Vectors & Tensor Notation

January-09-13 11:34 AM

Tensor Notation (Einstein Notation)

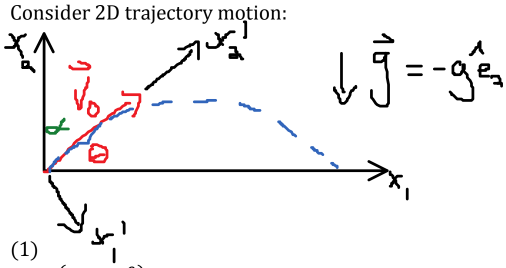
$$a_i b_i \equiv \sum_i a_i b_i$$

$$\vec{a} \cdot \vec{b} = \sum_i a_i b_i = a_i b_i$$

$$\vec{a} \otimes \vec{b} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = a_i b_j$$

$$\vec{a} \times \vec{b} = \epsilon_{ijk} a_j b_k$$

Consider 2D trajectory motion:



(1)

$$x_1 = (v_o \cos\theta)t$$

$$x_2 = (v_0 \sin\theta)t + \frac{1}{2} g t^2$$

These equations are valid (assuming no air resistance, etc.) but only for this specific choice of coordinate system

Now consider the similar problem with the projectile along an incline. The problem may be easier to analyze using the prime coordinates.

(2)

$$x_1' = (v_0 \cos(\alpha + \theta))t + \frac{1}{2}(g \sin\alpha)t^2$$

$$x_2' = (v_0 \sin(\alpha + \theta))t + \frac{1}{2}(g \cos\alpha)t^2$$

which, in the $\alpha \to 0$ limit become (1)

In other situations it may be advantageous to move the origin to give

(3)

$$x_1 = \left(x_{i_0}\right) + (v_0 \cos(\alpha + \theta))t + \frac{1}{2} t^2 (g \sin\alpha)$$

$$x_2 = \left(x_{2_0}\right) + (v_0\, sin(\alpha + \theta))t + \frac{1}{2} t^2 (g \cos\alpha)$$

In physics it is usually a good idea to ad a level of abstraction by removing the specific choice of origin from our equations. We do this by using vectors. (or, more generally, tensors). Eq. 3 can be written as

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{g} t^2$$

which makes no assumptions about the coordinate system.

This is an extremely important abstraction both conceptually and computationally as this equation is valid for any choice of coordinate system. We have the freedom the choose a coordinate system so as to make subsequent calculations easier.

Further benefits are:

- its compactness
- ease of computer implementation
- easy to extend to higher dimensions
- easy to compare to other systems
- we can ad further abstractions to understand even more interesting systems.

Because it will greatly simplify our work later in the term, we introduce a further abstraction by using for unit vectors $\hat{e}_i,\ i = 1,2,3$ instead of $\hat{\imath}, \hat{\jmath}, \hat{h}$ as this allows us to use tensor notation.

Rotations

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Notation

Doubled letters ($\mathbb{R}$) represent matrices.

Rotation Matrix

$$\mathbb{R}(-\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$\mathbb{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$\mathbb{R}(\alpha)$ rotates vectors by angle α in the counter-clockwise direction.

If $\{\hat{e}_1, \hat{e}_2\}$ is a coordinate system and $\{\hat{e}_1', \hat{e}_2'\}$ is another coordinate system rotated from the first by an angle of α counter-clockwise then

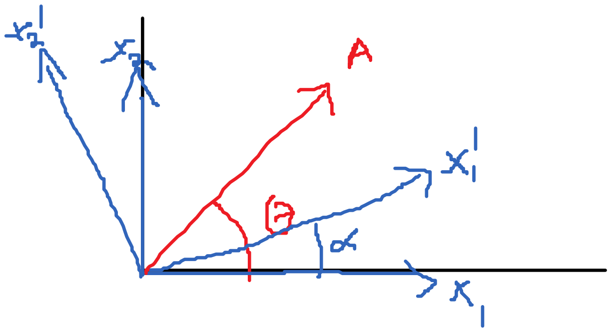
$$\mathbb{R}(\alpha) = \begin{pmatrix} \hat{e}_1 \cdot \hat{e}_1' & \hat{e}_1 \cdot \hat{e}_2' \\ \hat{e}_2 \cdot \hat{e}_1' & \hat{e}_2 \cdot \hat{e}_2' \end{pmatrix} = \hat{e}_i \cdot \hat{e}_j'$$

Note that $\hat{e}_i \cdot \hat{e}_j'$ is a 2×2 tensor in 2D (rank 2 tensor)

Tensor

Generalization of scalar/vector/matrix
The rank of a tensor is the number of indices used to indicate elements.

Consider a vector $\vec{A}$ and its representation in two coordinate systems.
$\vec{A} = A(\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2)$
$\vec{A}' = A(\cos(\theta-\alpha)\,\hat{e}_1' + \sin(\theta-\alpha)\,\hat{e}_2')$
$[\hat{e}_1', \hat{e}_2']$ is rotated with respect to $[\hat{e}_1, \hat{e}_2]$



$$\vec{A}' = A\big((cos\theta\cos\alpha + \sin\theta\sin\alpha)\hat{e}_1' + (\sin\theta\cos\alpha + \sin\alpha\cos\theta)\hat{e}_2'\big)$$
$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} A$$

Example

$$(1)\ \begin{cases} x_1 = (v_0\cos\theta)t \\ x_2 = (v_0\sin\theta)t - \frac{1}{2}gt^2 \end{cases}$$

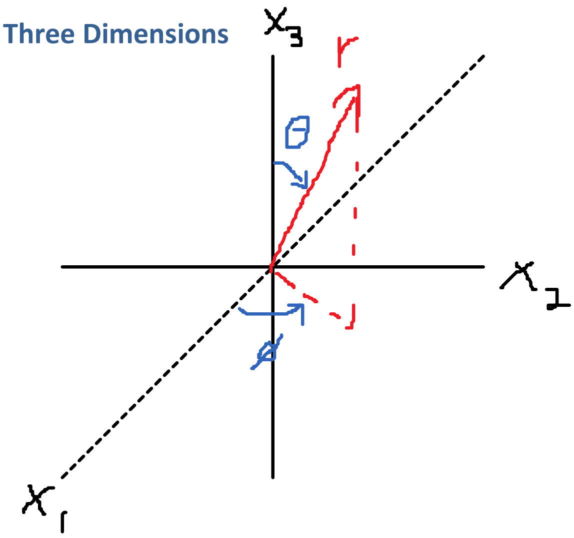
$$(2)\ \begin{cases} x_1' = (v_0\cos(\alpha+\theta))t + \frac{1}{2}(g\sin\alpha)t^2 \\ x_2' = (v_0\sin(\alpha+\theta))t + \frac{1}{2}(-g\cos\alpha)t^2 \end{cases}$$

$$\vec{r} = \begin{pmatrix} (v_0\cos\theta)t \\ (v_0\sin\theta)t - \frac{1}{2}gt^2 \end{pmatrix}$$

$$\vec{r}' = \mathbb{R}(\alpha)\vec{r}$$

$$\mathbb{R}(\alpha)\cdot\mathbb{R}(-\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}$$

Three Dimensions



Unit vectors

$$\hat{e}_i' = \begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix}$$

To get $\vec{r}$ from $\hat{e}_3$, rotate θ along x_2 axis then rotate by ϕ along x_3 axis

$$\hat{e}_i' = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\hat{e}_i = \mathbb{R}_3(\phi)\mathbb{R}_2(\theta)\hat{e}_i$$

$$= \begin{pmatrix} \cos\phi\cos\theta & -\sin\phi & \cos\phi\sin\theta \\ \sin\phi\cos\theta & \cos\phi & \sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \hat{e}_i$$

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Example (Problem 1.23)

Use the ϵ_{ijk} notation to derive the identity

$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = (\vec{A}\vec{B}\vec{D})\vec{C} - (\vec{A}\vec{B}\vec{C})\vec{D}$

Problem 1.11

where $\vec{A}\vec{B}\vec{C} = (\vec{A}\times\vec{B})\cdot\vec{C} = \vec{B}\vec{C}\vec{A} = \vec{C}\vec{A}\vec{B}$

$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \Rightarrow \vec{C}\cdot(\vec{A}\times\vec{B}) = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} =$ any cyclic permuation of rows

$= \vec{A}\cdot(\vec{B}\times\vec{C})$

$\vec{A}\vec{B}\vec{C} = C_i(\epsilon_{ijk}A_jB_k) = \epsilon_{ijk}A_jB_kC_i$

Problem 1.22

$\epsilon_{ijk}\epsilon_{krs} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$

$\vec{A}\times\vec{B} = \epsilon_{ijk}A_jB_k$

$\vec{C}\times\vec{D} = \epsilon_{lmn}C_mD_n$

$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = \epsilon_{pil}(\epsilon_{ijk}A_jB_k)(\epsilon_{lmn}C_mD_n)$

Using identity from problem 1.22

$= (\epsilon_{pil}\epsilon_{ijk})\epsilon_{lmn}A_jB_kC_mD_n = (\epsilon_{lpi}\epsilon_{ijk})\epsilon_{lmn}A_jB_kC_mD_n = (\delta_{lj}\delta_{pk} - \delta_{lk}\delta_{pj})\epsilon_{lmn}A_jB_kC_mD_n$

Vector Differentiation

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Scalars

Consider some scalar function of space (e.g. temperature or pressure) $\phi(s)$ evaluated along some curve $\Gamma(s)$, parameterized by s, which is also a scalar.

Since both ϕ and s are scalars (i.e.. they can't change under a coordinate transformation) we must have that $\phi(s) = \phi'(s') = \phi'(s)$
(Where x_1, x_2 is the original coordinate system and x_1', x_2' is another coordinate system)

So
$$\frac{d\phi}{ds} = \frac{d\phi'}{ds'} = \left(\frac{d\phi}{ds}\right)'$$

Vectors

Now consider a vector quantity which is a function of space, $\vec{A} = \vec{A}(\vec{r})$, which can also be evaluated along the curve $\Gamma(s)$. The components of $\vec{A}$ transform the usual way.
$\vec{A}' = \mathbb{R}\vec{A}$ or $A_i' = R_{ij}A_j$

So
$$\frac{d}{ds'}(\vec{A}') = \frac{d}{ds}(\vec{A}') = \frac{d}{ds}(\mathbb{R}\vec{A}) = \mathbb{R}\frac{d}{ds}(\vec{A})$$
or
$$\frac{dA_i'}{ds'} = R_{ij}\frac{dA_j}{ds}$$
and $\frac{d\vec{A}}{ds}$ also transforms as a vector.

The obvious situation where we will use this is when we find $\dot{\vec{r}} = \frac{d\vec{r}}{dE}$ and $\ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2}$
We can write:
$$\vec{r} = r_i\hat{e}_i = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3$$
$$\dot{\vec{r}} = \dot{r}_i\hat{e}_i, \qquad \ddot{\vec{r}} = \ddot{r}_i\hat{e}_i$$

Polar Coordinates

Things are more interesting in polar coordinates (r, θ) in which the unit vectors $\hat{r}$ and $\hat{\theta}$ are functions of position (and so time for a moving particle).

From the diagram (see O'Donovan's notes posted online) we see that:
$$\dot{\hat{r}} = \dot{\theta}\hat{\theta}, \qquad \dot{\hat{\theta}} = -\dot{\theta}\hat{r}$$
so in polar coordinates we have for velocity and acceleration:
$$\vec{r} = r\hat{r}$$
$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$
$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

Exercises

Exercise 7

$$\hat{r} \parallel \frac{\partial\vec{r}}{\partial r}, \qquad \hat{\theta} \parallel \frac{\partial\vec{r}}{\partial\theta}$$
$$\vec{r} = r(\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2)$$
$$\frac{\partial\vec{r}}{\partial r} = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2$$
$$\left|\frac{\partial\vec{r}}{\partial r}\right|^2 = \cos^2\theta + \sin^2\theta = 1$$
$$\hat{r} = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2$$

$$\frac{\partial\vec{r}}{\partial\theta} = r(-\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2)$$
$$\left|\frac{\partial\vec{r}}{\partial\theta}\right|^2 = r^2(\sin^2\theta + \cos^2\theta)$$
$$\hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|} = -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2$$

Exercise 8

$$\vec{r} = \vec{r}(r, \theta)$$
$$\dot{\vec{r}} = \frac{\partial\vec{r}}{\partial r}\cdot\frac{dr}{dt}$$

Vector Calculus

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Polar Gradient

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$$

Example

What is $\vec{\nabla}\frac{1}{r}$?

We have $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$r^2 = x_i x_i$

$$2r\frac{\partial r}{\partial x_j} = \frac{\partial x_i}{\partial x_j}x_i + x_i\frac{\partial x_i}{\partial x_j} = 2x_i\frac{\partial x_i}{\partial x_j} = 2x_i\delta_{ij}$$

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{r} = \frac{x_j}{\sqrt{x_k x_k}}$$

$r = \sqrt{x_i x_i}$

$$\frac{\partial r}{\partial x_j} = \frac{\frac{1}{2}}{\sqrt{x_k x_k}} \cdot \frac{\partial}{\partial x_j}(x_i x_i) = \frac{1}{2\sqrt{x_k x_k}} \cdot 2x_j = \frac{x_j}{\sqrt{x_k x_k}}$$

$$\vec{\nabla}\frac{1}{r} = \left(\hat{e}_i\frac{\partial}{\partial x_i}\right)\cdot\frac{1}{\sqrt{x_j x_j}} = \hat{e}_i\frac{\partial}{\partial x_i}(x_j x_j)\cdot\frac{-\frac{1}{2}}{(x_k x_k)^{\frac{3}{2}}} = -\frac{x_i}{\left(\sqrt{x_k x_k}\right)^3}\hat{e}_i = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$$

In spherical coordinates:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\vec{\nabla}\frac{1}{r} = \hat{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$

Example

Consider a function in polar coordinates, $f(r,\theta)$. What is its gradient $\vec{\nabla}$?

Use $r^2 = x_1^2 + x_2^2$ and $\tan\theta = x_2/x_1$ along with the chain rule we have

$r^2 = x_1^2 + x_2^2$

$$2r\frac{\partial r}{\partial x_i} = 2x_i$$

$$\frac{\partial r}{\partial x_2} = \frac{x_1}{r} = \cos\theta$$

$$\frac{\partial r}{\partial x_2} = \sin\theta$$

Also,

$$\tan\theta = \frac{x_2}{x_1}$$

$$\sec^2\theta\frac{\partial\theta}{\partial x_1} = -\frac{x_2}{x_1^2} = \frac{-r\sin\theta}{(r\cos\theta)^2}$$

$$\Rightarrow \frac{\partial\theta}{\partial x_1} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial x_2} = \frac{\cos\theta}{r}$$

Unit Vectors

$\hat{e}_1 = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

$\hat{e}_2 = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$

$\hat{r} = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2$

$\hat{\theta} = -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2$

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x_1} + \frac{\partial f}{\partial\theta}\cdot\frac{\partial\theta}{\partial x_1} = \frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial\theta}\cdot\frac{\sin\theta}{r}$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial\theta}\cdot\frac{\cos\theta}{r}$$

$$\vec{\nabla} f = \hat{e}_1\frac{\partial f}{\partial x_1} + \hat{e}_2\frac{\partial f}{\partial x_2} = \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)\left(\frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial\theta}\cdot\frac{\cos\theta}{r}\right) + (\dots)(\dots) = \dots = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}$$

Now we use the fact that the total derivative if Cartesian can be written as

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 = \left(\vec{\nabla} f\right)\cdot d\vec{r}$$

We have

$df = \vec{\nabla} f\cdot d\vec{r} = \vec{\nabla} f\cdot\left(dr\,\hat{r} + r\,d\theta\hat{\theta}\right)$ and

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta$$

$$\Rightarrow \vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}$$

Example Rotations

A rotation of $\frac{\pi}{2}$ about x_1 is x_3

$$\mathbb{R}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

A rotation of $\frac{\pi}{2}$ about x_2 is

$$\mathbb{R}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\mathbb{R}_2\mathbb{R}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\mathbb{R}_1\mathbb{R}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So rotations do not commute in general.
However, infinitesimal rotations do commute.

$$\frac{d\vec{\theta}}{dt} = \vec{\omega}$$

Newtonian Mechanics

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Newton's Laws

1. If $\vec{F} = 0$ then $\vec{v} = const.$
2. $\sum \vec{F} = m\vec{a} = \dot{\vec{p}}$ (N2L -Newton's 2nd Law)
3. Action + reaction
   $\vec{F}_{12} = -\vec{F}_{21}$

Reference Frames

In order to apply Newton's laws we will need a reference frame.
Specifically, N2L in its simplest form $\sum \vec{F} = m\vec{a}$ is only valid in an **inertial reference frame** — one that assumes that space and time are homogeneous and isotropic. A measurement of the acceleration in such frames yields zero in all places (homogeneous) and directions (isotropic).

Isotropic

Nothing differentiates any direction of space from the others

Homogeneous

Same everywhere in space and time.

Acceleration in a Frame

Acceleration of a reference frame at a point and time is the value $\Delta\vec{a}$ such that $\sum \vec{F} = m(\vec{a} - \Delta\vec{a})$ holds.
(My own definition)

Equations of Motion

In order to apply N2L we first need to select an inertial reference frame (IRF) (or a suitable approximation). Then, for each of N particles, determine all external forces acting upon it. The usual method is to draw a free body diagram (FBD) and then add the vector forces and resulting vector acceleration. Then we apply N2L to obtain the EoM (Equation of Motion)

$\ddot{\vec{r}}_n(t) = f(\vec{r}_p, \dot{\vec{r}}_p), \qquad n, p = 1 \dots N$
Each particle depends on state of all other particles

Problem Solving Steps

1. Read problem
2. Free body diagram
3. Coordinate system
4. Equations of motion (with coordinate vectors)
5. Solves equation of motion

Problem With Moving Charges

Charge q_2 moving in $\hat{e}_2$ direction
Charge q_1 moving in $\hat{e}_1$ direction

Induce magnetic fields with strengths B_2 and B_1 at each other.
$\vec{F}_{21} = q_2\vec{v}_2 \times \vec{B}_1 = q_2 v_2 B_1 \hat{e}_2 \times \hat{e}_3 = q_2 v_2 B_1 \hat{e}_1$
$\vec{F}_{12} = q_1\vec{v}_1 \times \vec{B}_2 = q_1 v_1 B_2 \hat{e}_1 \times \hat{e}_3 = -q_1 v_1 B_2 \hat{e}_2$

Why do these not obey Newton's Law #2?
Some momentum goes into the magnetic fields.

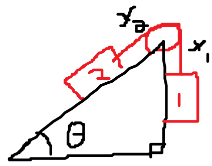
Example Equations of Motion

$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{g}t^2$
Resulted from $m\ddot{\vec{r}} = m\vec{g}$

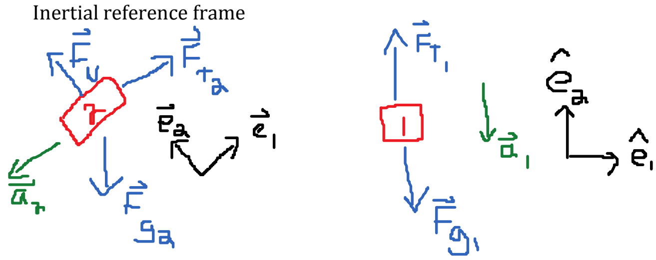
$\vec{r} = \vec{R}_0 \cos(\omega t + \phi)$
from
$\ddot{\vec{r}} = -\omega^2 \vec{r}$

Example

Consider two blocks, masses m_1 and m_2, connected by a light, inextensible string which passes over a light pulley as shown.



Inertial reference frame



$m_2\vec{a}_2 = \vec{F}_N + \vec{F}_T + \vec{F}_{g_2}, \qquad\qquad m_1\vec{a}_1 = \vec{F}_{T_1} + \vec{F}_{g_1}$

$-m_1 a_1 \hat{e}_2 = -m_1 g \hat{e}_2 + F\hat{e}_2$
$-m_2 a_2 \hat{e}_1 = F_N \hat{e}_2 + F_T \hat{e}_1 + m_2 g(-\sin\theta\,\hat{e}_1 - \cos\theta\,\hat{e}_2)$

$-m_1 a_1 = -m_1 g + F_T \Rightarrow F_T = m_1 g - m_1 a_1$
$-m_2 a_2 = 0 + F_T - m_2 g \sin\theta$
$0 = F_N + 0 - m_2 g\cos\theta \Rightarrow m_2 g \cos\theta$

Have 3 equations and 4 variables: F_T, F_N, a_1, a_2
Add constraint on length of string: $x_1 + x_2 + R\left(\frac{\pi}{2} + \theta\right) \Rightarrow \ddot{x}_1 = -\ddot{x}_2$

$\ddot{x}_1 = -\left(\frac{m_1 - m_2\sin\theta}{m_1 + m_2}\right) g \equiv a, \qquad$ constant
$x_1(t) = x_0 + v_0 t + \frac{1}{2}at^2$
Introduce $v = \dot{x}_1$
$\dot{v} = a = \frac{dv}{dt}$
$\int dv = a\int dt$
$v = at + C \Rightarrow v = v_0 + at$
$\frac{dx}{dt} = v - v_0 + at$
$\int dt = \int dt(v_0 + at)$
$x = v_0 t + \frac{1}{2}at^2 + C$
$x = x_0 + v_0 t + \frac{1}{2}at^2$

2D Mechanics

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2D Motion

N2L is

$$m\ddot{\vec{r}} -= \sum \vec{F}$$

Trajectory motion with drag $\vec{F}_D = -mk\vec{v}$
(Initial conditions: $\vec{r}(0) = \vec{0}$ and $\vec{v}(0) = \vec{v}_0$)
$m\ddot{\vec{r}} = \vec{F}_g + \vec{F}_D = m\vec{g} + mkv(-\hat{v})$
Choose $\vec{g} = -g\hat{e}_2$ and $\hat{e}_1$ horizontal
$m(\ddot{x}\hat{e}_1 + \ddot{x}_2\hat{e}_2) = -mg\hat{e}_2 - mk(\dot{x}_1\hat{e}_1 + \dot{x}\hat{e}_2)$
$m\ddot{x}_1 = -mk\dot{x}_1$
$m\ddot{x}_2 = -mg - mk\dot{x}_2$
and so the equations of motion are uncoupled (i.e. they can be solved independently of each other)
$v_1 = v_{10}e^{-kt}$
$x_1 = \frac{v_{10}}{k}\left(1 - e^{-kt}\right)$
$v_2 = -\frac{g}{k} + \left(\frac{g}{k} + v_{10}\right)e^{-kt}$
$x_2 = -\frac{gt}{k} + \left(\frac{g}{k^2} + \frac{v_{02}}{k}\right)\left(1 - e^{-kt}\right)$

The projectile will impact the horizontal plate at $z_2\left(t_f\right) = 0$
$0 = -\frac{gt_f}{k} + \left(\frac{g}{k^2} + \frac{v_{02}}{k}\right)\left(1 - e^{-kt_f}\right)$
$t_f = \frac{g + v_{02}k}{gk}\left(1 - e^{-kt_f}\right)$
Transcendental equation, can't solve exactly

Method 1

Fixed Point Iteration
$t_{i+1} = -\frac{gt_i}{k} + \left(\frac{g}{k^2} + \frac{v_{02}}{k}\right)\left(1 - e^{-kt_i}\right)$
with $t_0 = \frac{2v_{02}}{g}$ the $k = 0$ solution.
Use $k = 2, g = 10, v_0 = 10$ and $\theta = 45°$ to get
$t_0 = 1.4, t_1 = 0.67, \ldots, t_5 = 0.41, \ldots, t_{10} = 0.37, \ldots, t_{20} = 0.3692$

Method 2

Numerical Root Finding
Write
$f(t) = t - \frac{g + v_{02}k}{k^2}\left(1 - e^{-kt}\right)$
Could plot to get interval for the root, then use root finding algorithm to get root.

Example

Motion of a charged particle in a uniform $\vec{B}$ field with B.C.
$\vec{r}(0) = \vec{0}$
$\vec{v}(0) = \vec{v}_0$
$m\ddot{\vec{r}} = q\vec{v} \times \vec{B}$ choose $\vec{B} = B\hat{e}_2$
$\vec{r} = x_i\hat{e}_i$
$m\ddot{x}_i\hat{e}_i = q\dot{x}_i\hat{e}_i \times B\hat{e}_2 = qB\epsilon_{i2j}\hat{e}_j x_i = qB(\dot{x}_1\hat{e}_3 - \dot{x}_3\hat{e}_1)$
$\ddot{x}_1 = -\omega\dot{x}_3$
$\ddot{x}_2 = 0$
$\ddot{x}_3 = \omega\dot{x}_1$
where $\omega \equiv \frac{qB}{m}$ is the "cyclotron frequency"

$z = x_1 + ix_3$
$\ddot{z} = \ddot{x}_1 + i\ddot{x}_3 = -\omega\dot{x}_3 + i\omega\dot{x}_1 = i\omega(\dot{x}_1 + i\dot{x}_3) = i\omega z$
$z = Ae^{i\omega t} + C$

Conservation Theorems

January-28-13 11:43 AM

Linear Momentum

"The total linear momentum $\vec{p}$, of a particle is conserved when no net force acts on the particle."

$$\sum \vec{F} = 0 = 0 = \dot{\vec{p}} \Rightarrow \vec{p}(t) = \text{constant}$$

Note that this also holds for each component of $\vec{p}$

$$\dot{\vec{p}} \cdot \hat{e} = \sum \vec{F} \cdot \hat{e} = 0$$

$\Rightarrow$ component of $\vec{p}$ parallel to $\hat{e}$ is conserved.

Angular Momentum

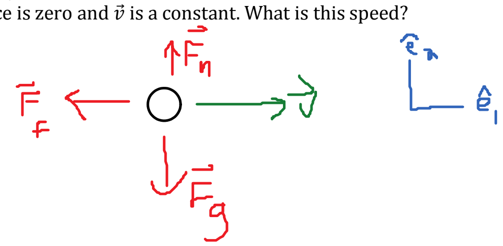
$$\dot{\vec{L}} = \sum \vec{N}$$

"The total angular momentum $\vec{L}$, of a particle is conserved when there is no net torque on the particle."

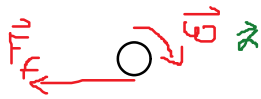
$$\sum \vec{N} =$$

Example

Consider a bowling ball (mass m, radius R) which initially has velocity $\vec{v}_0$ (and no rotation) as it slides along the horizontal alley. The kinetic friction causes the ball to slow and to rotate until it is rolling, at which point the frictional force is zero and $\vec{v}$ is a constant. What is this speed?



$m\vec{a} = \vec{F}_f + \vec{F}_N + \vec{F}_g$
$-ma\,\hat{e}_1 = -(\mu F_N)\hat{e}_1$
$0 = F_N\hat{e}_2 - (mg)\hat{e}_2$
$F_n = mg$
$ma = \mu mg$
$a = \mu g,$ constant
$v = v_0 - \mu gt$



$-I\,\alpha\hat{e}_3 = \vec{0} \times \vec{F}_N + \vec{0} \times \vec{F}_q + \vec{R} \times \vec{F}_f$

$-\left(\frac{2}{5}mR^2\right)\alpha\hat{e}_3 = -R(\mu mg)\hat{e}_3$

$\alpha = \frac{5\mu g}{2R}$

$\omega = 0 + \alpha t$

$\boxed{w = \frac{5\mu gt}{2R}}$

Rolling when $v_1 = \omega_1 R$

$v_0 - \mu g t_1 = \frac{5\mu g t_1}{2R}R$

$\frac{7\mu g t_1}{2} = v_0$

$t_1 = \frac{2v_0}{7\mu g}$ Time when ball is rolling not sliding

$v_0 = v_0 - \mu g\left(\frac{2v_0}{7\mu g}\right)$

$v_1 = \frac{5}{7}v_0$

With Conservation Laws

$$\dot{\vec{L}} = \sum \vec{N}$$

$\vec{N}$ - Torque

Chose origin along line of action $\vec{F}_f$ so
$\vec{N}_f = \vec{0} \times \vec{F}_f = 0$ and
$\vec{N}_g = -\vec{N}_N$

$$\sum \vec{N} = 0$$

$\vec{L}_0 = R\hat{e}_2 \times (mv_0\hat{e}_1) = -mv_0R\hat{e}_3$

$\vec{L}_1 = R\hat{e}_2 \times (mv_1\hat{e}_1) + I\vec{\omega} = -mv_0R\hat{e}_3 - \frac{2}{5}mR^2\frac{v_1}{R}\hat{e}_3$

$\vec{L}_0 = \vec{L}_1 \Rightarrow mv_0R = mv_1R + \frac{2}{5}mv_1R$

$v_1 = \frac{5}{7}v_0$

Special Relativity

January-30-13 12:02 PM

Similarly, we can choose any inertial reference frame we wish there is also no absolute kinetic energy.
As a result, only changes in energy can have any physical meaning in classical physics. As we shall see, in special relativity (SR) the rest frame of an object is special in which the absolute energy is mc^2. In any other frame is γmc^2 where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the "Lorentz factor" and v is the speed relative to observer's rest frame.

Interestingly, for small $v \ll c$ we can do a binomial expansion $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1+\frac{1}{2}\frac{v^2}{c^2}$
So $E = \gamma mc^2 \approx \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)mc^2 = mc^2 + \frac{1}{2}mv^2$ which is the rest energy plus the kinetic energy.

Returning to classical physics if we define energy as $E = T + U$ then $\dot{E} = \dot{T} + \dot{U}$ and from the WKE theorem
$dT = \vec{F}\cdot d\vec{r} \Rightarrow \dot{T} = \vec{F}\cdot\vec{v} = F_i\dot{x}_i$
and $U = U(\vec{x}, t)$
$$\dot{U} = \frac{\partial U}{\partial x_i}\dot{x}_i + \frac{dU}{dt} = \vec{\nabla}U\cdot\dot{\vec{r}} + \frac{\partial U}{\partial t}$$
so
$$\dot{E} = \vec{F}\cdot\dot{\vec{r}} + \vec{\nabla}U\cdot\dot{\vec{r}} + \frac{\partial U}{\partial t} = \left(\vec{F} + \vec{\nabla}U\right)\cdot\dot{\vec{r}} + \frac{\partial U}{\partial t}$$
but $\vec{F} = -\vec{\nabla}U$ for conservative forces
then
$$\dot{E} = \frac{\partial \mathrm{U}}{\partial \mathrm{t}}$$
and we can conclude that if the potential isn't an explicit function of time then $\dot{E} = 0$ and energy is conserved.

Lagrangian

$$L = T - U$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

Example: Harmonic Oscillator

$N2L$: $m\ddot{x} = -kx$
which has the known solution $x(t) = A\sin(\omega t + \phi)$
where $\omega^2 = \frac{k}{m}$ and A, ϕ are integration constants.
Multiply N2L by $\dot{x}$
$$m\ddot{x}\dot{x} = -kx\dot{x}$$
$$m\dot{v}v = -\frac{1}{2}k\frac{d}{dt}(x^2)$$
$$\frac{1}{2}m\frac{d}{dt}(v^2) = -\frac{1}{2}\frac{d}{dt}(x^2)$$
$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = 0$$

And we can conclude that $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ is a constant w.r.t. time.
We can rearrange this to
$$v = \dot{x} = \sqrt{\frac{2E}{m} - \frac{k}{m}x^2} = \frac{dx}{dt}$$
$$\frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}} = dt$$
$$\int_{x_0}^{x}\frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}} = \int_0^t dt = t$$
Introduce
$$x = \sqrt{\frac{2E}{k}}\sin\phi$$
$$dx = \sqrt{\frac{2E}{k}}\cos\phi\, d\phi$$
$$t = \int_{\phi_0}^{\phi}\frac{d\phi}{\sqrt{\frac{2E}{m}}}\cdot\frac{\sqrt{\frac{2E}{k}}\cos\phi}{\sqrt{1-\sin^2\phi}} = \sqrt{\frac{m}{k}}\int_{\phi_0}^{\phi}d\phi \;= \frac{1}{\omega}(\phi - \phi_0)$$
where $\omega^2 = \frac{k}{m}$

$$\phi = \omega t + \phi_0$$
$$x = \sqrt{\frac{2E}{k}}\sin(\omega t + \phi_0)$$
as expected.

We can do this for any 1D conservative force for which $F = -\frac{dU}{dx}$
$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + U(x)\right) = 0$$
$$E = \frac{1}{2}mv^2 + U(x) = \text{constant}$$
$$\int_{x_0}^{x}\frac{dx}{\sqrt{\frac{v}{m}}\,(E-U)} = t$$

Central Forces

February-01-13 11:32 AM

Last day we were examining 1d conservative problems and found that we could write the solution as

$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}(E-V)}}$$

where $F = -\frac{dV}{dx}$

Central Forces

$\vec{F} = F(r)\hat{r}$

For central force problems there is no net torque

$\vec{\tau} = \vec{r} \times \vec{F} = F\vec{r} \times \hat{r} = 0$

and so the angular momentum is conserved.

$\vec{L} = \vec{r} \times m\vec{v} \Rightarrow \dot{\vec{L}} = \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\dot{\vec{v}}$

but $\vec{v} = \dot{\vec{r}}$

and $\dot{\vec{v}} = \vec{a} = \frac{\vec{F}}{m} = \frac{F}{m}\hat{r}$

so $\dot{\vec{L}} = 0$

Furthermore, since

$\vec{L} = \vec{r} \times \vec{p}$ we have $\vec{L} \cdot \vec{r} = 0$ and so $\vec{r}$ is tin the pane perpendicular to $\vec{L}$. (aka the "orbital plane")

So our 3D problem is 2D with the right choice of coordinates. (e.g. $\vec{L} = L\hat{e}_3$)

And because we have a constant of the motion (i.e. $\vec{L}$), this will allow us to eliminate another coordinate.

Origins of Special Relativity

February-04-13 11:29 AM

In 1864, James Clerk Maxwell unified electricity and magnetism. In vacuum his equations are:

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \cdot \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\vec{E}$$

These equations can be combined using $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla} \cdot \left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A}$

$$\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{E}$$

$$\nabla^2 \vec{B} = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{B}$$

In on space dimension, these are

$$\frac{\partial^2 \vec{E}}{\partial z^2} = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{E}$$

which is a wave equation with wave speed $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ and solutions

$$\vec{E}(z,t) = \vec{E}_0 \sin(kz - \omega t)$$

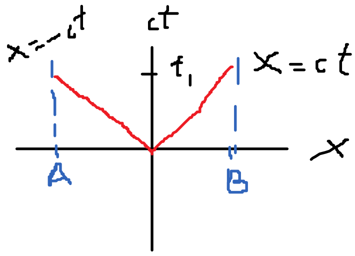
where $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$

Maxwell realized that his equations where not invariant with respect to Galilean transformations. (i.e. they change in different inertial reference frames.)

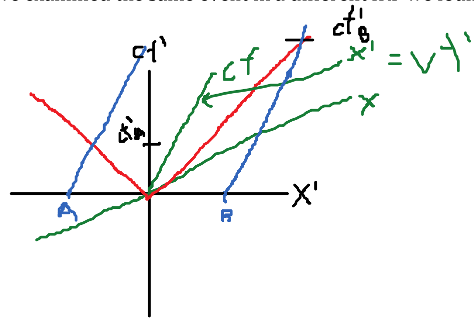
Relativity of Simultaneity

February-06-13 11:30 AM

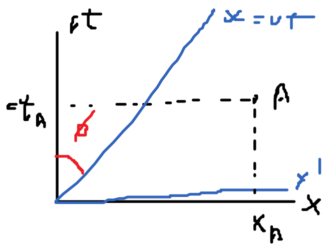
Last day we were considering "events" - a flash of light at the origin of space-time diagrams.



But when we examined the same event in a different IRF we found that the flash did not arrive simultaneously at A and B.



Lets look at a "Newton Diagram"



$\tan\phi = \frac{vt}{ct} = \frac{v}{c}$

Event A has coordinates

(x_A, t_a) and in printed IRF

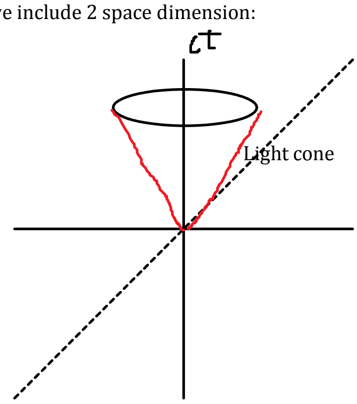
$(x'_A, t'_A) = (xA - vt_A, tA)$

or

$\begin{pmatrix} x \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

Causality

If we include 2 space dimension:



The light cone is the world-line of photons emitted at the origin.

Lorentz Transformation

Replace Galilean

$x = x' - vt$

$t = 0 + t'$

with

$x = a_{11}x' + a_{12}t'$

$t = a_{21}x' + a_{22}t'$

$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

Note> we will eventually introduce $x_4 = ict$ so we have x_μ with $\mu = 1, 2, 3, 4$ and $x_\mu x_\mu = x_1 x_1 + \cdots$)

Consider the motion of a point fixed in the primed system.

(i.e. $x' =$ constant) we have
$dx = a_{12}dt'$
$dt = a_{22}dt'$
$$\frac{dx}{dt} = \frac{a_{12}}{a_{22}} = v$$
where $v = \frac{dx}{dt} = -\frac{dx'}{dt'}$
is the relative speed of the two IRFs.

Lorentz Transform

February-11-13 11:33 AM

As discussed last week, we require the transformation between inertial reference frames to be inertial (so that un-accelerated straight line motion is still un-accelerated straight line motion and such transformations are additive: i.e. $T(v_1 + v_2) = T(v_1)T(v_2)$)

For reference the Galilean transformation is $x' = x - vt,\ t' = t$
the Lorentz transformation must have the linear form:

$$x = a_{11}x' + a_{12}ct'$$
$$ct = a_{21}x' + a_{22}ct'$$

where the a_{ij} are dimensionless constants.

Consider the motion of a point fixed in the primed IRF. (i.e. $x' =$ constant). We have that

$$dx = a_{12}cdt'$$
$$cdt = a_{22}cdt'$$
$$\frac{dx}{dt} = \boxed{c\frac{a_{12}}{a_{22}} = v}$$

where $v = \dfrac{dx}{dt}$ is the relative speed of the two IRFs

Now consider a pulse of light travelling in the $+x$ direction. For convenience we emit this pulse at $t = 0$ from the origin when the two IRFs are coincident. The position of the pulse is then

$$x = ct, \qquad x' = ct'$$

(by the second postulate).

$$x = ct = a_{11}x' + a_{12}ct' = (a_{11} + a_{12})ct'$$
$$ct = x = a_{21}x' + a_{22}ct' = (a_{21} + a_{22})ct'$$
$$\text{so } \boxed{a_{11} + a_{12} = a_{21} + a_{22}}$$

Similarly, in the -x direction we get

$$\boxed{a_{11} - a_{12} = -a_{21} + a_{22}}$$

Combining the boxed equations (2) and (3) we get
$a_{11} = a_{22}$ and $a_{12} = a_{21}$
Using (1), $a_{12} = \frac{v}{c}a_{22}$
$x = a_{11}x' + a_{12}ct' = \gamma(x' + vt')$
$ct = a_{21}x' + a_{22}ct' = \gamma(vx' + ct')$

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \gamma \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

This can be inverted to

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma^{-1} \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

Now we exploit the first postulate and swap the two IRFs and replace v with $-v$

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$\Rightarrow \gamma = \frac{1}{\gamma\left(1 - \left(\frac{v}{c}\right)^2\right)} \Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

is the **Lorentz Factor**

Recall that for a Galilean transformation

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

and now we have

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

Try two Galilean transforms

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & u+v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

Addition of Velocities

February-13-13 11:33 AM

Addition of Velocities

Consider an object traveling with a constant speed u in the primed system. Its position is then

$$x' = ut'$$

$$\gamma(x - vt) = \frac{u}{c}\gamma\left(-\frac{v}{c}x + ct\right)$$

$$x\left(1 + \frac{uv}{c^2}\right) = (u + v)t$$

$$x = \left(\frac{u + v}{1 + \frac{uv}{c^2}}\right)t$$

so the position of the object in the unprimed IRF is $x = wt$ where

$$w = \frac{u + v}{1 + \frac{uv}{c^2}}$$

Example

Show that if Bob is moving with velocity $u\hat{e}_1$ relative to Alice and Eve has velocity $v\hat{e}_1$ relative to Bob, that, using a Galilean transformation,

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

that Eve has velocity $(u + v)\hat{e}_1$ relative to Alice

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & u + v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

Repeat with a Lorentz transformation

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma_u\begin{pmatrix} 1 & \frac{u}{c} \\ \frac{u}{c} & 1 \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \gamma_v\begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix}\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma_v\begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix}\gamma_u\begin{pmatrix} 1 & \frac{u}{c} \\ \frac{u}{c} & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} = \gamma_v\gamma_u\begin{pmatrix} 1 + \frac{uv}{c^2} & \frac{u + v}{c} \\ \frac{u + v}{c} & 1 + \frac{uv}{c^2} \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u^2}{c^2}\right)}}\begin{pmatrix} 1 + \frac{uv}{c^2} & \frac{u + v}{c} \\ \frac{u + v}{c} & 1 + \frac{uv}{c^2} \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix}$$

Consider the square of the 1-1 term

$$\frac{\left(1 + \frac{uv}{c^2}\right)^2}{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u^2}{c^2}\right)} = \frac{1 + \frac{2uv}{c^2} + \left(\frac{uv}{c^2}\right)^2}{\left(1 + \frac{2uv}{c^2} + \left(\frac{uv}{c^2}\right)^2\right) - \frac{u^2 + v^2}{c^2} - \frac{2uv}{c^2}}$$

$$a_{11}^2 = \frac{\left(1 + \frac{uv}{c^2}\right)^2}{\left(1 + \frac{uv}{c^2}\right)^2 - \frac{1}{c^2}(u^2 + 2uv + v^2)} = \frac{1}{1 - \frac{w^2}{c^2}} = \gamma_w^2, \qquad \frac{w^2}{c^2} \equiv \frac{\left(\frac{u + v}{c}\right)^2}{\left(1 + \frac{uv}{c^2}\right)^2}$$

So

$$a_{11}^2 = \gamma_w^2 = a_{22}^2$$

$$a_{12}^2 = w^2$$

and

$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \gamma_w\begin{pmatrix} 1 & \frac{w}{c} \\ \frac{w}{c} & 1 \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix}$$

$$w \equiv \frac{u + v}{1 + \frac{uv}{c^2}}$$

is Eve's speed relative to Alice.

Now use the rapidity α such that $\tanh\alpha = \frac{v}{c}$
and show that two successive Lorentz transforms is also a Lorentz transform.
If $\tanh\alpha = \frac{v}{c}$ then $\cosh\alpha = \gamma$ and $\sinh\alpha = \frac{v}{c}\gamma$
and

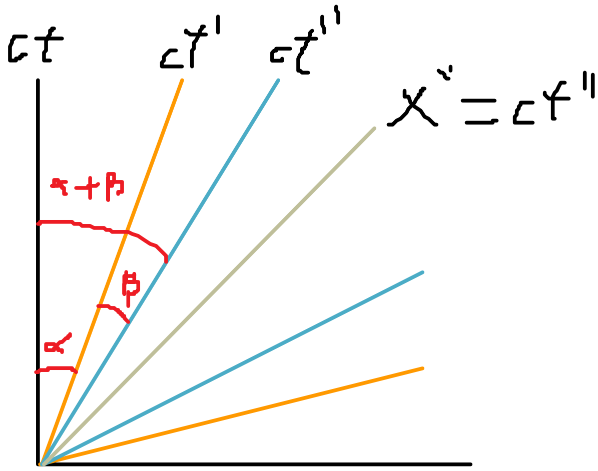
$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ ct'' \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh(\alpha+\beta) & \sinh(\alpha+\beta) \\ \sinh(\alpha+\beta) & \cosh(\alpha+\beta) \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

and so

$$\frac{w}{c} = \tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}$$

Draw the space-time diagram



Proper Time

Consider the quantity

$$c^2\tau^2 \equiv c^2t^2 - x^2 = \left(\gamma\left(\frac{v}{c}x' + ct'\right)\right)^2 - \left(\gamma(x' + vt')\right)^2$$

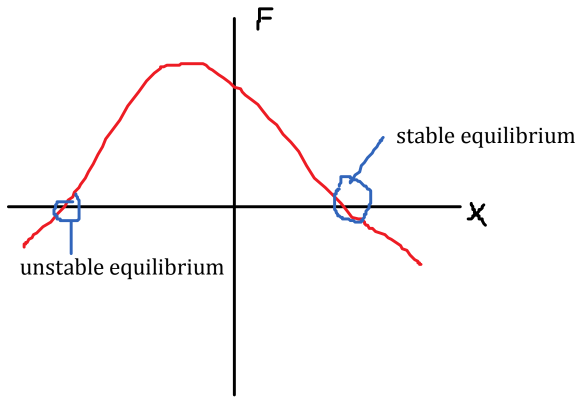
$$= \gamma^2\left(\left(\frac{v}{c}x'\right)^2 + 2vx't' + (ct')^2 - (x')^2 - 2vx't' + (vt')^2\right)$$

$$= \gamma^2\left(-(x')^2\left(1 - \frac{v^2}{c^2}\right) + c^2\left(1 - \frac{v^2}{c^2}\right)(t')^2\right) = (ct')^2 - (x')^2 = c^2\tau'^2$$

So $\tau' = \tau$ is invariant under Lorentz transforms.

Oscillations

February-25-13 11:31 AM

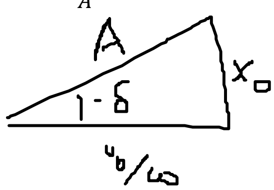


$F(x) = F_0 + \frac{dF}{dx}\Big|_0 x + \frac{1}{2}\frac{d^2F}{dx^2}\Big|_0 x^2 + \cdots$
define $k \equiv -\frac{dF}{dx}|_0$, if $F_0 = 0$ then
Hooke's law: $F = -kx$

N2L: $m\ddot{x} = -kx$
$x(t) = A\sin(\omega t - \delta)$
$w^2 = \frac{k}{m}$, A & δ are integration constants

The kinetic energy is $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(\omega A\cos(\omega t - \delta))^2 = \frac{1}{2}kA^2\cos^2(\omega t - \delta)$
The potential energy is
$U = -\Delta W = -\int F dx = -\int -kx dx = \frac{1}{2}kx^2$
$E = T + U = \frac{1}{2}kA^2(\cos^2(\omega t - \delta) + \sin^2(\omega t - \delta)) = \frac{1}{2}kA^2$
So the total energy is constant.
$\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$
$A^2 = x^2 + \frac{v^2}{\omega^2}$

Example
Find $x(t)$ for $x(0) = x_0$ & $\dot{x}(0) = v_0$
$x(0) = x_0 = A\sin(0 - \delta) \Rightarrow \sin(-\delta) = \frac{x_0}{A}$
but $A^2 = x_0^2 + \frac{v_0^2}{\omega^2}$



so
$x(t) = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}\sin\left(\omega t - \tan^{-1}\left(\frac{x_0\omega}{v_0}\right)\right)$
$x(t) = A\sin(\omega t - \delta) = A(\sin(\omega t)\cos(-\delta) + \cos(\omega t)\sin(-\delta)) = A\left(\sin(\omega t)\frac{v_0}{A\omega} + \cos(\omega t)\frac{x_0}{A}\right)$
$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$

Again, the complex way
N2L: $\ddot{x} + \omega_0^2 x = 0$

Use the Ansantz
$x(t) = Re(Ae^{-\omega t})$
$Re\left((-i\omega)^2 x + \omega_0^2 x\right) = 0$
$Re(-\omega^2 x + \omega_0^2 x) = 0$, $\quad (-\omega^2 + \omega_0^2)Re(x) = 0$
$\Rightarrow \omega_\pm = \pm\omega_0$
$x(t) = Re\left(A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t}\right)$
$= Re(A_+)\cos(\omega_0 t) - Im(A_+)\sin(\omega_0 t) + Re(A_-)\cos(-\omega_0 t) - Im(A_-)\sin(-\omega_0 t)$
$= Re(A_+ + A_-)\cos(\omega_0 t) - Im(A_+ - A_-)\sin(\omega_0 t)$

Example - As before
$x(0) = x_0$. $\dot{x}(0) = v_0$

$x(0) = x_0 = Re(A_+ + A_-)$
$\dot{x}(t) = -\omega_0 Re(A_+ + A_-)\sin(\omega_0 t) - \omega_0 Im(A_+ - A_-)\cos(\omega_0 t)$
$\dot{x}(0) = v_0 = -\omega_0 Im(A_+ - A_-)$
$x(t) = x_0 \cos(\omega_0 t) + \dfrac{v_0}{\omega_0}\sin(\omega_0 t)$
as before

Damped Oscillations

$F_D = -b\dot{x}$
$m\ddot{x} = -b\dot{x} - kx$
$\ddot{x} + 2\beta\dot{x} + \omega_0 x = 0$
where $\omega_0^2 = \frac{k}{m}$ and $2\beta = \frac{b}{m}$

Ansatz
$x(t) = Ae^{-i\alpha t}$
$(-i\alpha)^2 x + 2\beta(-c\alpha)x + \omega_0^2 x = 0$
$(-\alpha^2 - 2i\beta\alpha + \omega_0^2)x = 0$
$\alpha_\pm = -i\beta \pm \sqrt{-\beta^2 + \omega_0^2}$

Damped & Driven Oscillator

February-27-13 11:30 AM

Last day we looked at the harmonic oscillator
$\ddot{x} + \omega_0^2 = 0$
and used the ansatz
$x = Re\left(Ae^{-i\omega t}\right)$
to find $x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$

Damped Oscillator

Damping force proportional to velocity, $-b\dot{x}$
N2L:

$$m\ddot{x} = -kx - b\dot{x}$$

or

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

where $2\beta = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$, both with units of frequency.

Use the ansatz:

$$x = Re\left(Ae^{-i\alpha t}\right)$$

as before to get

$$Re(-\alpha^2 x - 2i\beta\alpha x + \omega_0^2 x) = 0$$

$$(\alpha^2 + 2i\beta\alpha - \omega_0^2)Re(x) = 0$$

$$\alpha_\pm = -i\beta \pm \sqrt{\omega_0^2 - \beta^2} = -i\beta \pm \omega, \qquad \omega^2 \equiv \omega_0^2 - \beta^2$$

$$x(t) = e^{-\beta t} Re\left(A_+ e^{-i\omega t} + A_- e^{-i\omega t}\right) = e^{-\beta t}\left(Re(A_+ + A_-)\cos(\omega t) - Im(A_+ - A_-)\sin(\omega t)\right)$$

Example

$x(0) = x_0, \qquad \dot{x}(0) = v_0$

$\Rightarrow x(0) = x_0 = Re(A_+ + A_-)$

$\dot{x}(t) = Re\left(-\beta e^{-\beta t}\left(A_+ e^{i\omega t} + A_- e^{-i\omega t}\right) + e^{-\beta t} i\omega\left(A_+ e^{-i\omega t} - A_- e^{-i\omega t}\right)\right)$

$\dot{x}(0) = Re\left(-\beta(A_+ + A_-) + i\omega(A_+ - A_-)\right)$

$v_0 = -\beta x_0 - \omega Im\,(A_+ - A_-)$

$Im(A_+ - A_-) = -\dfrac{v_0 + \beta x_0}{\omega}$

Note

For

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$$

Can rewrite

$$x(t) = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}\left(\cos\delta\cos(\omega t) + \sin\delta\sin(\omega t)\right), \qquad \delta = \operatorname{atan}\left(\frac{v_0}{x_0\omega}\right)$$

$$x(t) = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}\cos(\omega t - \delta)$$

Back to example

$$x(t) = e^{-\beta t}\left(x_0 \cos\left(\sqrt{\omega_0^2 - \beta^2 t}\right) + \frac{v_0 + \beta x_0}{\sqrt{\omega_0^2 - \beta^2}}\sin\left(\sqrt{\omega_0^2 - \beta^2 t}\right)\right)$$

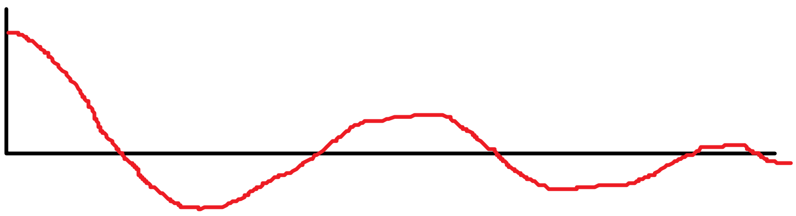
Possibilities

Underdamped: $\omega_0 > \beta$

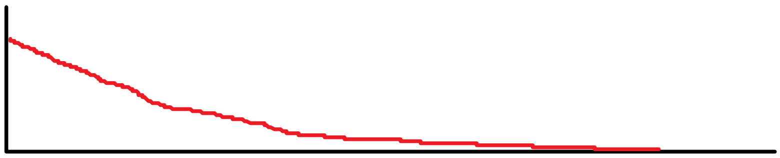
-

Possibilities

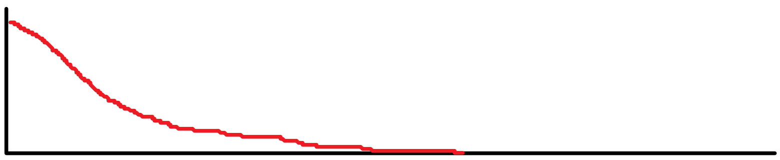
Underdamped: $\omega_0 > \beta$



Overdamped: $\omega_0 < \beta$



Critically Damped: $\omega_0 = \beta$



Driven Oscillations

$m\ddot{x} = -km - b\dot{x} + F$

Will use $F\cos(\Omega t)$

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos(\Omega t)$

$$2\beta \equiv \frac{b}{m}, \qquad \omega_0^2 \equiv \frac{k}{m}, \qquad A \equiv \frac{F}{m}$$

The complementary solution is the solution to the homogeneous equation and the particular solution is

$x_p(t) = B_1\cos(\Omega t) + B_2\sin(\Omega t)$

$$B_1 = \frac{(\omega_0^2 - \Omega^2)A}{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}$$

$$B_1 = \frac{2\beta\Omega A}{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}$$

Driven Oscillations

March-01-13 11:31 AM

Last day we were looking at driven oscillations,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F}{m}\cos(\omega t) \ \ (1)$$

and we had the complementary solution the same as the un-driven oscillator and particular solution

$$x_p(t) = A' \cos(\omega t - \delta)$$

where

$$A' = \frac{A}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}}$$

$$\tan\delta = \frac{2\beta\Omega}{\omega_0^2 - \Omega^2}$$

which we will no derive

$$x_p(t) = B_1\cos(\Omega t) + B_2\sin(\Omega t)$$

Plugging into (1)

$$-\Omega^2 x_p + 2\beta\Omega(-B_1\sin(\Omega t) + B_2\cos(\Omega t)) + \omega_0^2 x_p = \frac{F}{m}\cos(\Omega t)$$

$$(\omega_0^2 - \Omega^2)(B_1\cos(\Omega t) + B_2\sin(\Omega t)) + 2\beta\Omega(-B_1\sin(\Omega t) + B_2\cos(\Omega t)) = \frac{F}{m}\cos(\Omega \mathrm{t})$$

$$\cos(\Omega t)\left((\omega_0^2 - \Omega^2)B_1 + 2\beta\Omega B_2\right) + \sin(\Omega t)\left((\omega_0^2 - \Omega^2)B_2 - 2\beta\Omega B_1\right) = \frac{F}{m}\cos(\Omega t)$$

$\Rightarrow$

$$(\omega_0^2 - \Omega^2)B_2 = 2\beta\Omega B_1$$

$$(\omega_0^2 - \Omega^2)B_1 + 2\beta\Omega B_2 = \frac{F}{m}$$

$$B_1 = \frac{(\omega_0^2 - \Omega^2)A}{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}$$

$$B_2 = \frac{2\beta\Omega A}{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}$$

So

$$x_p(t) = A\cos(\Omega t + \delta)$$

Resonance

In some situations, A' can be large. It is a maximum at "resonance" which occurs at $\Omega = \Omega_R$ where

$$\left.\frac{d}{d\Omega}A'\right|_{\Omega=\Omega_R} = 0, \qquad \Omega_R = \sqrt{\Omega_0^2 - 2\beta^2}$$

General Driving Force

We have

$$\left(\frac{d}{d^2} + 2\beta\frac{d}{dt} + \omega_0^2\right)x = F(t)$$

Introduce

$$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \text{\textbackslash omeva}_0^2$$

a linear differential operator

$$D\big(a_1 f_1(t) + d_2 f_2(t)\big) = a_1 D f_1 + d_2 D f_2$$

So if we can solve our EoM (equation of motion) for each driving function $f_i(x)$ then the solution for the driving function

$$f(t) = \sum_i a_i f_i(t)$$

will be the sum of the solutions.

Fourier Series

March-01-13 12:14 PM

If we have some function $f(t)$, that is defined on an interval $[0, T]$ or which is periodic with period T then we can write it as a series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)$$

and the coefficients a_n and b_n can be found by operating with $\frac{1}{T}\int_0^T dt \cos\left(\frac{2\pi mt}{T}\right)$ and $\frac{1}{T}\int_0^T dt \sin\left(\frac{2\pi mt}{T}\right)$ respectively.

There may be errors in equations after this point.

$$\omega = \frac{2\pi}{T}$$

$$\frac{1}{T}\int_0^T \big(dt \cos(n\omega t)\, f(t)\big)$$

$$= \frac{1}{2}a_0 \frac{1}{T}\int_0^T dt \cos(m\omega t)$$

$$+ \frac{1}{T}\sum_{n=1}^{\infty}\left(a_n \int_0^T dt \cos(m\omega t)\cos(n\omega t) + b_n \int_0^T dt \cos(m\omega t)\sin(n\omega t)\right)$$

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

$$\int_0^T \big(dt \cos(n\omega t)\, f(t)\big)$$

$$= \frac{a_0}{2} + \frac{1}{2T}\sum_{n=1}^{\infty} a_n \int_0^T dt\,\big(\cos((m+n)\omega t) + \cos((m-n)\omega t)\big)$$

$$\frac{1}{T}\int_0^T \cos(m\omega t)\, f(t) = \frac{1}{2T}a_m$$

$$a_m = \frac{2}{T}\int_0^T dt \cos(m\omega t)\, f(t)$$

similarly,

$$b_m = \frac{2}{T}\int_0^T dt \sin(m\omega t)\, f(t)$$

$$a_0 = \frac{1}{T}\int_0^T dt\, f(t)$$

Example

Square wave

$$f(t) = \begin{cases} A & \text{if } 0 < t < \frac{T}{2} \\ -A & \text{if } \frac{T}{2} < t < 0 \end{cases}$$

$$a_0 = \frac{2}{T}\int_0^T dt\, f(t) = 0$$

$$a_n = \frac{2}{T}\int_0^T dt \cos(n\omega t)\, f(t) = \frac{2}{T}\int_0^{\frac{T}{2}} dt \cos(n\omega t)\,(A) - \frac{2}{T}\int_{\frac{T}{2}}^T dt \cos(n\omega t)\,(A)$$

$$= \frac{2A}{T}\left(\left[\frac{\sin(n\omega t)}{n\omega}\right]_0^{\frac{T}{2}} - \left[\frac{\sin(n\omega t)}{n\omega}\right]_{\frac{T}{2}}^T\right)$$

$$= \frac{2A}{nT\omega}\left(\sin\left(\frac{n\omega T}{2}\right) - 0 - \sin(n\omega T) + \sin\left(\frac{n\omega T}{2}\right)\right) = 0$$

$$b_n = \frac{2}{T}\int_0^T dt \sin(n\omega t)\, f(t) = \cdots$$

$$= -\frac{A}{2\pi n}(\cos(n\pi) - 1 - \cos(2n\pi) + \cos(n\pi)\} = \frac{A}{\pi n}(1 - (-1)^n)$$

$$= \begin{cases} \frac{A}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f(t) = \sum_{n \text{ odd}} \frac{A}{n\pi}\sin(n\omega t)$$

Fourier Series Cont.

March-06-13 11:39 AM

Last day we were looking at Fourier series,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

and found that the Fourier coefficients were given by

$$a_n = \frac{2}{T}\int_0^T dt \cdot \cos(n\omega t)\, f(t)$$

$$b_n = \frac{2}{T}\int_0^T dt \cdot \sin(n\omega t)\, \mathrm{f(t)}$$

where $\omega = \frac{2\pi}{T}$

We did an example of the square wave and found that

$$a_n = 0, \qquad b_n = \frac{4A}{2\pi n}(1 - (-1)^n)$$

so that

$$f(t) = \frac{4A}{\pi}\left(\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \cdots\right)$$

We can instead write

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{in\omega t}$$

where

$$f_n = \frac{1}{t}\int_0^T dt\, e^{-in\omega t} f(t)$$

It is easy to show that $f_0 = \frac{1}{2}a_0$

$f_n + f_{-n} = a_n, \qquad i(f_n - f_{-n}) = b_n$

Superposition Principle

If we have a linear ODE

$e.g.\ \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

then the response to a driving force $F(t) = F_1(t) + F_2(t)$

is just the sum of the responses to the individual driving forces $f_n(t)$

$\ddot{x_1} + 2\beta\dot{x_1} + \omega_0^2 x_1 = F_1(t)$

$\ddot{x}_2 + 2\beta\dot{x_2} + \omega_0^2 x_2 = F_2(x)$

Gravitation

March-08-13 11:32 AM

Newton's Law of universal gravitation

$$\vec{F} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

where $G = 6.674 \times 10^{-11} N \frac{m^2}{kg^2}$

$$r_{12} = |\vec{r}_{12}| \text{ and } \hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$$

For a continuous distribution of density $\rho(\vec{r})$ then $dm = \rho(\vec{r})dV$
and the net force on a small test mass is

$$\vec{F}(\vec{r}) = -\int_M \frac{Gmdm}{|\vec{r}-\vec{r}'|^2} = -Gm\int_V \frac{\rho(\vec{r}'^2)}{|\vec{r}-\vec{r}'|^2}\widehat{r-r'}dV$$

$\vec{r}$ is the location of the test mass, $\vec{r}'$ is the dummy variable for the position of a dV

It is useful to define the gravitational field $\vec{g}$ as

$$\vec{g} = \frac{\vec{F}}{m} = -G\int_V dV\,\rho(\vec{r}')\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

and the gravitational potential Φ such that its negative gradient is $\vec{g}$

$$\vec{g} = -\vec{\nabla}\Phi$$

We can do this because the gravitational force is conservative.

Work

$$W = \int d\vec{r}\cdot\vec{F}$$

$$W = \oint_C d\vec{r}\cdot\vec{F} = \int_A d\vec{A}\cdot\vec{\nabla}\times\vec{F}$$

(Stokes theorem)
The gravitational field due to a mass m at the origin is

$$\vec{g}(\vec{r}) = -\frac{Gm\hat{r}}{r^2} = -\frac{Gm\vec{r}}{r^3}$$

and the corresponding potential is

$$\Phi(\vec{r}) = \Phi(r) = -\frac{Gm}{r}$$

Recall that

$$\vec{\nabla}r^n = \left(\hat{e}_i\frac{}{\partial x_i}\right)(x_jx_j)^{\frac{}{2}} = \hat{e}_i\left(\frac{}{\partial x_i}x_j + x_j\,\partial\frac{}{x_i}\right)\frac{}{2}(x_kx_k)^{\frac{}{2}-1} = \hat{e}_i(2\delta_{ij}x_j)\frac{}{2}(r^2)^{\frac{}{2}-1} = \hat{e}_ix_in\,r^{n-2} = \vec{r}nr^{n-2}$$

so for $n = 1$ we get

$$\vec{\nabla}\frac{1}{r} = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$$

So

$$\vec{\nabla}\Phi = -Gm\vec{\nabla}\left(\frac{1}{r}\right) = \frac{Gm\hat{r}}{r^2} = -\vec{g}$$

and we have, by convention, chosen

$$\lim_{r\to\infty}\Phi(r) = 0$$

For a continuous density distribution, $\rho(\vec{r})$, the gravitational potential is

$$\Phi(\vec{r}) = -G\int_V dV\,\frac{\rho(\vec{r})}{|\vec{r}-\vec{r}'|}$$

The potential energy, $U(\vec{r})$, of a mass m in this field is

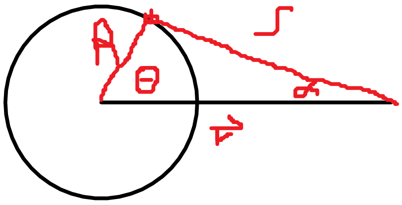
$$U(\vec{r}) = m\Phi(\vec{r})$$

and the gravitational force is

$\vec{F} = -\vec{\nabla}U = -m\vec{\nabla}\Phi = m\vec{g}$

Example: Spherical shell

An important problem is the gravitational field due to a spherical shell of mass m and radius R
Point a distance $\vec{r}$ from center of shell. s is distance from dm to test mass.
α is angle between dm and center of sphere relative to test mass.
θ gives angle between dm and the test mass relative to the center of the shell.



$d\vec{g}(\vec{r}) = G\sigma \cdot \frac{R}{s^2} d\theta \cdot R \sin\theta \cos\alpha\,(-\hat{r})$

where $\sigma = \frac{m}{4\pi R^2}$ the mass per unit area

Cosine rules

$s^2 = R^2 + r^2 - 2rR\cos\theta$

$R^2 = s^2 + r^2 - 2rs\cos\alpha$

so

$\vec{g} = -\hat{r}\sigma G R^2 \int_0^\pi d\theta \sin\theta \frac{s^2 + r^2 - R^2}{2rs\, s^2}$

$2s\,ds = 2rR\sin\theta\,d\theta$

$\vec{g} - \hat{r}\sigma R^2 \frac{2\pi G}{2r^2 R} \int_{-R}^{R} ds \left(\frac{r^2 - R^2}{s^2} + 1\right) = \cdots = -\hat{r}\frac{\pi\sigma R G}{r^2} 4R$

but $\sigma = \frac{m}{4\pi R^2}$ so

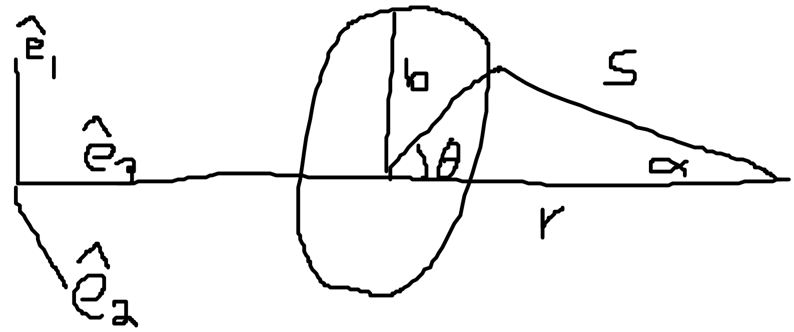
$\vec{g} = -\frac{Gm}{r^2}\hat{r}$

So the shell is equivalent to having a point mass at its center of the same mass.

More Gravitation

March-11-13 11:49 AM

Ring

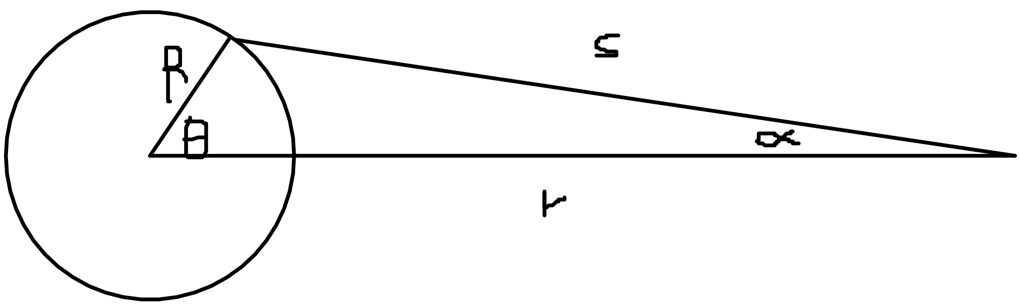


$$d\vec{g} = G\frac{dm}{s^2}\left(-\cos\alpha\,\hat{e}_3 + \sin\alpha\,(\sin\phi\,\hat{e}_1 + \cos\phi\,\hat{e}_2)\right)$$

$$\vec{g} = \int_0^{2\pi}\frac{m}{2\pi}d\phi\frac{G}{s^2}\left(-\cos\alpha\,\hat{e}_3 + \sin\alpha\,(\sin\phi\,\hat{e}_1 + \cos\phi\,\hat{e}_2)\right) = -\frac{Gm}{2\pi s^2}\cos\alpha\,\hat{e}_3\int_0^{2\pi}d\phi$$

$$= -\frac{Gm}{s^2}\cos\alpha\,\hat{e}_3$$

Sphere



$$dm = \frac{m}{4\pi R^2}(2\pi R\sin\theta)(Rd\theta) = \frac{1}{2}m\sin\theta\,d\theta$$

$$d\vec{g} = G\frac{dm}{s^2}\cos\alpha\,\hat{e}_3$$

$$\vec{g} = \int_0^{\pi}\frac{G}{s^2}\cos\alpha\,\hat{e}_3\frac{1}{2}m\sin\theta\,d\theta = \frac{m}{2}G\,\hat{e}_3\int_{-R}^{R}\frac{\cos\alpha\sin\theta\,d\theta}{s^2} = \cdots = -\frac{Gm}{r^2}\hat{e}_3$$