11 (dual) One operator is the dual of another operator if it negates the result when applied to the negated operands. The zero-operand operators $\top$ and $\bot$ are each other's duals. If $op_0\ \neg a = \neg\ op_1\ a$ then op_0 and op_1 are duals. If $(\neg a)\ op_0\ (\neg b) = \neg(a\ op_1\ b)$ then op_0 and op_1 are duals. And so on for more operands.

(a) Of the 4 one-operand binary operators, there is 1 pair of duals, and 2 operators that are their own duals. Find them.

(b) Of the 16 two-operand binary operators, there are 6 pairs of duals, and 4 operators that are their own duals. Find them.

(c) What is the dual of the three-operand operator **if then else fi** ? Express it using only the operator **if then else fi** .

(d) The dual of a binary expression without variables is formed as follows: replace each operator with its dual, adding parentheses if necessary to maintain the precedence. Explain why the dual of a theorem is an antitheorem, and vice versa.

(e) Let P be a binary expression without variables. From part (d) we know that every binary expression without variables of the form

$$\text{(dual of } P\text{)} = \neg P$$

is a theorem. Therefore, to find the dual of a binary expression with variables, we must replace each operator by its dual and negate each variable. For example, if a and b are binary variables, then the dual of $a{\wedge}b$ is $\neg a \vee \neg b$. And since

$$\text{(dual of } a{\wedge}b\text{)} = \neg(a{\wedge}b)$$

we have one of the Duality Laws:

$$\neg a \vee \neg b = \neg(a \wedge b)$$

The other of the Duality Laws is obtained by equating the dual and negation of $a{\vee}b$. Obtain five laws that do not appear in this book by equating a dual with a negation.

(f) Dual operators have theorem tables that are each other's vertical mirror reflections. For example, the theorem table for $\wedge$ (below left) is the vertical mirror reflection of the theorem table for $\vee$ (below right).

$$\wedge:\ \begin{array}{c|c} \top\top & \top \\ \top\bot & \bot \\ \bot\top & \bot \\ \bot\bot & \bot \end{array} \qquad\qquad \vee:\ \begin{array}{c|c} \top\top & \top \\ \top\bot & \top \\ \bot\top & \top \\ \bot\bot & \bot \end{array}$$

Design symbols (you may redesign existing symbols where necessary) for the 4 one-operand and 16 two-operand binary operators according to the following criteria.

(i) Dual operators should have symbols that are vertical mirror reflections (like $\wedge$ and $\vee$). This implies that self-dual operators have vertically symmetric symbols, and all others have vertically asymmetric symbols.

(ii) If $a\ op_0\ b = b\ op_1\ a$ then op_0 and op_1 should have symbols that are horizontal mirror reflections (like $\Rightarrow$ and $\Leftarrow$). This implies that symmetric operators have horizontally symmetric symbols, and all others have horizontally asymmetric symbols.

After trying the question, scroll down to the solution.

(a) Of the 4 one-operand binary operators, there is 1 pair of duals, and 2 operators that are their own duals. Find them.

§ To answer this question, I'll use the symbols I introduce in part (f). The pair of duals is: $\bar{\top}$ (always $\top$) and $\underline{\bot}$ (always $\bot$). The two self-duals are: $\mathtt{I}$ (identity) and $\ne$ (negation).

(b) Of the 16 two-operand binary operators, there are 6 pairs of duals, and 4 operators that are their own duals. Find them.

§ To answer this question, I'll use the symbols I will introduce in part (f). The six dual pairs are: $\bar{\triangle}\,\underline{\nabla}$, $\vee\ \wedge$, $\geqslant\ \eqslantgtr$, $\leqslant\ \eqslantless$, $\underline{\triangle}\,\bar{\nabla}$, $\triangle\ \nabla$. The four self-duals are: $<$, $>$, $\triangleright$, $\triangleleft$.

(c) What is the dual of the three-operand operator **if then else fi** ? Express it using only the operator **if then else fi** .

§ Its theorem table is

$\top\top\top$	$\top\top\bot$	$\top\bot\top$	$\top\bot\bot$	$\bot\top\top$	$\bot\top\bot$	$\bot\bot\top$	$\bot\bot\bot$
$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$

The dual of **if** a **then** b **else** c **fi** is equivalent to **if** a **then** c **else** b **fi** .

(d) The dual of a binary expression without variables is formed as follows: replace each operator with its dual, adding parentheses if necessary to maintain the precedence. Explain why the dual of a theorem is an antitheorem, and vice versa.

§ I will show that for every expression P without variables, (dual of P) $= \neg P$. I do so by induction on the structure of expression P . The two binary values give us two base cases.

$$\begin{array}{lll} & (\text{dual of } \top) & \text{use the dual-forming rules} \\ = & \bot & \\ = & \neg\top & \end{array}$$

$$\begin{array}{lll} & (\text{dual of } \bot) & \text{use the dual-forming rules} \\ = & \top & \\ = & \neg\bot & \end{array}$$

There is an induction step for each of the binary operators. Suppose (this is an induction hypothesis) that (dual of P) $= \neg P$. Then

$$\begin{array}{lll} & (\text{dual of } \neg P) & \text{use the dual-forming rules} \\ = & \neg(\text{dual of } P) & \text{use the induction hypothesis} \\ = & \neg\neg P & \end{array}$$

Suppose that (dual of P) $= \neg P$ and (dual of Q) $= \neg Q$. Then

$$\begin{array}{lll} & (\text{dual of } P \wedge Q) & \text{use the dual-forming rules} \\ = & (\text{dual of } P) \vee (\text{dual of } Q) & \text{use the induction hypotheses} \\ = & \neg P \vee \neg Q & \text{use duality law} \\ = & \neg(P \wedge Q) & \end{array}$$

And similarly for all other operators.

§(e) From $a=b$ we get $\neg a \ne \neg b = \neg(a=b)$
From **if** a **then** b **else** c **fi** we get **if** $\neg a$ **then** $\neg c$ **else** $\neg b$ **fi** $=$ $\neg$**if** a **then** b **else** c **fi**
From $a=b \wedge c$ we get $\neg a \ne \neg b \vee \neg c = \neg(a=b \wedge c)$
From $a=b \vee c$ we get $\neg a \ne \neg b \wedge \neg c = \neg(a=b \vee c)$
From $a = (b \wedge c)$ we get $\neg a \ne (\neg b \vee \neg c) = \neg(a = (b \wedge c))$

§(f)

old			$\neg$	
new	$\bar{\top}$	$\mathtt{I}$	ǂ	$\underline{\bot}$
$\top$	$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\bot$

old		$\vee$	$\Leftarrow$		$\Rightarrow$		$=$	$\wedge$		ǂ						
new	$\bar{\vartriangle}$	$\vee$	$\geqslant$	$<$	$\leqslant$	$>$	$\underline{\vartriangle}$	$\wedge$	$\vartriangle$	$\bar{\triangledown}$	$\vartriangleright$	$\eqslantgtr$	$\vartriangleleft$	$\eqslantless$	$\triangledown$	$\underline{\triangledown}$
$\top\ \top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top\ \bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
$\bot\ \top$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\bot\ \bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$