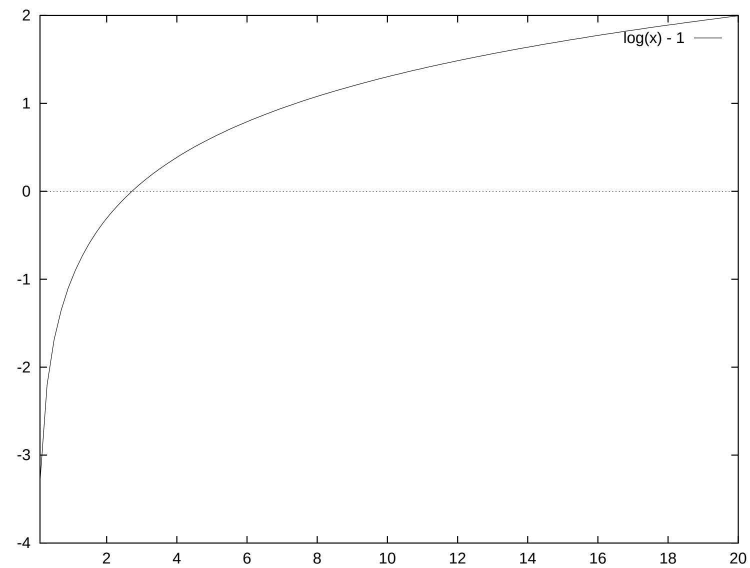
Question 1. [11 MARKS]

Recall Newton's method and the bisection method for finding roots of functions. Suppose you need to find the value of $e = \exp(1)$. You decide to do this by finding the root of $f(x) = \ln(x) - 1$. You reason that $\ln(\exp(1)) - 1 = 0$, so $\exp(1)$ is the root of the equation. Here's the situation in graphical form:



Part (a) [3 MARKS]

State an advantage the bisection method has over Newton's method in this case. State two different possible criteria for successfully terminating the bisection process.

1. Bisection guarantees convergence for easy-to-choose x_0 and x_1.
2. Stop when $|x_n - x_{n+1}|$ is smaller than some predetermined tolerance.
3. Stop when $|f(x)|$ is smaller than some given tolerance.
4. Stop after a predetermined number of iterations.

Part (b) [4 MARKS]

State an advantage Newton's method has over the bisection method in this case. State two different possible criteria for successfully terminating Newton's method. State a possible criteria for unsuccessfully terminating (abandoning) Newton's method.

1. Newton's method converges faster (quadratic) when it converges.
2. Stop when $|x_n - x_{n+1}|$ is smaller than some predetermined tolerance.
3. Stop when $|f(x)|$ is smaller than some given tolerance.
4. Stop (abandon) if $x_n \leq 0$.

Part (c) [4 MARKS]

1. For the bisection method above you need initial guesses x_0 and x_1. State values for each of x_0 and x_1 that will work, and justify your choice.

   Guess $x_0 = 1$, and $x_1 = 4$, since $\ln 1$ is 0, so $f(1)$ is -1, and $\ln 4$ is greater than 1 (assuming we know that e is somewhere between 2 and 3). So $f(x_0)f(x_1) < 0$, and the bisection method is guaranteed to work

2. For Newton's method above you need one initial guess, x_0. State a value for x_0 that will work, and justify your choice. A (possibly) helpful fact is that the first derivative of $\ln x$ is $(1/x)$, and the second derivative is $-(1/x^2)$.

   Guess $x_0 = 2$. Then use the criteria from the course Readings to determine whether $(f'(x))^2 > |(\ln x - 1)f''(x)|$ in a region containing the root and 2.

   $$\frac{1}{x^2} > |\frac{-1}{x^2}(\ln x - 1)| \Longleftrightarrow 1 > |1 - \ln x|$$

   This inequality will be true in the interval $(1, e^2)$, which certainly includes both 2 and the root, e.

Question 2. [14 MARKS]

Suppose `STEP_NUM` is a global integer constant greater than 2, and the following global declaration has been made:

```
/* a continuous function */
extern double cont_func(double x);
```

Part (a) [10 MARKS]

Complete the C code for the definition of `integrate`, using one of the numerical integration techniques discussed in our course, to integrate `cont_func()` from `lower` to `upper`. Be sure to evaluate `cont_func` exactly `STEP_NUM` times. You may assume that `lower` is no bigger than `upper`.

```
extern double integrate(double lower, double upper)
{
    int i;
    double step, sum;
    step = (upper - lower) / (STEP_NUM - 2);

    /* intermediate points appear twice in trapezoidal rule */
    sum = (cont_func(lower) + cont_func(upper)) / 2;
    for (i = 1; i < STEP_NUM - 1; i++)
        sum += cont_func(lower + i * step);
    return sum * step;
}
```

Part (b) [4 MARKS]

Discuss two possible sources of numerical errors in the previous part, and suggest ways to reduce them. You may have already implemented some of your suggestions, but there is no need to do so if you haven't.

1. Calculation of `step` involves both floating point subtraction and division. These sources of error would be compounded if I repeatedly added `step` to get the new argument for `cont_func()`, so I multiply by the index instead.

2. The trapezoids are worse approximations of the integral when the step size is larger (that is, `STEP_NUM` is smaller). Increasing `STEP_NUM` will reduce this source of error.

Question 3. [15 MARKS]

Inspect the adjacency matrix for an undirected graph below.

	A	B	C	D	E	F	G
A	0	1	1	1	1	0	0
B	1	0	1	1	1	0	0
C	1	1	0	1	1	0	0
D	1	1	1	0	1	1	0
E	1	1	1	1	0	0	1
F	0	0	0	1	0	0	1
G	0	0	0	0	1	1	0

Part (a) [10 MARKS]

Draw a graphical representation — draw the vertices as circles labelled A through G with lines connecting those vertices that have an edge between them. Leave room in the circles for a number representing a colour.

Part (b) [5 MARKS]

Colour the graph you drew above by labelling the vertices with the colours 0, 1, 2, ... You should not colour two adjacent vertices with the same colour. You should use as few colours as you can.

Question 4. [10 MARKS]

Consider the undirected, weighted graph below, with vertex A distinguished as the source. Fill in the minimum distances (first row of the table below) from source A to each node in the graph (by convention the minimum distance from A to A is 0). Fill in the predecessor of each vertex (second row of the table below), that is, the vertex immediately preceding the vertex heading each column in a path of minimum distance from A. Since A itself has no predecessor, it has corresponding entry *.

	A	B	C	D	E	F	G
Minimum distance from A	0	6	4	6	2	2	3
Predecessor	*	F	E	G	A	A	F