CSC165, Summer 2014
Induction examples

1 Introduction: the Principle of Simple Induction

The principle of simple induction is as follows. Assume that P is a predicate. Then:

$$[P(0) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))] \Rightarrow [\forall n \in \mathbb{N}, P(n)]$$

In other words, if $P(0)$ is true ($n = 0$ is called the *base case*) and $P(n)$ implies $P(n+1)$ (this implication is called the *induction step*, then $P(n)$ is true for any n. Why is this true? One way to think about it is that we can "cook up" a proof that P(n) is true for any n by repeatedly applying the induction step, starting from the claim that $P(n)$ is true: $P(0)$ is true, then $P(1)$ is true (# implication), so $P(2)$ is true (# implication), so $P(3)$ is true (#implication)... so $P(n)$ is true (# implication). You can imagine writing a Python script to prove that $P(n)$ is true for any given n, so we know that $P(n)$ is true since we can prove it.

2 Sum of an arithmetic series

We prove that $0 + 1 + 2 + ... + n = \frac{n(n+1)}{2}$ for any natural n using induction. Note that $0 + 1 + 2 + ... + n$ can be written as

$$\sum_{i=0}^{n} i$$

First, we define the predicate:

$$P(k) := \sum_{i=0}^{k} i = \frac{k(k+1)}{2}$$

Note that $P(k)$ being true for all $k \in \mathbb{N}$ is exactly what we are trying to prove. We just restated the assertion in the form of a predicate.

We first prove the base case:

$\sum_{i=0}^{0} = 0 = \frac{0(0+1)}{2}$ # algebra

Then $P(0)$ is true # substitute k=0 into the definition of P(k)

We can now prove the induction step:
Assume $k \in \mathbb{N}$
Assume $P(k)$ is true
Then $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ # substitution of the definition of P
Then $\sum_{i=0}^{k+1} i = (\sum_{i=0}^{k} i + (k+1)) = \frac{k(k+1)}{2} + k + 1$ # algebra
Then $\sum_{i=0}^{k+1} i = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2} = \frac{(k+1)((k+1)+1)}{2}$ # algebra
Then $P(k+1)$ is true # substitution
Then $P(k) \Rightarrow P(k+1)$ # introduce implication
Then $\forall k, P(k) \Rightarrow P(k+1)$ # introduce universal

We can now conclude
$P(0)$ # proven above
Also, $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$ # proven above
Then $\forall n \in \mathbb{N}, P(n)$ # principle of simple induction
Then $\forall n \in \mathbb{N}, 0 + 1 + 2 + 4 + ... + n = \frac{n(n+1)/2}{2}$ # substitution
This proves the claim.

3 The number of rows in a truth table

We would like to prove that if a truth table has n variables, then the table will have 2^n rows (i.e., possible values for all the variables.) For example, if there is only one variable in the truth table, P, it can have two values and so the table will just have two rows: $P =$ true and $P =$ false. (The number of columns might be larger: maybe one column is P, another is $\neg P$, and another is $P \wedge P$. The first and third columns are equal, but part of the point of truth tables is that some columns may be equal, so that's okay.

As was pointed out in class, if there are zero variables, arguably the truth table has 0 rows, but $2^0 = 1$. So it makes sense to think about numbers of variables that are at least 1. We can still, of course, start counting from zero by setting up the predicate as:

Q(n) := the number of rows in a truth table with $n + 1$ variables is 2^{n+1}

With this predicate, $Q(0)$ means that the number of rows in a truth table with one variable is two, which is true.

We generally prefer to implicitly generalize the principle of simple induction a little bit, and implicitly use the fact that

$$\forall n_0 \in \mathbb{N}, [P(n_0) \wedge (\forall n \in \{n_0, n_0 + 1, n_0 + 2, ...\}, P(n) \Rightarrow P(n+1))] \Rightarrow [\forall n \in \{n_0, n_0 + 1, n_0 + 2, ...\}, P(n)]$$

This fact is also referred to as the principle of simple induction. The base case will now be $P(n_0)$ for whatever n_0 we pick.

We define the predicate as:

P(n) := the number of rows in a truth table with n variables is 2^n

$P(0)$, unlike $Q(0)$, is false, so we have to use $n_0 = 1$ as the base case:
Assume T is a truth table with $n_0 = 1$ variable
Assume WLOG the variable is called P
Then the truth table T contains a row for P = true and a row for P = false and no other row # only possible values
The the truth table T contains two rows # we listed all of the two possibilities
Then all truth tables with a single variable contains two rows. # introduce universal

We can now prove the induction step. The basic idea is to observe that we can list all the lines of a truth table with $(n + 1)$ variables by gluing together two truth tables for n variables, and assigning True to the n-th variables the first time, and False the second time.

Assume $n \in \{1, 2, 3, ...\}$

Assume $P(n)$ is true

Assume T is a truth table with (n+1) variables (WLOG, call them P_1, P_2, ..., P_n, P_{n+1}. Assume WLOG that all the truth tables under consideration only have variables (and not expressions) as columns (since additional columns don't matter)

Then we can build the rows of T by taking the rows of the truth table for P_2, ..., P_{n+1} twice, setting the value of P_1 as true the first time, and as false the second time # the possible values of $P_2 ... P_{n+1}$ are the same in the smaller table as in the larger table, and the only possible values of P_1 are true and false

Then T is twice as large as the truth table for $P_2 ... P_{n+1}$ # T is built by taking that truth table and writing it twice, varying the values of P_1 The size of the truth table for P_2, ..., P_{n+1} is 2^n # P(n) is true by assumption

Then the size of T is $22^n = 2^{n+1}$ # algebra

Then $P(n+1)$ # substitution all tables with the same number of variables have the same number of rows

Then $P(n) \Rightarrow P(n+1)$ # introduce implication

Then $\forall n \in \{1, 2, 3, ...\}.P(n) \Rightarrow P(n+1)$ # introduce universal

We can now finish the proof:

$P(1)$ # proven above

Also, $\forall k \in \{1, 2, 3, ...\}, P(k) \Rightarrow P(k+1)$ # proven above

Then $\forall n \in \{1, 2, 3, ...\}, P(n)$ # principle of simple induction

Then for all natural $n \geqslant 1$, a truth table with n variables has 2^n rows # substitution