153 Let i be an integer variable. Add time according to the recursive measure, and then find the strongest P you can such that

(a) $P \Leftarrow$ **if** *even* i **then** $i:= i/2$ **else** $i:= i+1$ **fi**.
if $i=1$ **then** *ok* **else** P **fi**

(b) $P \Leftarrow$ **if** *even* i **then** $i:= i/2$ **else** $i:= i-3$ **fi**.
if $i=0$ **then** *ok* **else** P **fi**

After trying the question, scroll down to the solution.

(a) $P \Leftarrow$ **if** $even\ i$ **then** $i:= i/2$ **else** $i:= i+1$ **fi**.
if $i=1$ **then** ok **else** P **fi**

§ Adding time and rewriting a little,

$P \Leftarrow i=2 \land i'=1 \land t'=t$ (x)
$\lor\ i \neq 2 \land even\ i \land (i:= i/2.\ t:= t+1.\ P)$ (y)
$\lor\ odd\ i \land (i:= i+1.\ t:= t+1.\ P)$ (z)

This is a theorem when

$P = i'=1$ (0)
$\land\ (i<1 \Rightarrow t'=\infty)$ (1)
$\land\ (i=1 \Rightarrow t' = t+1)$ (2)
$\land\ (i=2 \Rightarrow t'=t\ )$ (3)

The proof is by cases and by parts.

(x0) $i'=1 \ \Leftarrow\ i=2 \land i'=1 \land t'=t$ by specialization
$= \top$

(y0) $i'=1 \ \Leftarrow\ i \neq 2 \land even\ i \land (i:= i/2.\ t:= t+1.\ i'=1)$ substitutions, specialization
$= \top$

(z0) $i'=1 \ \Leftarrow\ odd\ i \land (i:= i+1.\ t:= t+1.\ i'=1)$ substitutions, specialization
$= \top$

(x1) $(i<1 \Rightarrow t'=\infty \ \Leftarrow\ i=2 \land i'=1 \land t'=t)$ portation, base
$= \top$

(y1) $(i<1 \Rightarrow t'=\infty \ \Leftarrow\ i \neq 2 \land even\ i \land (i:= i/2.\ t:= t+1.\ i<1 \Rightarrow t'=\infty))$
portation, substitution twice
$= t'=\infty \ \Leftarrow\ i<1 \land even\ i \land (i<2 \Rightarrow t'=\infty)$ discharge, specialization
$= \top$

(z1) $(i<1 \Rightarrow t'=\infty \ \Leftarrow\ odd\ i \land (i:= i+1.\ t:= t+1.\ i<1 \Rightarrow t'=\infty))$ portation, substitutions
$= t'=\infty \ \Leftarrow\ i<1 \land odd\ i \land\ (i<0 \Rightarrow t'=\infty)$
discharge (since $i<1 \land odd\ i = i<0$), specialization
$= \top$

(x2) $(i=1 \Rightarrow t' = t+1 \ \Leftarrow\ i=2 \land i'=1 \land t'=t)$ portation, base
$= \top$

(y2) $(i=1 \Rightarrow t' = t+1 \ \Leftarrow\ i \neq 2 \land even\ i \land (i:= i/2.\ t:= t+1.\ i=1 \Rightarrow t' = t+1))$
portation, base
$= \top$

(z2) $(i=1 \Rightarrow t' = t+1 \ \Leftarrow\ odd\ i \land (i:= i+1.\ t:= t+1.\ i=1 \Rightarrow t' = t+1))$
portation, substitutions
$= t' = t+1 \ \Leftarrow\ i=1 \land (i=0 \Rightarrow t' = t+2)$ stuck

This one doesn't work! That doesn't mean that P is wrong, only that this proof attempt didn't work. Just as in the Fast Exponentiation problem, we cannot prove all the parts of the timing separately. The $i=1$ part requires the $i=2$ part. So I'll try

(z23) $(i=1 \Rightarrow t' = t+1) \land (i=2 \Rightarrow t'=t\ )$
$\Leftarrow\ odd\ i \land (i:= i+1.\ t:= t+1.\ (i=1 \Rightarrow t' = t+1) \land (i=2 \Rightarrow t'=t\ ))$ substitutions
$= (i=1 \Rightarrow t' = t+1) \land (i=2 \Rightarrow t'=t\ )$
$\Leftarrow\ odd\ i \land (i=0 \Rightarrow t' = t+2) \land (i=1 \Rightarrow t'=t+1)$ distribution
$= (i=1 \Rightarrow t' = t+1 \ \Leftarrow\ odd\ i \land (i=0 \Rightarrow t' = t+2) \land (i=1 \Rightarrow t'=t+1))$
$\land\ (i=2 \Rightarrow t'=t \ \Leftarrow\ odd\ i \land (i=0 \Rightarrow t' = t+2) \land (i=1 \Rightarrow t'=t+1))$
portation and discharge in the first conjunct; portation and base in the second
$= \top$

Good. There are still two more.

(x3) $(i=2 \Rightarrow t'=t \ \Leftarrow\ i=2 \land i'=1 \land t'=t)$ portation, specialization
$= \top$

(y3) $(i=2 \Rightarrow t'=t \;\;\Longleftarrow\;\; i \ne 2 \wedge even\ i \wedge (i{:=}\ i/2.\ \ t{:=}\ t+1.\ \ i=2 \Rightarrow t'=t))$ portation, base

$= \;\; \top$

I can go on adding conjuncts to P for particular values of i, but I would like something that covers all values of i. I conjecture that it's a theorem when

$$\begin{aligned} P \;=\;\; & i'=1 \\ & \wedge\ (i<1 \Rightarrow t'=\infty) \\ & \wedge\ (i=1 \Rightarrow t' = t+1) \\ & \wedge\ (i>1 \Rightarrow t + log\ i - 1 \le t' \le t + 2 \times log\ i) \end{aligned}$$

but I am unable to prove it. (Even putting all cases and parts together in one huge expression fails.) Straight from the program, the bound for $i \ge 1$ is exactly $b\ i$, defined as $b\ 1 = 1$, $b\ 2 = 0$, $b(2\times i+1) = 2+b(i+1)$, $b(2\times i+2) = 1+b(i+1)$, and it is logarithmic.

(b) $P \Longleftarrow$ **if** $even\ i$ **then** $i{:=}\ i/2$ **else** $i{:=}\ i-3$ **fi**.
if $i=0$ **then** ok **else** P **fi**

§ Adding time and rewriting a little,

$$\begin{aligned} P \;\Longleftarrow\;\; & i{:}\ 0,3 \wedge i'=0 \wedge t'=t && \text{(x)} \\ \vee\;\; & i\ne 0 \wedge even\ i \wedge (i{:=}\ i/2.\ \ t{:=}\ t+1.\ \ P) && \text{(y)} \\ \vee\;\; & i \ne 3 \wedge odd\ i \wedge (i{:=}\ i-3.\ \ t{:=}\ t+1.\ \ P) && \text{(z)} \end{aligned}$$

I conjecture that this is a theorem when

$$\begin{aligned} P \;=\;\; & i'=0 \\ & \wedge\ (i<0 \Rightarrow t'=\infty) \\ & \wedge\ (\neg\ i{:}\ 3\times nat \Rightarrow t'=\infty) \\ & \wedge\ (i{:}\ 0,3 \Rightarrow t'=t\ ) \\ & \wedge\ (i{:}\ 3\times(nat+2) \Rightarrow t + log\ i - 2 \le t' \le t + 2 \times log\ i) \end{aligned}$$

but I am unable to prove it. The best I can do is

$$\begin{aligned} P \;=\;\; & i'=0 && \text{(0)} \\ & \wedge\ (i<0 \Rightarrow t'=\infty) && \text{(1)} \\ & \wedge\ (\neg\ i{:}\ 3\times nat \Rightarrow t'=\infty) && \text{(2)} \\ & \wedge\ (i{:}\ 0,3 \Rightarrow t'=t) && \text{(3)} \end{aligned}$$

The proof is by cases and by parts.

(x0) $i'=0 \;\Leftarrow\; i{:}\ 0,3 \wedge i'=0 \wedge t'=t$ by specialization

$= \;\; \top$

(y0) $i'=0 \;\Leftarrow\; i\ne 0 \wedge even\ i \wedge (i{:=}\ i/2.\ \ t{:=}\ t+1.\ \ i'=0)$ substitutions, specialization

$= \;\; \top$

(z0) $i'=0 \;\Leftarrow\; i\ne 3 \wedge odd\ i \wedge (i{:=}\ i-3.\ \ t{:=}\ t+1.\ \ i'=0)$ substitutions, specialization

$= \;\; \top$

(x1) $(i<0 \Rightarrow t'=\infty \;\;\Longleftarrow\;\; i{:}\ 0,3 \wedge i'=0 \wedge t'=t)$ portation, base

$= \;\; \top$

(y1) $(i<0 \Rightarrow t'=\infty \;\;\Longleftarrow\;\; i\ne 0 \wedge even\ i \wedge (i{:=}\ i/2.\ \ t{:=}\ t+1.\ \ i<0 \Rightarrow t'=\infty))$

portation, substitution twice

$= \;\; t'=\infty \;\Leftarrow\; i<0 \wedge even\ i \wedge (i<0 \Rightarrow t'=\infty)$ discharge, specialization

$= \;\; \top$

(z1) $(i<0 \Rightarrow t'=\infty \;\;\Longleftarrow\;\; i\ne 3 \wedge odd\ i \wedge (i{:=}\ i-3.\ \ t{:=}\ t+1.\ \ i<0 \Rightarrow t'=\infty))$

portation, substitutions

$= \;\; t'=\infty \;\Leftarrow\; i<0 \wedge odd\ i \wedge\ (i<3 \Rightarrow t'=\infty)$ discharge (since $i<0 \Rightarrow i<3$), specialization

$= \;\; \top$

(x2) $(\neg\ i{:}\ 3\times nat \Rightarrow t'=\infty \;\;\Longleftarrow\;\; i{:}\ 0,3 \wedge i'=0 \wedge t'=t)$ portation, base

$= \;\; \top$

(y2) $(\;\; \neg\ i{:}\ 3\times nat \Rightarrow t'=\infty$
$\Longleftarrow\;\; i\ne 0 \wedge even\ i \wedge (i{:=}\ i/2.\ \ t{:=}\ t+1.\ \ \neg\ i{:}\ 3\times nat \Rightarrow t'=\infty)\ )$

portation, substitutions

$= \;\; t'=\infty \;\Leftarrow\; \neg\ i{:}\ 3\times nat\ \wedge even\ i \wedge (\neg\ i{:}\ 6\times nat \Rightarrow t'=\infty)$

discharge (since $\neg\ i\colon 3\times nat \ \Rightarrow\ \neg\ i\colon 6\times nat$), specialization

$=\quad \top$

(z2) $(\neg\ i\colon 3\times nat \Rightarrow t'=\infty \ \Longleftarrow\ i\ne 3 \wedge odd\ i \wedge (i{:=}\ i-3.\ \ t{:=}\ t+1.\ \neg\ i\colon 3\times nat \Rightarrow t'=\infty))$

portation, substitutions

$=\quad t'=\infty \ \ \Leftarrow\ \ \neg\ i\colon 3\times nat \ \wedge\ odd\ i \ \wedge\ (\neg\ i\colon 3\times(nat+1) \Rightarrow t'=\infty))$

discharge (since $\neg\ i\colon 3\times nat \ \Rightarrow\ \neg\ i\colon 3\times(nat+1)$), specialization

$=\quad \top$

(x3) $(i\colon 0{,}3 \Rightarrow t'=t \ \Longleftarrow\ i\colon 0{,}3 \wedge i'=0 \wedge t'=t)$ portation, specialization

$=\quad \top$

(y3) $(i\colon 0{,}3 \Rightarrow t'=t \ \Longleftarrow\ i\ne 0 \wedge even\ i \wedge (i{:=}\ i/2.\ \ t{:=}\ t+1.\ \ i\colon 0{,}3 \Rightarrow t'=t\ ))$ portation, base

$=\quad \top$

(z3) $(i\colon 0{,}3 \Rightarrow t'=t \ \Longleftarrow\ i\ne 3 \wedge odd\ i \wedge (i{:=}\ i-3.\ \ t{:=}\ t+1.\ \ i\colon 0{,}3 \Rightarrow t'=t\ ))$ portation, base

$=\quad \top$