

Test Problem 1 (Linear) [6, p.14]

$$\epsilon y'' - y = 0, \quad y(0) = 1, \quad y(1) = 0.$$

Exact solution is $y(x) = (\exp(-x/\sqrt{\epsilon}) - \exp((x-2)/\sqrt{\epsilon})) / (1 - \exp(-2/\sqrt{\epsilon}))$.

The solution has a boundary layer of width $O(\sqrt{\epsilon})$ at $x = 0$.

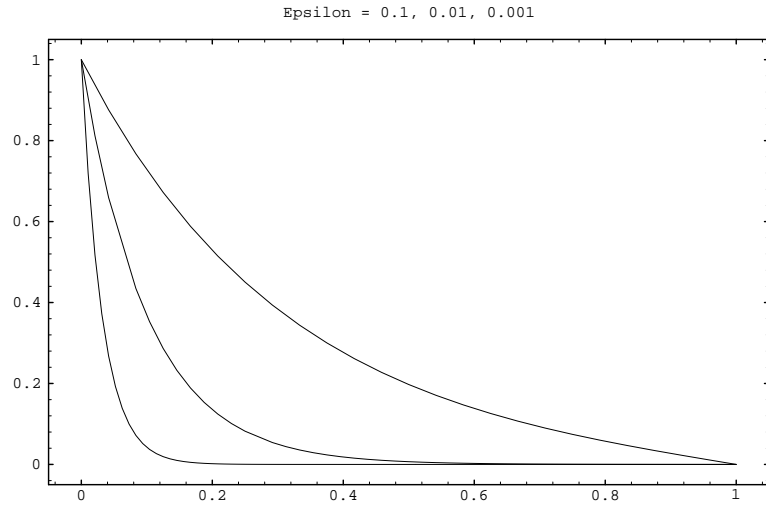


Figure 1: Test Problem 1, $y(x)$ versus x

Test Problem 2 (Linear) [6, p.14]

$$\epsilon y'' - y' = 0, \quad y(0) = 1, \quad y(1) = 0.$$

Exact solution is $y(x) = (1 - \exp((x - 1)/\epsilon)) / (1 - \exp(-1/\epsilon))$.

The solution has a boundary layer of width $O(\epsilon)$ at $x = 1$.

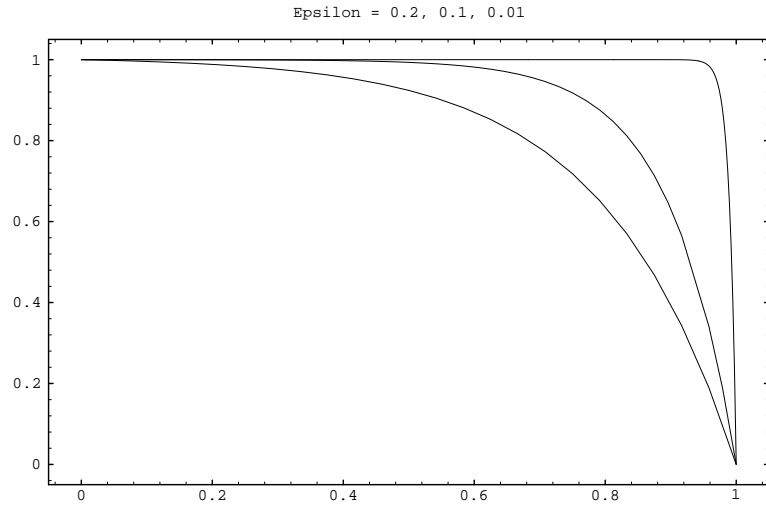


Figure 2: Test Problem 2, $y(x)$ versus x

Test Problem 3 (Linear) [6, p.133]

$$\epsilon y'' + (2 + \cos(\pi x))y' - y = -(1 + \epsilon \pi^2) \cos(\pi x) - (2 + \cos(\pi x))\pi \sin(\pi x),$$
$$y(-1) = y(1) = -1.$$

Exact solution is $y(x) = \cos(\pi x)$.

The problem has no turning points and the solution is smooth.

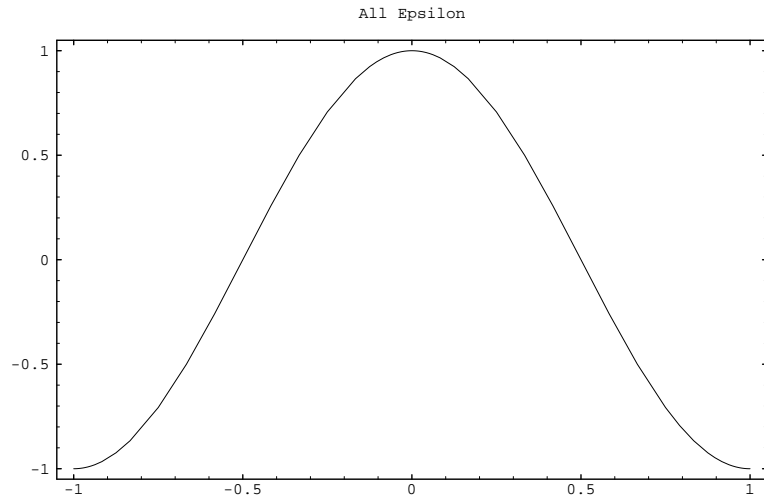


Figure 3: Test Problem 3, $y(x)$ versus x

Test Problem 4 (Linear) [6, p.136]

$$\epsilon y'' + y' - (1 + \epsilon)y = 0, \quad y(-1) = 1 + \exp(-2), \quad y(1) = 1 + \exp(-2(1 + \epsilon)/\epsilon).$$

Exact solution is $y(x) = \exp(x - 1) + \exp(-(1 + \epsilon)(1 + x)/\epsilon)$.

The solution has a boundary layer of width $O(\epsilon)$ at $x = -1$.

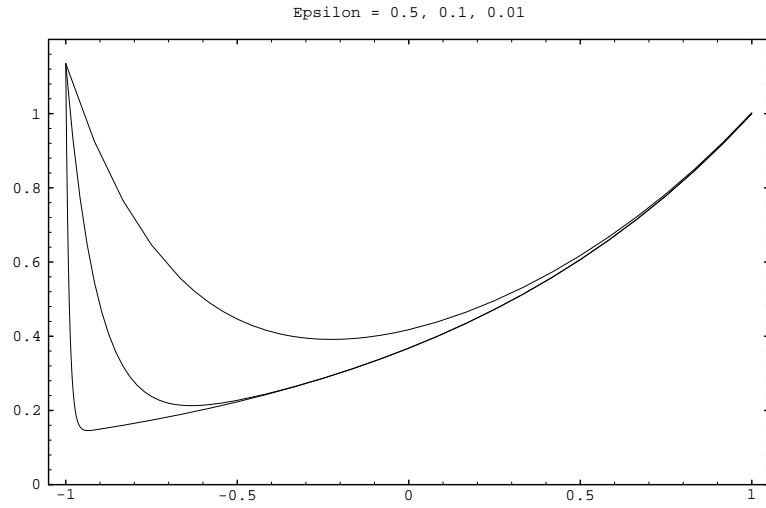


Figure 4: Test Problem 4, $y(x)$ versus x

Test Problem 5 (Linear) [6, p.138]

$$\epsilon y'' - xy' - y = -(1 + \epsilon\pi^2) \cos(\pi x) + \pi x \sin(\pi x), \quad y(-1) = y(1) = -1.$$

Exact solution is $y(x) = \cos(\pi x)$.

The problem has a turning point at $x = 0$ but the solution is smooth.

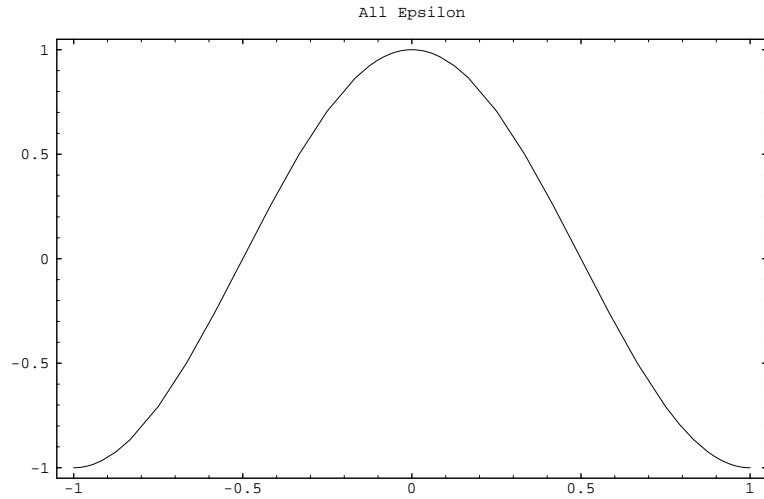


Figure 5: Test Problem 5, $y(x)$ versus x

Test Problem 6 (Linear) [6, p.138]

$$\epsilon y'' + xy' = -\epsilon\pi^2 \cos(\pi x) - \pi x \sin(\pi x), \quad y(-1) = -2, y(1) = 0.$$

Exact solution is $y(x) = \cos(\pi x) + \operatorname{erf}(x/\sqrt{2\epsilon})/\operatorname{erf}(1/\sqrt{2\epsilon})$.

The solution has a shock layer in the turning point region near $x = 0$.

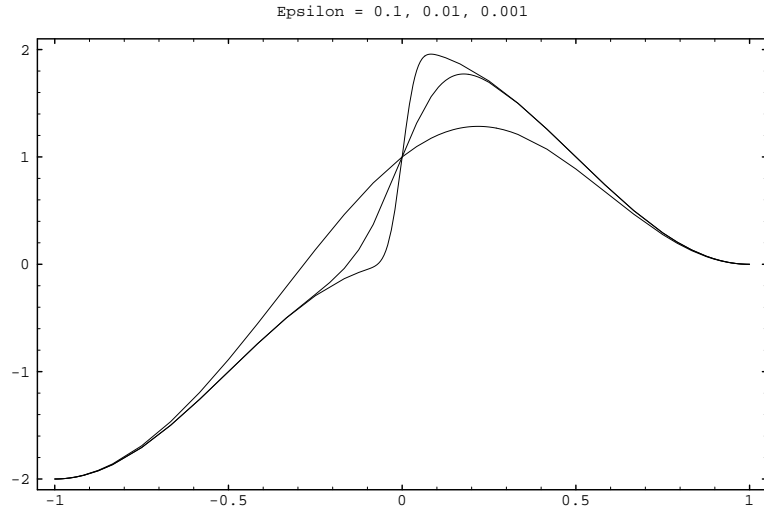


Figure 6: Test Problem 6, $y(x)$ versus x

Test Problem 7¹ (Linear) [6, p.142]

$$\epsilon y'' + xy' - y = -(1 + \epsilon\pi^2) \cos(\pi x) - \pi x \sin(\pi x), \quad y(-1) = -1, \quad y(1) = 1.$$

Exact solution is $y(x) = \cos(\pi x) + x + \frac{x \operatorname{erf}(x/\sqrt{2\epsilon}) + \sqrt{2\epsilon/\pi} \exp(-x^2/2\epsilon)}{\operatorname{erf}(1/2\epsilon) + \sqrt{2\epsilon/\pi} \exp(-1/2\epsilon)}$.

The solution has a corner layer in the turning point region near $x = 0$.

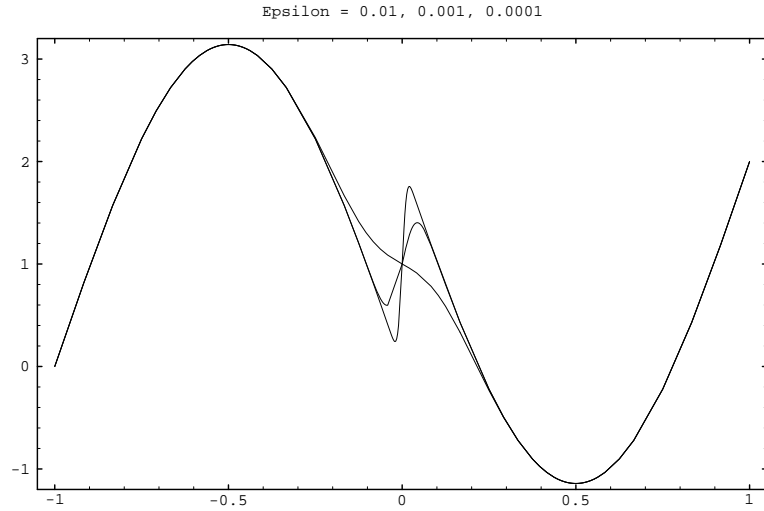


Figure 7: Test Problem 7, $y'(x)$ versus x

¹The layer region for problem 7 is illustrated more clearly by $y'(x)$ than by $y(x)$.

Test Problem 8 (Linear) [6, p.14]

$$\epsilon y'' + y' = 0, \quad y(0) = 1, \quad y(1) = 2.$$

Exact solution is $y(x) = (2 - \exp(-1/\epsilon) - \exp(-x/\epsilon))/(1 - \exp(-1/\epsilon))$.

The solution has a boundary layer of width $O(\epsilon)$ at $x = 0$.

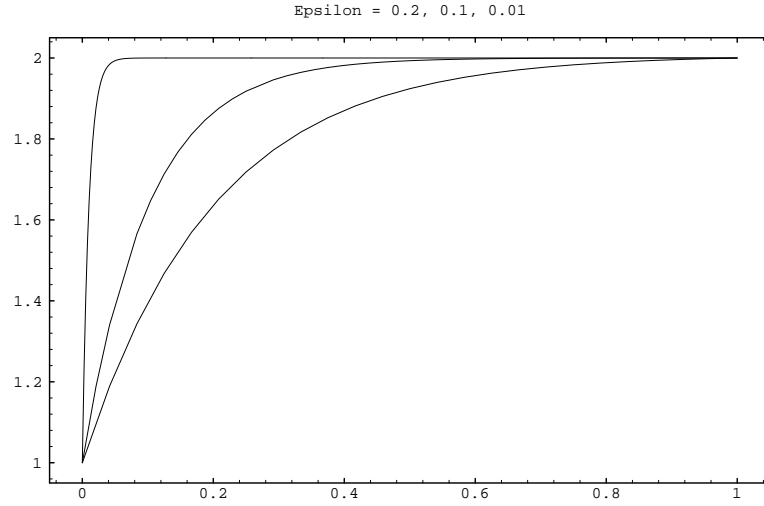


Figure 8: Test Problem 8, $y(x)$ versus x

Test Problem 9 Modified Runge Problem (Linear) [2]
 $(\epsilon + x^2)y'' + 4xy' + 2y = 0, \quad y(-1) = y(1) = 1/(1 + \epsilon).$
Exact solution is $y(x) = 1/(\epsilon + x^2)$.
The solution has a spike of height $1/\epsilon$ at $x = 0$.

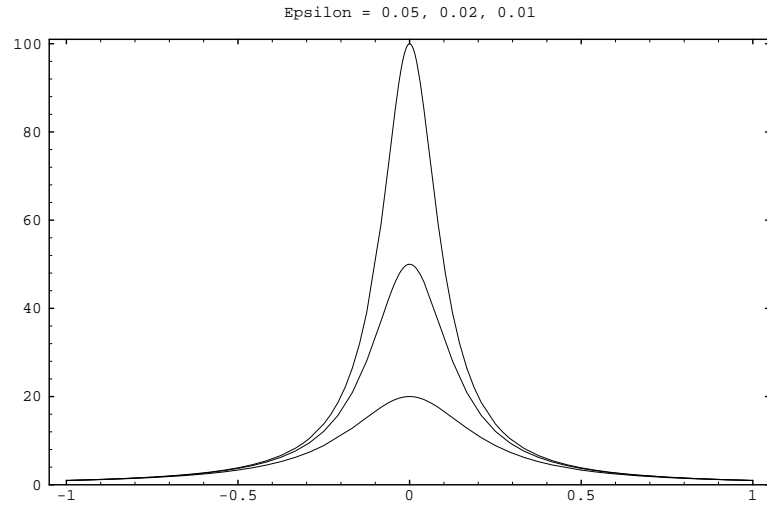


Figure 9: Test Problem 9, $y(x)$ versus x

Test Problem 10 (Linear) [6, p.14]

$$\epsilon y'' + xy' = 0, \quad y(-1) = 0, \quad y(1) = 2.$$

Exact solution is $y(x) = 1 + \operatorname{erf}(x/\sqrt{2\epsilon})/\operatorname{erf}(1/\sqrt{2\epsilon})$.

The solution has a turning point of width $O(\sqrt{\epsilon})$ at $x = 0$.

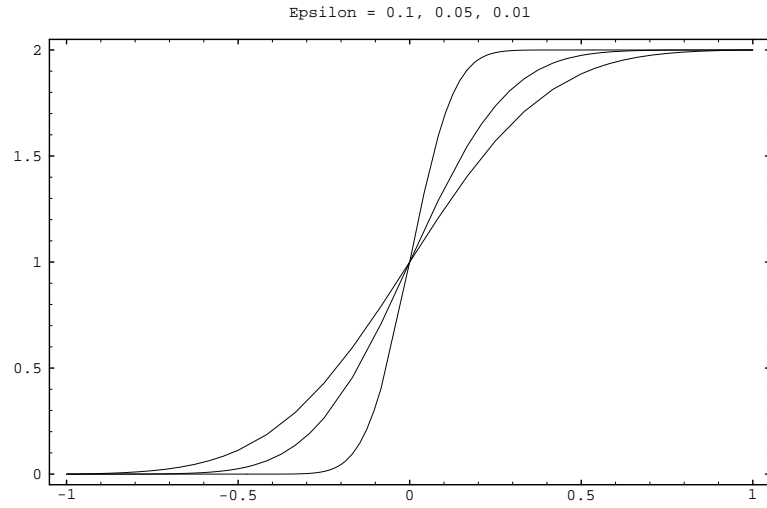


Figure 10: Test Problem 10, $y(x)$ versus x

Test Problem 11 (Linear) [2]

$$\epsilon y'' - y = -(\epsilon\pi^2 + 1) \cos(\pi x), \quad y(-1) = y(1) = -1.$$

Exact solution is $y(x) = \cos(\pi x)$.

The solution is smooth.

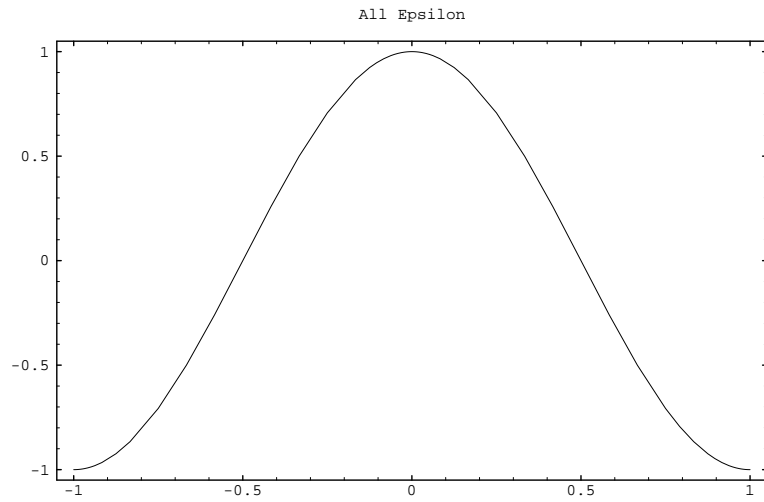


Figure 11: Test Problem 11, $y(x)$ versus x

Test Problem 12 (Linear) [2]

$$\epsilon y'' - y = -(\epsilon\pi^2 + 1) \cos(\pi x), \quad y(-1) = -1, \quad y(1) = 0.$$

Exact solution is $y(x) = \cos(\pi x) + \exp((x-1)/\sqrt{\epsilon})$.

The solution has a boundary layer of width $O(\sqrt{\epsilon})$ near $x = 1$.

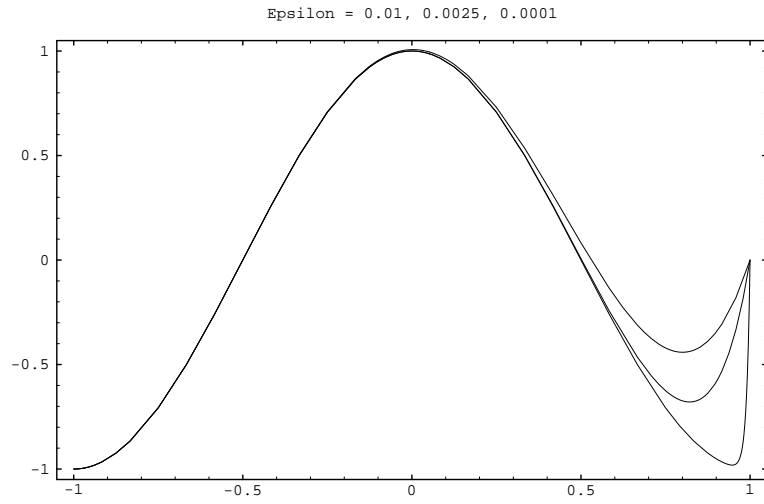


Figure 12: Test Problem 12, $y(x)$ versus x

Test Problem 13 (Linear) [2]

$$\epsilon y'' - y = -(\epsilon\pi^2 + 1) \cos(\pi x), \quad y(-1) = 0, \quad y(1) = -1.$$

Exact solution is $y(x) = \cos(\pi x) + \exp(-(x+1)/\sqrt{\epsilon})$.

The solution has a boundary layer of width $O(\sqrt{\epsilon})$ near $x = -1$.

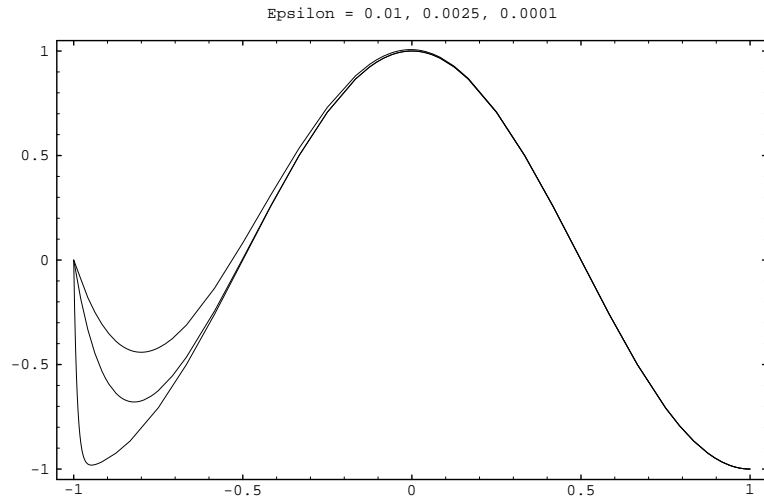


Figure 13: Test Problem 13, $y(x)$ versus x

Test Problem 14 (Linear) [2]

$$\epsilon y'' - y = -(\epsilon\pi^2 + 1) \cos(\pi x), \quad y(-1) = y(1) = 0.$$

Exact solution is $y(x) = \cos(\pi x) + \exp((x-1)/\sqrt{\epsilon}) + \exp(-(x+1)/\sqrt{\epsilon})$.

The solution has boundary layers of width $O(\sqrt{\epsilon})$ near $x = -1$ and $x = 1$.

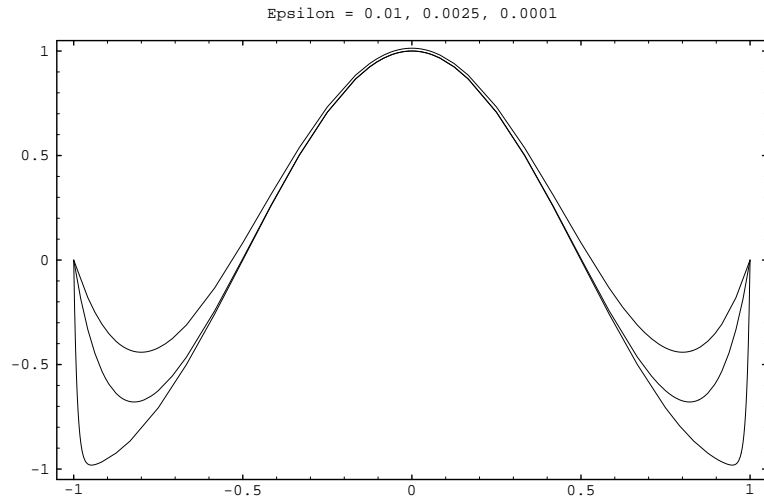


Figure 14: Test Problem 14, $y(x)$ versus x

Test Problem 15 (Linear) [6, p.14]

$$\epsilon y'' - xy = 0, \quad y(-1) = y(1) = 1.$$

Exact solution is $y(x) = AA_i(x\epsilon^{-1/3}) + BB_i(x\epsilon^{-1/3})$.

Here $A_i(x)$, $B_i(x)$ are Airy functions. The solution has an oscillating part coupled to a smooth solution which is in turn coupled to a boundary layer.

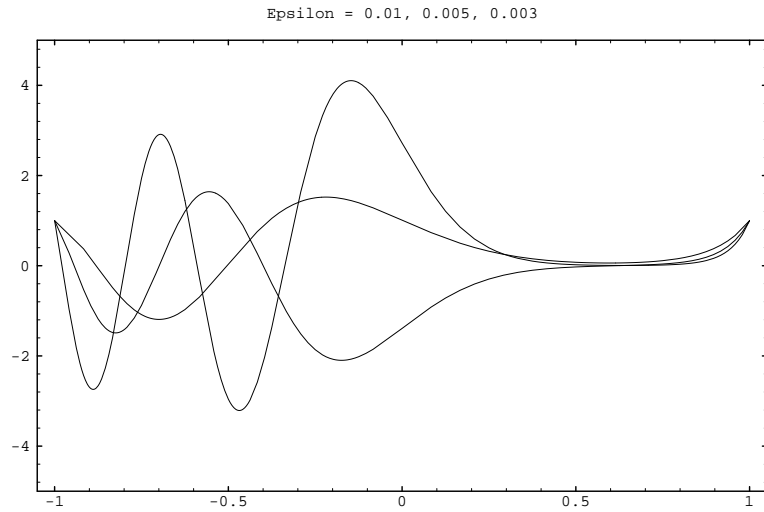


Figure 15: Test Problem 15, $y(x)$ versus x

Test Problem 16 (Linear) [2]

$$\epsilon^2 y'' + \pi^2 y/4 = 0, \quad y(0) = 0, \quad y(1) = \sin(\pi/2\epsilon), \quad 1/\epsilon \text{ odd.}$$

Exact solution is $y(x) = \sin(\pi x/2\epsilon)$.

The solution oscillates rapidly for small ϵ .

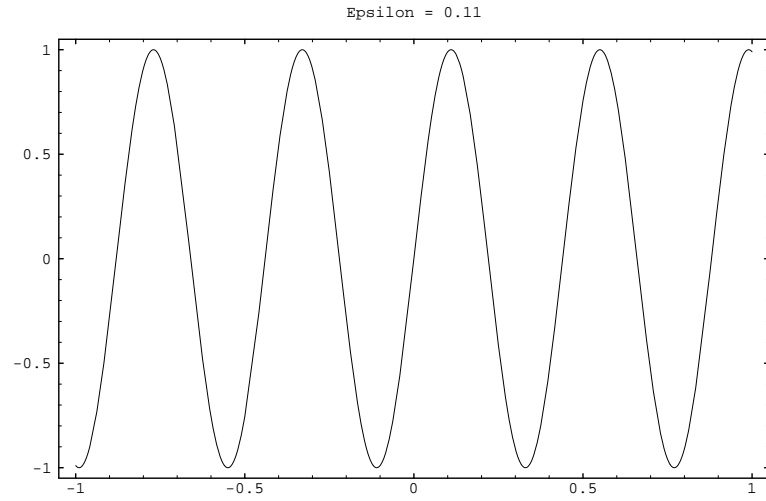


Figure 16: Test Problem 16, $y(x)$ versus x

Test Problem 17 (Linear) [7]

$$y'' = -3\epsilon y/(\epsilon + x^2)^2, \quad y(0.1) = -y(-0.1) = 0.1/\sqrt{\epsilon + 0.01}.$$

Exact solution is $y(x) = x/\sqrt{\epsilon + x^2}$.

The solution has an interior layer of width $O(\sqrt{\epsilon})$ at $x = 0$.

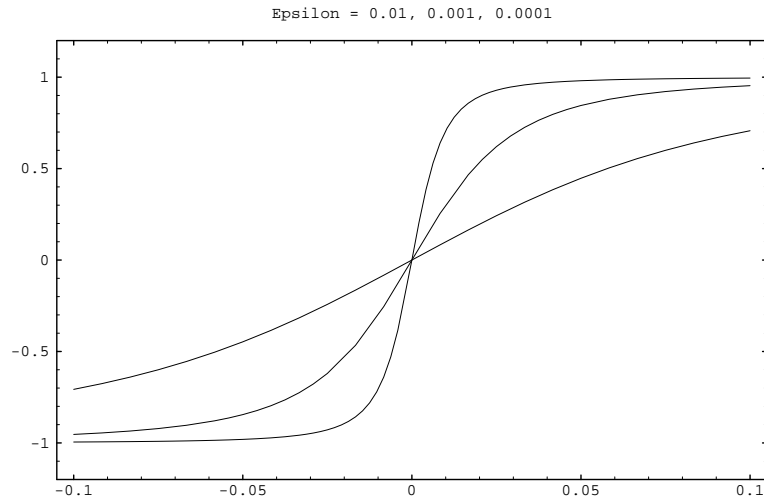


Figure 17: Test Problem 17, $y(x)$ versus x

Test Problem 18 (Linear) [6, p.14]

$$\epsilon y'' = -y', \quad y(0) = 1, \quad y(1) = \exp(-1/\epsilon).$$

Exact solution is $y(x) = \exp(-x/\epsilon)$.

The solution has a boundary layer of width $O(\epsilon)$ at $x = 0$.

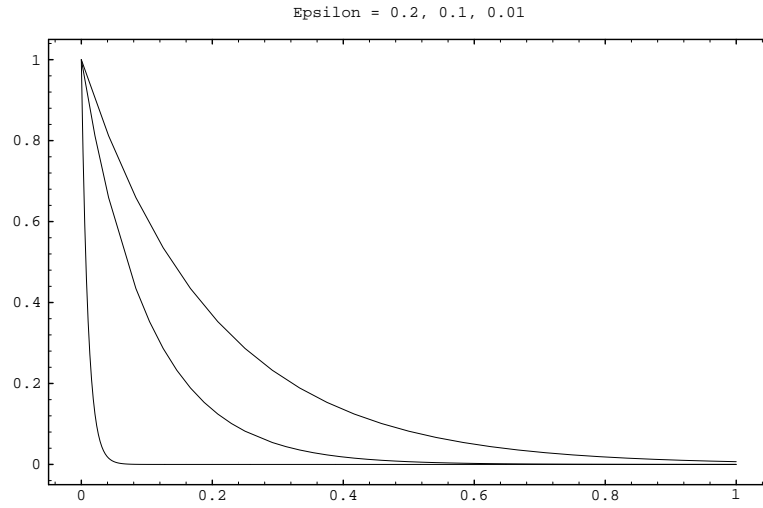


Figure 18: Test Problem 18, $y(x)$ versus x

Test Problem 19 (Nonlinear) [6, p.154]

$$\epsilon y'' + \exp(y)y' - \frac{\pi}{2} \sin(\pi x/2) \exp(2y) = 0, \quad y(0) = y(1) = 0.$$

The asymptotic solution for $\epsilon \rightarrow 0$ of this problem is given by O'Malley [9, p.123]. The solution is $y(x) = -\ln((1 + \cos(\pi x/2))(1 - \frac{1}{2} \exp(-x/2\epsilon))) + O(\epsilon)$. The solution has a boundary layer at $x = 0$.

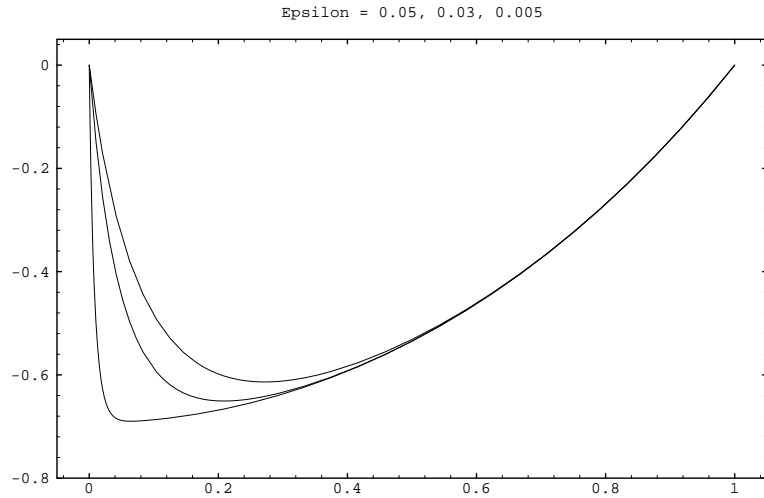


Figure 19: Test Problem 19, $y(x)$ versus x

Test Problem 20 (Nonlinear) [10], [12] and [6, p.158]
 $\epsilon y'' + (y')^2 = 1, \quad y(0) = 1 + \epsilon \ln \cosh(-0.745/\epsilon), \quad y(1) = 1 + \epsilon \ln \cosh(0.255/\epsilon).$
The exact solution is $y(x) = 1 + \epsilon \ln \cosh((x - 0.745)/\epsilon).$
The solution has a corner layer at $x = 0.745$.

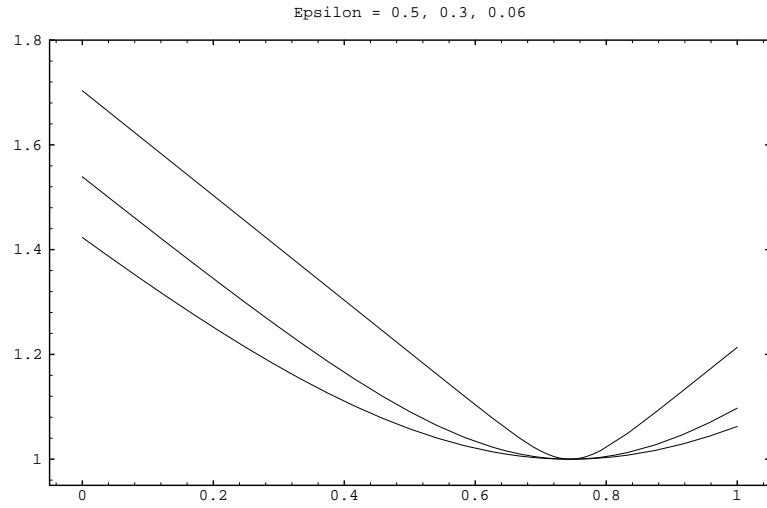


Figure 20: Test Problem 20, $y(x)$ versus x

Test Problem 21 (Nonlinear) [3]

$$\epsilon y'' = y + y^2 - \exp(-2x/\sqrt{\epsilon}), \quad y(0) = 1, \quad y(1) = \exp(-1/\sqrt{\epsilon}).$$

The exact solution is $y(x) = \exp(-x/\sqrt{\epsilon})$.

The solution has a boundary layer of width $O(\sqrt{\epsilon})$ at $x = 0$.

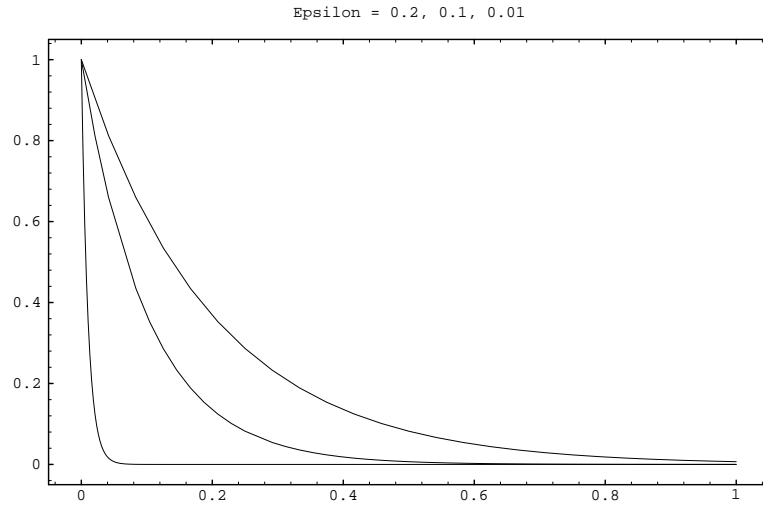


Figure 21: Test Problem 21, $y(x)$ versus x

Test Problem 22 (Nonlinear) [8]
 $\epsilon y'' + y' + y^2 = 0, \quad y(0) = 0, \quad y(1) = \frac{1}{2}.$
The solution has a boundary layer near $x = 0$.

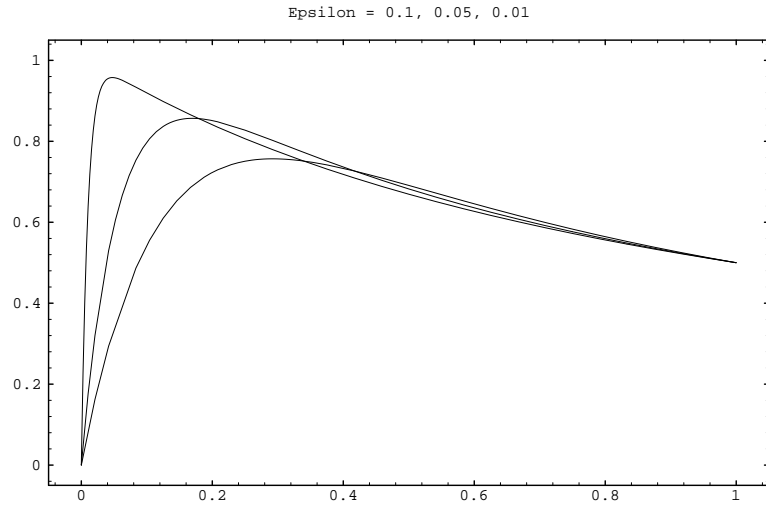


Figure 22: Test Problem 22, $y(x)$ versus x

Test Problem 23 Troesch's Equation (Nonlinear) (see e.g. [11])
 $y'' = \mu \sinh(\mu y), \quad y(0) = 0, \quad y(1) = 1.$
The solution has a boundary layer near $x = 1$.

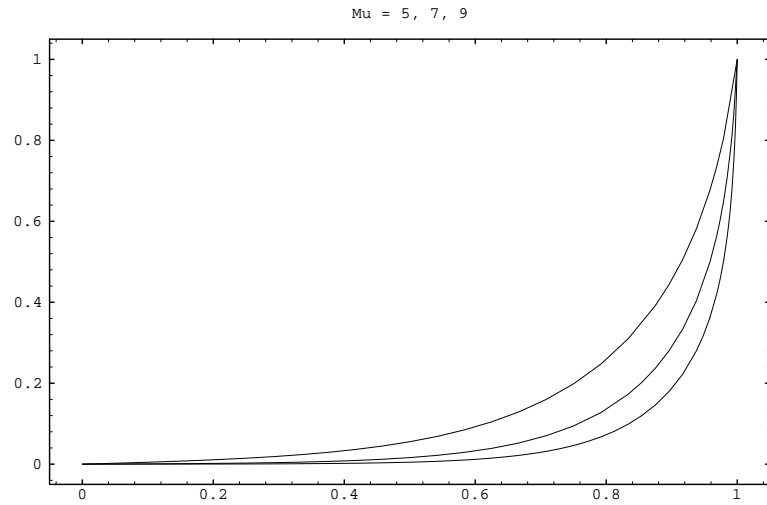


Figure 23: Test Problem 23, $y(x)$ versus x

Test Problem 24 (Nonlinear) [1, p.21], [10] and [6, p.162]

$$\epsilon A(x)yy'' - \left(\frac{1+\gamma}{2} - \epsilon A'(x)\right)yy' + \frac{y'}{y} + \frac{A'(x)}{A(x)} \left(1 - \left(\frac{\gamma-1}{2}\right)y^2\right) = 0,$$
$$y(0) = 0.9129, y(1) = 0.375.$$

In our tests $A(x) = 1 + x^2$ and $\gamma = 1.4$.

This equation represents a shock wave in a one dimensional nozzle flow.

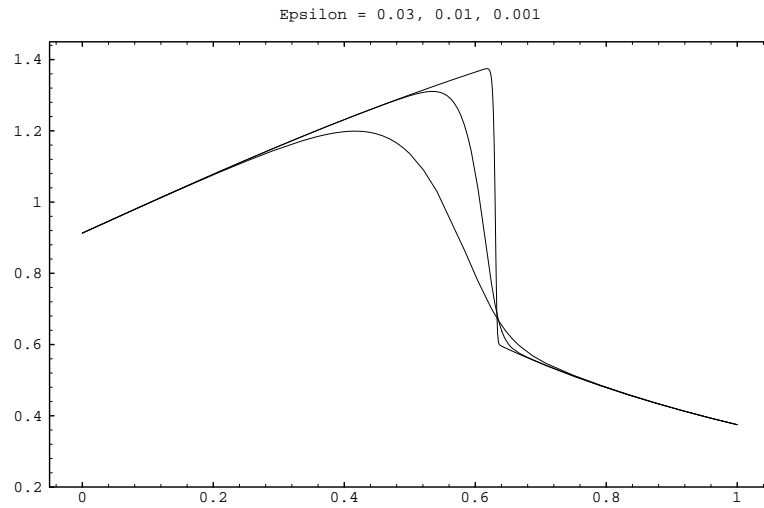


Figure 24: Test Problem 24, $y(x)$ versus x

Test Problem 25 (Nonlinear) (see e.g. [4, pp.29-38] and [6, p.159])
 $\epsilon y'' + yy' - y = 0, \quad y(0) = -\frac{1}{3}, \quad y(1) = \frac{1}{3}.$
The solution has corner layers at $x = \frac{1}{3}$ and $x = \frac{2}{3}$.

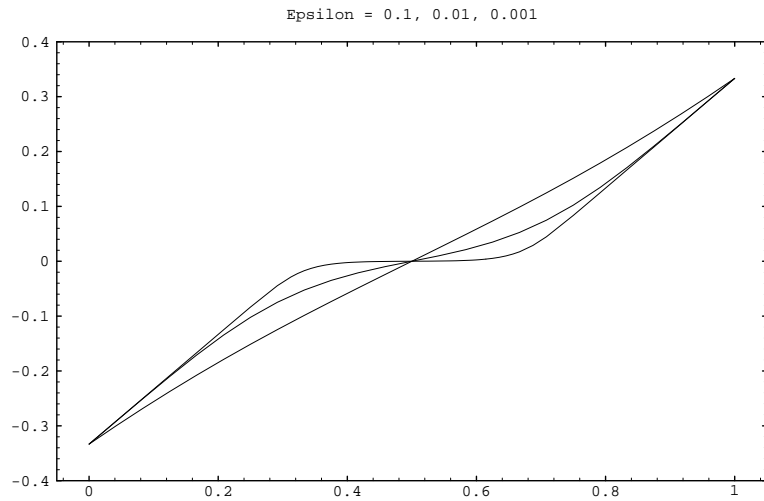


Figure 25: Test Problem 25, $y(x)$ versus x

Test Problem 26 (Nonlinear) (see e.g. [4, pp.29-38] and [6, p.159])
 $\epsilon y'' + yy' - y = 0, \quad y(0) = 1, \quad y(1) = -\frac{1}{3}.$
The solution has boundary layers at $x = 0$ and $x = 1$.

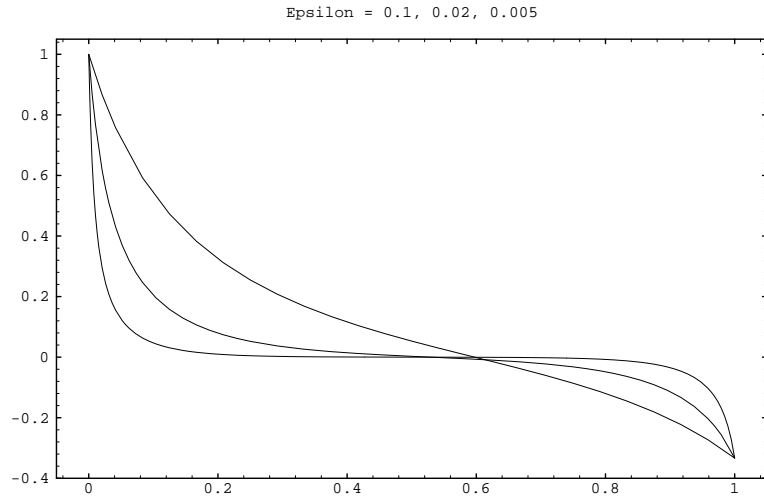


Figure 26: Test Problem 26, $y(x)$ versus x

Test Problem 27 (Nonlinear) (see e.g. [4, pp.29-38] and [6, p.159])

$$\epsilon y'' + yy' - y = 0, \quad y(0) = 1, \quad y(1) = \frac{1}{3}.$$

The solution has a boundary layer at $x = 0$ and a corner layer at $x = \frac{2}{3}$.

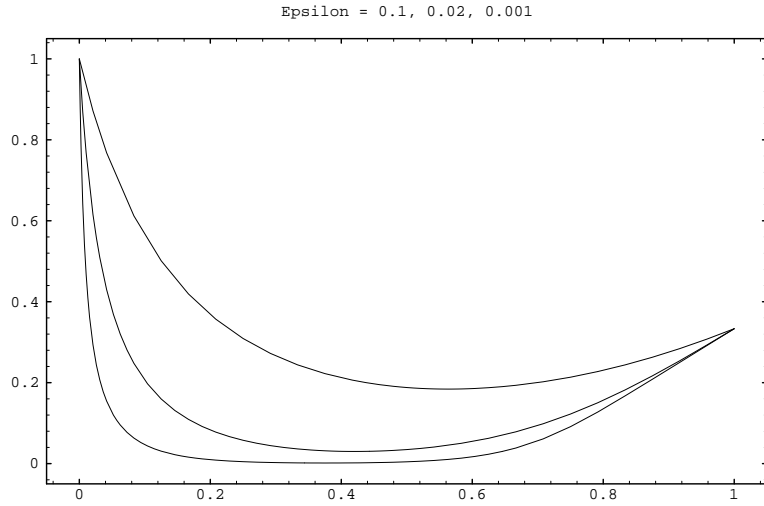


Figure 27: Test Problem 27, $y(x)$ versus x

Test Problem 28 (Nonlinear) (see e.g. [4, pp.29-38] and [6, p.159])
 $\epsilon y'' + yy' - y = 0, \quad y(0) = 1, \quad y(1) = \frac{3}{2}$.
The solution has a boundary layer at $x = 0$.

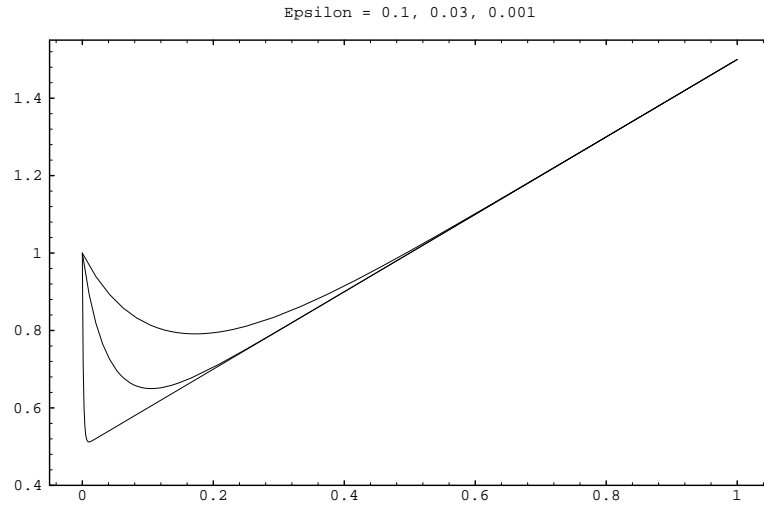


Figure 28: Test Problem 28, $y(x)$ versus x

Test Problem 29 (Nonlinear) (see e.g. [4, pp.29-38] and [6, p.159])
 $\epsilon y'' + yy' - y = 0, \quad y(0) = 0, \quad y(1) = \frac{3}{2}$.
The solution has a boundary layer at $x = 0$.

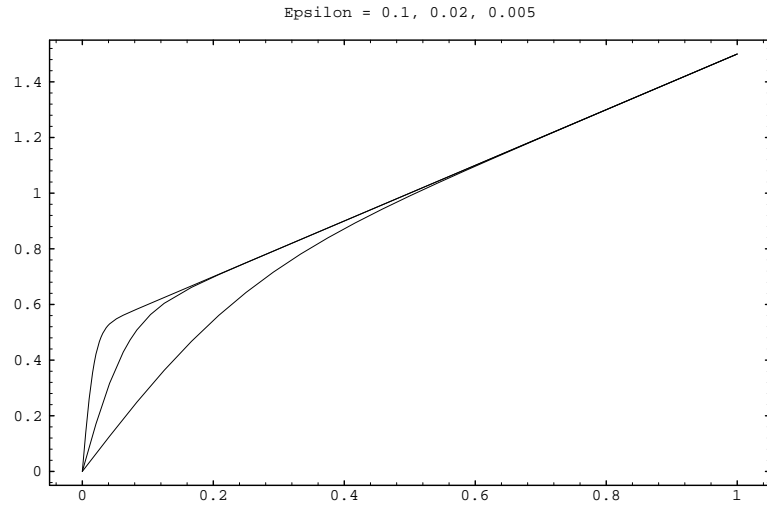


Figure 29: Test Problem 29, $y(x)$ versus x

Test Problem 30 (Nonlinear) (see e.g. [4, pp.29-38] and [6, p.159])
 $\epsilon y'' + yy' - y = 0, \quad y(0) = -\frac{7}{6}, \quad y(1) = \frac{3}{2}.$
The solution has a shock layer at $x = \frac{1}{3}$.

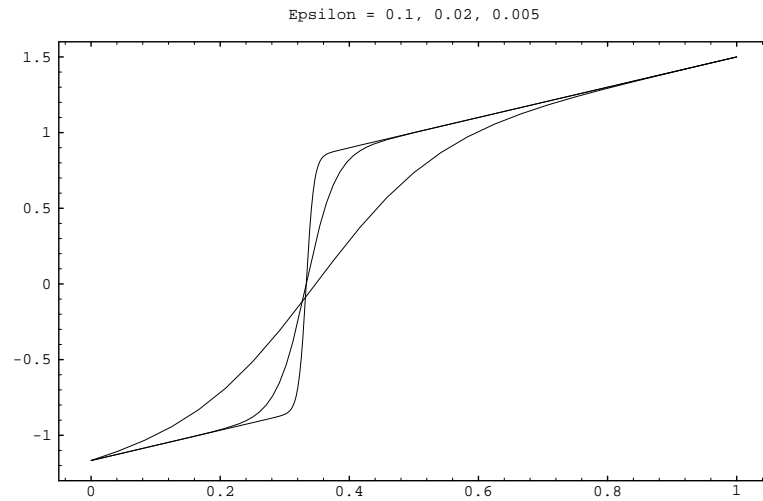


Figure 30: Test Problem 30, $y(x)$ versus x

Test Problem 31 (Nonlinear) [1, p.18]

$$y' = \sin \theta, \quad \theta' = M, \quad \epsilon M' = -Q, \quad \epsilon Q' = (y - 1) \cos \theta - MT, \quad T = \sec \theta + \epsilon Q \tan \theta,$$

$$y(0) = y(1) = 0, \quad M(0) = M(1) = 0.$$

This equation models nonlinear elastic beams.

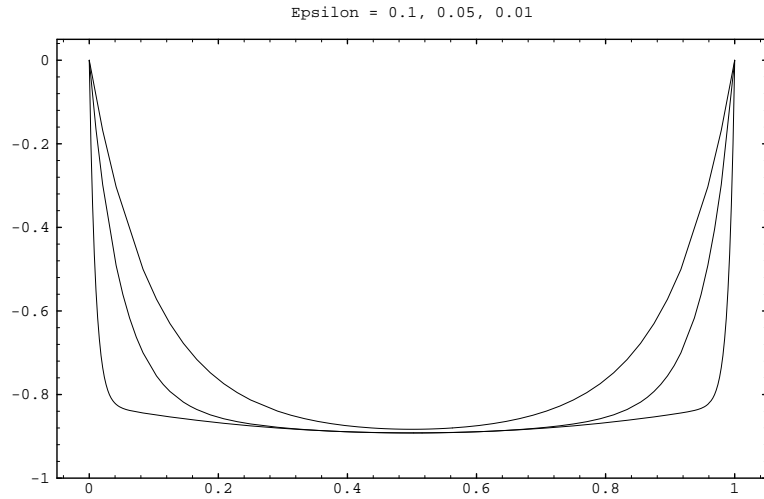


Figure 31: Test Problem 31, $M(x)$ versus x

Test Problem 32 (Nonlinear) [1, p.8]

$$y'''' = R(y'y'' - yy'''), \quad y(0) = y'(0) = 0, \quad y(1) = 1, \quad y'(1) = 0.$$

This problem arises from fluid injection through one side of a long vertical channel.

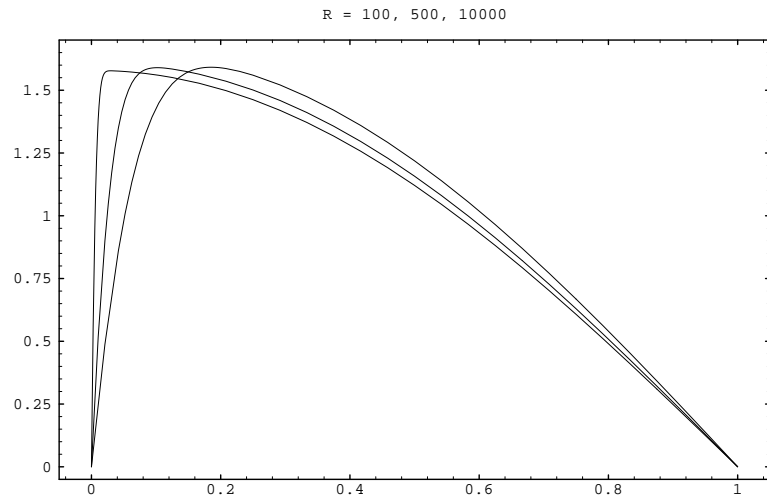


Figure 32: Test Problem 32, $y'(x)$ versus x

References

- [1] U. Ascher, R.M.M. Mattheij, and R.D. Russell. *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*. Prentice-Hall, Englewood Cliffs, N.J., 1988.
- [2] J.R. Cash. A comparison of some global methods for solving two-point boundary value problems. *Appl. Math. Comput.*, 31:449–462, 1989.
- [3] J.R. Cash and M.H. Wright. A deferred correction method for nonlinear two-point boundary value problems: Implementation and numerical evaluation. *SIAM J. Sci. Stat. Comput.*, 12:971–989, 1991.
- [4] J.D. Cole. *Perturbation Methods in Applied Mathematics*. Blaisdell, Waltham, MA, 1968.
- [5] C. de Boor and R. Weiss. SOLVEBLOK: A package for solving almost block diagonal linear systems. *ACM Trans. Math. Software*, 6:80–87, 1980.
- [6] P.W. Hemker. *A Numerical Study of Stiff Two-Point Boundary Value Problems*. Mathematisch Centrum, Amsterdam, 1977.
- [7] M. Lentini and V. Pereyra. An adaptive finite difference solver for nonlinear two-point boundary value problems with mild boundary layers. *SIAM J. Numer. Anal.*, 14:91–111, 1977.
- [8] M.R. Maier. Numerical solution of singularly perturbed boundary value problems using a collocation method with tension splines. In U. Ascher and R.D. Russell, editors, *Numerical Boundary Value Ordinary Differential Equations*, pages 207–225. Birkhäuser, Boston, MA, 1985.
- [9] R.E. O'Malley, Jr. *Introduction to Singular Perturbations*. Academic Press, New York, 1974.
- [10] C.E. Pearson. On nonlinear ordinary differential equations of boundary layer type. *J. Math. Phys.*, 47:351–358, 1968.
- [11] M.R. Scott and H.A. Watts. Computational solution of linear two-point boundary value problems via orthonormalization. *SIAM J. Numer. Anal.*, 14:40–70, 1977.
- [12] W. Wasow. The capriciousness of singular perturbations. *Nieuw Arch. Wisk.*, 18:190–210, 1970.