} (\beta_{ij} - \alpha_{ij})^2} \leq \sqrt{\sum_{i,j=1}^{n} \lambda^2} = n\lambda = \delta$$

$\Rightarrow |P_n(\beta_{11}, \dots, \beta_{nn}) - P_n(\alpha_{11}, \dots, \alpha_{nn})| < \frac{\varepsilon}{2} \Rightarrow |\det(N) - \det(M)| < \frac{\varepsilon}{2} \Rightarrow$

$-\frac{\varepsilon}{2} < \det(N) - \varepsilon < \frac{\varepsilon}{2} \Rightarrow \frac{\varepsilon}{2} < \det(N) < \frac{3\varepsilon}{2} \Rightarrow \det(N) \neq 0$

So N is invertible ■

Proof of Proposition 18.3

Denote $(Jf)(\vec{a}) = M = [\alpha_{ij}]_{1\leq i,j\leq n}$

So $\alpha_{i,j} = \left(\partial_j f^{(i)}\right)(\vec{a})\ \forall 1 \leq i,j \leq n$. Lemma 18.2 says $\exists \lambda > 0$ such that if $N = [\beta_i, j]_{1\leq i,j\leq n}$ has $|\alpha_{ij} - \beta_{ij}| < \lambda\ \forall 1 \leq i,j \leq n$ then N is invertible.

Due to continuity of partial derivatives $\partial_j f^{(i)}$ at $\vec{a}$ we can find $r > 0$ such that $B(\vec{a}; r) \subseteq A$ and such that $\left|\left(\partial_j f^{(i)}\right)(\vec{b}) - \left(\partial_j f^{(i)}\right)(\vec{a})\right| < \lambda, \forall 1 \leq i,j \leq n, \forall \vec{b} \in B(\vec{a}; r)$

We will prove that this r satisfies the claim.

Fix $\vec{x} \neq \vec{y}$ in $B(\vec{a}; r)$. Must prove that $f(\vec{x}) \neq f(\vec{y})$. Assume by contradiction that $f(\vec{x}) = f(\vec{y})$, that is $f^{(i)}(\vec{x}) = f^{(i)}(\vec{y})\ \forall 1 \leq i \leq n$

For every $1 \leq i \leq n$, we apply MVT in direction $\vec{v}$ to the function $f^{(i)} \in C^1(A, \mathbb{R})$ where $\vec{v} = \vec{y} - \vec{x}$.

Get a point $\vec{b} \in Co(\vec{x}, \vec{y})$ such that $0 = f^{(i)}(\vec{y}) - f^{(i)}(\vec{x}) = \langle (\nabla f^{(i)})(\vec{b}_i), \vec{v} \rangle$

Consider the matrix $N = [\beta_{ij}]_{1\leq i,j\leq n} = \begin{bmatrix} (\nabla f^{(1)})(\vec{b_1}) \\ \vdots \\ (\nabla f^{(n)})(\vec{b_n}) \end{bmatrix}$

With $\beta_{ij} = \left(\partial_j f^{(i)}\right)(\vec{b})\ \forall 1 \leq i,j \leq n$ get $\left|\left(\partial_j f^{(i)}\right)(\vec{b_i}) - \left(\partial_j f^{(i)}\right)(\vec{a})\right| < \lambda$

Therefore N is invertible.

But $\langle (\nabla f^{(i)})(\vec{b_i}), \vec{v} \rangle = 0\ \forall 1 \leq i \leq n \Rightarrow \vec{v} \in \ker N$ so N is not invertible. Contradiction

QED

18.6 Remark Proof

a) Was done in Prop 18.3
b) We will accept (part with V being open is itself a theorem called the "open mapping theorem")
c) Easy, do it now. Holds in fact for any C^1-diffeomorphism.
Consider composed function $h: U \to U, h = g \circ f$
$h(\vec{x}) = g\left(f(\vec{x})\right), \quad \forall \vec{x} \in U$
Chain rule says $(Jh)(\vec{a}) = (Jg)(\vec{b}) \cdot (Jf)(\vec{a})$
But on the other hand have, $h(\vec{x}) = \vec{x}\ \forall \vec{x} \in U$
So $h(\vec{x}) = \left(h^{(1)}(\vec{x}), \dots, h^{(n)}(\vec{x})\right) = \left(x^{(1)}, \dots, x^{(n)}\right)$
Hence $\left(\partial_j h^{(i)}\right)(\vec{x}) = \begin{cases} 0\ if\ j \neq i \\ 1 if\ j = i \end{cases} \Rightarrow (Jh)(\vec{a}) = I_n$
So chain rule gives $I_n = (Jg)(\vec{b}) \times (Jf)(\vec{a}) \Rightarrow (Jg)(\vec{b}) = \left((Jf)(\vec{a})\right)^{-1}$

Change of Variables

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18.7 Definition

$A \subseteq \mathbb{R}^n, f \in C^1(A, \mathbb{R}^n), \vec{a} \in A$
The **Jacobian** of f at $\vec{a}$ is defined as
$|J|(\vec{a}) \coloneqq |\det((Jf)(\vec{a}))|$ where $(Jf)(\vec{a}) \in M_{n \times n}$
is the **Jacobian matrix** of f at $\vec{a}$

18.8 Remark

$A \subseteq \mathbb{R}^n$ open, $f \in C^1(A, \mathbb{R}^n)$
Have new function $|J|_f : A \to \mathbb{R}$
This is continuous.
If $\vec{a}_k \to_{k \to \infty} \vec{a}$ in A then $(\partial_j f^{(i)})(\vec{a}_k) \to_{k\to\infty} (\partial_j f^{(i)})(\vec{a})$
$\Rightarrow (Jf)(\vec{a}_k) \to (Jf)(\vec{a}) \Rightarrow \det((Jf)(\vec{a}_k)) \to \det((Jf)(\vec{a}))$
Because det is polynomial hence continuous
$\therefore |J|_f(\vec{a}_k) \to |J|_f(\vec{a})$ so $|J|_f$ respects sequences $\Rightarrow$ continuous.

18.9 Theorem (Change of Variable)

$A, B \subseteq \mathbb{R}^n$ open and bounded $T: A \to B$ a C^1-diffeomorphism.
Suppose in addition that $|J|_f$ is bounded on A
$(\exists c > 0 \; s.t. |J|_T(\vec{x}) < c, \forall \vec{x} \in A)$

Let $g \in Int_b(B, \mathbb{R})$. Put $f = g \circ T$ so $f: A \to \mathbb{R}, f(\vec{x}) = g(T(\vec{x})),\ \vec{x} \in A$
Then $f \in Int_b(A, \mathbb{R})$ and $\int_B g(\vec{y}) d\vec{y} = \int_A f(\vec{x}) \cdot |J|_T(\vec{x}) d\vec{x}, \qquad [C - V]$

18.10 Remark (how to remember [C-V]

Do the substitution $\vec{y} = T(\vec{x}),\ (\vec{y} \in B, \vec{x} \in A)$
$d\vec{y} = |J|_T(\vec{x}) d\vec{x}$
$\int_B g(\vec{y}) d\vec{y} = \int_A g(T(\vec{x})) |J|_T d\vec{x} = \int_A |J|_T f(\vec{x}) d\vec{x}$
This is analogous to substitution in one variable
$y = T(x), \qquad dy = T'(x) dx$

18.12 Remark

Why does the formula $(C - V)$ hold?
$|J|_T$ keeps track of how volumes are distorted by T

Take again the case of $T: R \to A$ from example 18.11
Take a division $R = \bigcup_{i=1}^k P_i, \qquad A = \bigcup_{i=1}^k Q_i, \qquad Q_i = T(P_i)$
Then $\int_R f \approx \sum_{i=1}^k \sup_{P_i}(f) \cdot vol(P_i)$
$\int_A g \approx \sum_{i=1}^k \sup_{Q_i}(g) \cdot vol(Q_i)$
$\forall 1 \le i \le k$ we have $\sup_{P_i}(f) = \sup_{P_i} g(T(\vec{x})) = \sup_{Q_i}(g)$
But not true that $vol(Q_i) = vol(P_i)$
In fact have $\frac{vol(Q_i)}{vol(P_i)} \approx value\ of\ |J|_T\ on\ P_i$
Since $|J|_T$ is continuous, it is approximately constant for small P_i

On this specific example
$vol(P_i) = (r' - r)(\theta' - \theta)$
$vol(R_i) = \frac{r'^2 - r^2}{2}(\theta' - \theta) \Rightarrow \frac{vol(Q_i)}{vol(P_i)} = \frac{r + r'}{2} \approx r$

18.11 Example

Take $R = (r_1, r_2) \times (0, 2\pi),\ A = \{(s,t) \in \mathbb{R}^2 \,|r_1 < \sqrt{s^2 + t^2} < r_2\} \setminus$
$\{(s, 0) | r_1 < s < r_2\}$
$T((r, \theta)) = (r\cos\theta, r\sin\theta) = (T^{(1)}(r,\theta), T^{(2)}(r,\theta))$
$(\nabla T^{(1)})(r, \theta) = (\cos\theta, -r\sin\theta)$
$(\nabla T^{(2)})(r, \theta) = (\sin\theta, r\cos\theta)$
$(JT)(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$
$|J|_T(r, \theta) = r\cos^2\theta + r\sin^2\theta = r$

Formula (C-V) says if $g \in Int_B(A, \mathbb{R})$ then $f = g \circ T \in Int_B(R, \mathbb{R})$ with
$\int_A g((s,t)) d(s,t) = \int_R f((r,\theta)) \cdot r \cdot d(r, \theta) = \int_{r_1}^{r_2} \int_0^{\pi} g(r\cos\theta, r\sin\theta) r \, d(r, \theta)$