Computer Science 236F
St. George Campus

October 14, 2010
University of Toronto

Homework Assignment #2
Due: November 4, 2010 at 6pm

On the cover page of your assignment, you must write ***and sign*** *the following statement: "I have read and understood the homework collaboration policy described in the Course Information handout." Without such a signed statement your homework will not be marked.*

---

1. (20 marks) In this question you should prove your assertions using only the logical equivalences of Section 5.6 (i.e. without using truth tables).

   (a) (5 marks) Prove that $x \wedge (\neg y \leftrightarrow z)$ is logically equivalent to $((x \to y) \vee \neg z) \to (x \wedge \neg(y \to z))$

   (b) (5 marks) Find a CNF formula that is logically equivalent to $\neg y \to (x \leftrightarrow z)$. Prove your assertion.

   (c) (10 marks) Classify each formula below as either a tautology, a contradiction or a contingency (satisfiable but not a tautology). Justify your answers.

      i. $((x \vee y) \to z) \vee (z \to (x \vee y))$
      ii. $(x \to y) \wedge (\neg x \to y)$
      iii. $(x \to y) \vee (x \to \neg y)$

2. (20 marks)

   (a) (10 marks) Prove that $\{\neg, \to\}$ is a complete set of connectives.

   (b) (10 marks) Prove that $\{\oplus, \vee\}$ is not a complete set of connectives.

3. (10 marks)

   Consider truth assignments containing only the propositional variables $x_0, x_1,\ x_2, x_3$ and y_0, y_1, y_2, y_3, y_4. Every such truth assignment gives a value of 1 (representing true) or 0 (representing false) to each variable. Therefore we can think of a truth assignment τ as determining a 4-bit integer x_τ, where the most significant bit is x_3 and the least significant bit is x_0. In particular, $x_\tau = \tau(x_0) + 2\tau(x_1) + 4\tau(x_2) + 8\tau(x_3)$. Similarly the assignment τ determines a 5-bit integer $y_\tau = \tau(y_0) + 2\tau(y_1) + 4\tau(y_2) + 8\tau(y_3) + 16\tau(y_4)$.

   Write a formula that is satisfied by exactly those truth assignments τ for which $y_\tau = x_\tau + 1$. You may use any of the Boolean connectives discussed in the notes. Justify your answer.

   Hint: You need to express as a formula the condition under which the 5-bit number $y_4y_3y_2y_1y_0$ is one more than the 4-bit number $x_3x_2x_1x_0$. Think of what this formula would have to say about $y_0, ..., y_4$ first in case $x_0 = 0$, then in case $x_0 = 1$ but $x_1 = 0$ and so on and so forth.

4. (10 marks) Consider the first order language of arithmetic described in Section 6.2 and the structures $\mathcal{N}$ and $\mathcal{Z}$ described on page 152 in the notes. For each sentence below state whether it is true in $\mathcal{N}$, $\mathcal{Z}$, both or neither. Justify your answer by translating each sentence into a statement in precise English about numbers and then explain why that statement is true or false for the natural numbers or for the integers.

   (a) $\forall x\ \exists y\ L(y, x)$

   (b) $\forall x\ \forall y\ ((L(x, 0) \wedge P(x, x, y)) \to L(0, y))$

   (c) $\forall x\ \exists y\ \forall z\ \neg P(y, z, x)$

   (d) $\exists x\ \forall y (\neg \approx (0, y) \to L(x, y))$

5. (4 marks) Find an equivalent formula F' to

$$F : \exists x R(x, y) \rightarrow \exists x(P(x) \vee \neg(\exists y)Q(x, y))$$

such that F' is in PRENEX form and F' has the same free variables as F.

6. (16 marks) For each of the following formulas determine whether it is valid, satisfiable or unsatisfiable. Justify your answer either with a formal proof using logical equivalences and/or by defining appropriate structures.

   (a) $(\forall x Q(x) \rightarrow \neg\exists y \exists z R(y, z)) \leftrightarrow \exists x \forall y \forall z(Q(x) \rightarrow \neg R(y, z))$

   (b) $(\forall x P(x) \rightarrow \exists y \neg Q(y)) \rightarrow (\forall x Q(x) \rightarrow \exists y \neg P(y))$

   (c) $(\forall x P(x) \rightarrow \neg\forall x Q(x)) \leftrightarrow \exists x \forall y(P(x) \rightarrow \neg Q(y))$

   (d) $(\forall x(P(x) \rightarrow \exists y \neg Q(y))) \rightarrow \neg(\forall x Q(x) \rightarrow \exists y \neg P(y))$