CSC236, Summer 2005, Assignment 4 Hints 2

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1. Here are some more correct values for the number of formulae (Question 1):

   n=3 10935

   n=4 275562

   n=5 7440174

   n=6 210450636

   n=7 6155681103

2. If you're having trouble with 2(a), it may make things easier to observe that the sub-formula $((Q \to R) \vee (R \to Q))$ is a tautology. Therefore you should be able to reduce $P \wedge ((Q \to R) \vee (R \to Q))$ to just P (since for any tautology T and formula P, $P \wedge T$ LEQV P).

3. The last part of Question 3 asks you to prove whether some set of connectives (which you can choose your own symbols for) is a complete set or not. To prove that some set of connectives S is complete, it is enough to show that every formula using only connectives in a "known" complete set C has a logically equivalent formula using only connectives in S.

   Conversely, to show that S is incomplete it is enough to show that every formula using only connectives in S has a logically equivalent formula using only connectives in some "known" incomplete set I.

   The following sets of connectives are "known" to be incomplete:

   - $\{\wedge, \to, \vee\}$
   - $\{\neg, \oplus, \leftrightarrow\}$

   The following sets of connectives are "known" to be complete:

   - $\{\neg, \vee\}$
   - $\{\neg. \wedge\}$
   - $\{\oplus, \to\}$

   Of course, any subset of an incomplete set of connectives is incomplete, and any superset of a complete set of connectives is also complete.

4. For question 4 (c), assume that you already know how to express (from part b) the fact that there is an edge between vertices a and b. Part of what your formula for part (c) needs to express is that every path in G is unique (G is acyclic). To show that every *shortest* path is unique, consider the fact that if two shortest paths between a and b exist, they must *diverge* at some point and then come back together (at or before b). So there must be two edges (x, y) and (x, y') which each lie on a shortest path between a and b such that y and y' are distinct. Your formula must disallow this.

   Once you have prohibited duplicate *shortest* paths you will have prohibited all shortest cycles of even degree (why? draw a picture). To prohibit shortest cycles of odd degree, consider that in a cycle of odd degree there is always an edge (a, b) and a (distinct) vertex c such that c is equally close to both a and b. This means that none of a,b, and c is on a shortest path to the others (why? draw a picture). Your formula must therefore prohibit the existence of such a,b, and c.

If G has any cycles it has a shortest cycle. Therefore to express that G is acyclic it is sufficient to express that (to sum up the above):

- G has no duplicate shortest paths (shortest paths are unique) which is to say that there are no edges (x, y) and (x, y') on a shortest path between the same vertices such that y and y' are distinct.
- G has no shortest odd cycle, which is to say there are no distinct a, b and c such that none of the three is on a shortest path between the other two.

All that's left is to express that G is connected (there is a (shortest) path between any two vertices). This should be straightforward. In your answer you do not have to mention the above arguments, you can simply express the restrictions described in this hint.