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UNIVERSITY OF TORONTO
Faculty of Arts and Science

AUGUST 2014 EXAMINATIONS

CSC 165 H1Y
Instructor: M. Guerzhoy

Duration — 3 hours

Examination Aids: aid sheet distributed with the exam

Student Number: |_|_|_|_|_|_|_|_|_|_|

Family Name(s): ____________________

Given Name(s): ____________________

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This final examination paper consists of 6 questions on 18 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy is complete and fill in the identification section above.*

Answer each question directly on the exam paper, in the space provided, and use the rough work pages for rough work. If you need more space for one of your solutions, use a blank page, *indicate clearly the part of your work that should be marked, and indicate the number of the page where your answer is on the page where the question appears.*

You will receive 10% of the marks for any question which you answer with "I don't know."

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

If you obtain less than 40% on this exam, your mark in the course will be at most 47%.

Marking Guide

1: ______/15

2: ______/15

3: ______/15

4: ______/15

5: ______/15

6: ______/ 8

TOTAL: ______/83

Good Luck!

Use the space for answers that did not fit in elsewhere. ***Clearly label each answer with the appropriate question and part number, and indicate where your answers are on the page where the question appears.***

Question 1. [15 marks]

Recall that a real number x is rational (i.e., $x \in \mathbb{Q}$) if $\exists p \in \mathbb{Z}, \exists q \in \mathbb{Z}^*, x = p/q$. Prove the following claim. Give a detailed structured proof, justifying every step.

$$\forall x \in \mathbb{R}, [x \in \mathbb{Q} \Rightarrow (x^2 + 5) \in \mathbb{Q}]$$

Use the space for answers that did not fit in elsewhere. ***Clearly label each answer with the appropriate question and part number, and indicate where your answers are on the page where the question appears.***

Question 2. [15 MARKS]

Recall that $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$. Assume that $D \subset \mathbb{N}$ and $D \neq \varnothing$. State whether the following claim is true, and then prove or disprove it. Give a detailed structured proof, justifying every step.

$$[\forall x \in D, \exists y \in \mathbb{N}, y < x] \Leftrightarrow [0 \notin D]$$

Use the space for answers that did not fit in elsewhere. ***Clearly label each answer with the appropriate question and part number, and indicate where your answers are on the page where the question appears.***

Question 3. [15 MARKS]

Let $\mathbb{R}^+$ be the set of positive real numbers and $\mathbb{N}^+$ be the set of positive natural numbers. Prove the following claim. You may not use, without proof, any properties of big-Oh, other than its definition. Give a detailed structured proof, justifying every step.

$$\forall a \in \mathbb{R}^+, \forall b \in \mathbb{R}^+, \forall p_1 \in \mathbb{N}^+, \forall p_2 \in \mathbb{N}^+, [p_1 > p_2] \Rightarrow an^{p_1} \notin \mathcal{O}(bn^{p_2})$$

Use the space for answers that did not fit in elsewhere. ***Clearly label each answer with the appropriate question and part number, and indicate where your answers are on the page where the question appears.***

Question 4. [15 MARKS]

State whether the following claim is true, and then prove or disprove it. Give a detailed structured proof, justifying every step.

$$\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, \exists z \in \mathbb{N}, [x > (y - z)]$$

Use the space for answers that did not fit in elsewhere. ***Clearly label each answer with the appropriate question and part number, and indicate where your answers are on the page where the question appears.***

Question 5. [15 MARKS]

Let $\mathcal{F}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{R}^+$. For functions $f \in \mathcal{F}$ and $g \in \mathcal{F}$, let $(g+h)$ be a function such that $\forall n \in \mathbb{N}, (g+h)(n) = g(n) + h(n)$. State whether the following claim is true or false, and then prove or disprove it. You may not use, without proof, any properties of big-Oh, other than its definition. Give a detailed structured proof, justifying every step.

$$\forall f \in \mathcal{F}, \forall g \in \mathcal{F}, \forall h \in \mathcal{F}, [[g \in \mathcal{O}(f) \wedge h \in \mathcal{O}(f)] \Rightarrow [(g+h) \in \mathcal{O}(f)]]$$

Use the space for answers that did not fit in elsewhere. ***Clearly label each answer with the appropriate question and part number, and indicate where your answers are on the page where the question appears.***

Question 6. [8 MARKS]

Assume that $x \in \mathbb{R}$ and $(x + 1/x) \in \mathbb{Z}$. Using induction, prove the following claim. Use the detailed structured proof format and justify every step.

$$\forall n \in \mathbb{N}, (x^n + \frac{1}{x^n}) \in \mathbb{Z}$$

Use the space for answers that did not fit in elsewhere. ***Clearly label each answer with the appropriate question and part number, and indicate where your answers are on the page where the question appears.***

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