140 Let n and s be natural variables. The program

$$R \Leftarrow s:= 0.\ Q$$
$$Q \Leftarrow \textbf{if } n=0 \textbf{ then } ok \textbf{ else } n:= n-1.\ s:= s+n.\ Q \textbf{ fi}$$

adds up the first n natural numbers. Define R and Q appropriately, and prove the two refinements.

After trying the question, scroll down to the solution.

§ Define

$R \;=\; s' = \Sigma i\colon 0,..n\cdot\, i$ or $R \;=\; s' = \Sigma\, [0;..n]$

$Q \;=\; s' = s + \Sigma i\colon 0,..n\cdot\, i$ or $Q \;=\; s' = s + \Sigma\, [0;..n]$

Proof of R refinement:

	$s:= 0.\;\; Q$	expand Q
$=$	$s:= 0.\;\; s' = s + \Sigma\, [0;..n]$	substitution law and simplify
$=$	$s' = \Sigma\, [0;..n]$	
$=$	R	

Proof of Q refinement, first case:

	$n=0 \wedge ok$	expand ok
$=$	$n=0 \wedge n'=n \wedge s'=s$	
$\Longrightarrow$	$s' = s + \Sigma\, [0;..n]$	
$=$	Q	

Proof of Q refinement, last case:

	$n\neq 0 \wedge (n:= n-1.\;\; s:= s+n.\;\; Q)$	expand Q
$=$	$n\neq 0 \wedge (n:= n-1.\;\; s:= s+n.\;\; s' = s + \Sigma\, [0;..n])$	substitution law twice
$=$	$n\neq 0 \;\wedge\; s' = s + (n-1) + \Sigma\, [0;..n-1]$	simplify and specialize
$\Longrightarrow$	$s' = s + \Sigma\, [0;..n]$	
$=$	Q	