CSC236, Summer 2005, Assignment 4 Hints 1

Joe Whitney

1. Question 1 is somewhat similar to questions on assignments 1 and 2; you should be able to solve it using the same inductive counting techniques you have used several times. Look at Definition 5.1 in the course notes to see how propositional formulae are defined inductively. Notice that each application of the inductive step increases the number of connectives by 1.

2. When proving equivalencies between formulae in question 2 (using the named laws in section 5.6) you should ideally mention every rule which is used at each step of transforming the first formula into the second. This will not only ensure full marks but will help you to check your own work and make sure there are no errors.

3. Question 3 talks about propositional connectives and logical equivalence. Logical equivalence is a *semantic* notion and is determined by the *truth-function* (defined by a truth table) represented by a connective. All binary connectives can be recognized by their truth tables, and the properties *trivial* and *one-sided* can characterized in terms of truth tables/functions as well.

4. Question 4 (a) asks you for a formula that is satisfiable iff (expresses that) there are exactly 3 *distinct* vertice using the system $\mathcal{LG}$s. Since $\approx$ is interpreted as equality, the fact that two vertices a and b are *distinct* can be represented by the formula $\neg(\approx(a, b))$. To show that there are exactly three vertices, your formula must express that there are three distinct vertices, and that there are no *other* vertices.

5. Question 4 (b) asks for a formula expressing the fact that some vertex a is connected by an edge to some vertex b. If a and b are connected by an edge, how many vertices must there be on a shortest simple path from a to b?

6. Question 4(c): To ensure that there is only one path between two vertices a and b, your formula must be satisfiable only when

   - There is only one shortest path between any two vertices.
   - There is no path between any two vertices *other than* the shortest path. That is, G has no cycles.

   To express the first property, your formula must be satisfiable only when every shortest path is unique. If a shortest path $v_1, v_2, \ldots, v_k$ is unique, then for every i between 1 and k, there is a *unique* v_i. That is, there is exactly 1 vertex that is *ith* on a shortest path from v_1 to v_k. Also, if v_2 is the unique 2nd vertex on a shortest path to v_k, then it is also the unique 1st vertex on a path from v_2 to v_k (unless $k = 2$).

   To express the second property is a bit trickier because $\mathcal{LG}$ cannot express anything about *non-shortest* paths directly. But it can help to recognize the following facts: If G has any cycles, then it has a shortest cycle and (since we are only considering *simple* cycles) that cycle must have at least three vertices a, b, and c; furthermore, the shortest path between any two of these vertices *must* be entirely on the cycle. Try drawing some pictures of cycles with a, b, and c connected by paths of different lengths. Take note of the cases in which one vertex is on a shortest path between the other two, and cases which violate the unique shortest path property.