CSC236, Summer 2005, Assignment 2

Due: Thursday June 23rd, 10 am

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Instructions

Please work on all questions. Turn in the outline and structure of a proof, even if you cannot provide every step of the proof, and we will try to assign some part marks. However, if there is any question you cannot see how to even begin, leave it blank you will receive 20% of the marks for that question.

Be sure to give full credit to any sources you consult (other than course notes, TAs, and the instructor) in preparing this problem set. If you try to pass off somebody else's work as your own for credit, you are committing an academic offense, and that can entail serious consequences. Any ideas that you do not attribute to someone else are assumed to be the ideas of the author(s) listed below, and will be evaluated for grading.

Write your name(s) and student number(s) (maximum of two names and two student numbers) in the space below.

Name ______________________________

Student # ______________________________

Name ______________________________

Student # ______________________________

1. Manipulate a stack: Suppose you have a sequence of n distinct characters, and a LIFO (Last In, First Out) stack that allows exactly two operations:

   (a) push: If the sequence is nonempty, remove the first element from the sequence and add it to the top of the stack. Otherwise do nothing.

   (b) popp: If the stack is nonempty, remove the top element and print it to output. Otherwise do nothing.

   If you begin with a sequence of $n = 2$ distinct characters, then you can produce exactly 2 distinct outputs. Suppose your sequence is $\langle xy \rangle$, then you can produce

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   xy: push popp push popp
   yx: push push popp popp
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   How many different outputs can you produce with a sequence xyz, of length 3? How about of length n? Prove your claims.

2. Here is a recursive definition for $\mathcal{T}^*$, a subset of the family of ternary strings. Let $\mathcal{T}^*$ be the smallest set such that:

   Basis: 0 is in $\mathcal{T}^*$.

   Induction step: If $x, y \in \mathcal{T}^*$, then so are $x0y$, $1x2$, and $2x1$.

   (a) Prove that if $k \in \mathbb{N}$, then there is no string in $\mathcal{T}^*$ with exactly $3^k + 1$ zeros.

   (b) Prove that if $k \in \mathbb{N}$, then there is no string in $\mathcal{T}^*$ that has exactly 2^{k+1} digits.

   (c) Prove that there is no string in $\mathcal{T}^*$ whose digits sum to 97.

3. In lecture we discussed the recursive formula for $G(n)$, the number of binary strings of length n that do not have adjacent zeros.

   (a) Using the expression from class, derive a closed form for $G(n)$, the number of binary strings of length n that do not have adjacent zeros.

   (b) Using the approach from class, develop a recursive formula (but not a closed form) for $H(n)$, the number of binary strings of length n that do not have 3 adjacent zeros. Justify your formula.

   (c) Find a closed form for $J(n)$, which is defined for $n \in \mathbb{N}$ as:

$$J(n) = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ J(n-1) + 2J(n-2), & n > 1 \end{cases}.$$

4. Hack some algebra:

   (a) The binomial coefficient $\binom{n}{k}$ is defined for nonnegative integers $0 \le k \le n$ by:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

   and it represents the number of ways of choosing k elements from a set of n elements. Use the definition of $\binom{n}{k}$ to prove that if $0 < k < n$, then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(b) Prove that if $1 \le k \le n$, then

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(c) Suppose $x, y \in \mathbb{R}$. Use induction on n and part (a) to prove that:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(d) Prove that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

(e) Suppose n is a positive integer. Use the previous parts and some manipulation of the sum to prove that:

$$\sum_{k=0}^{n} k\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k} = 1.$$