155 Find a finite function f of natural variables i and j to serve as an upper bound on the execution time of the following program, and prove

$$
\begin{aligned}
t' \le t + f\,i\,j \;\Longleftarrow\;\; & \textbf{if } i{=}0 \land j{=}0 \textbf{ then } ok \\
& \textbf{else } \textbf{if } i{=}0 \textbf{ then } i{:=}\, j{\times}j.\;\; j{:=}\, j{-}1.\;\; t{:=}\, t{+}1.\;\; t' \le t + f\,i\,j \textbf{ fi} \\
& \qquad\;\, \textbf{else } i{:=}\, i{-}1.\;\; t{:=}\, t{+}1.\;\; t' \le t + f\,i\,j \textbf{ fi}
\end{aligned}
$$

After trying the question, scroll down to the solution.

§ One such function is $f\,i\,j = i+j+j^3$. Proof by cases:

first case

$$
\begin{array}{lll}
& t' \le t+i+j+j^3 \;\Leftarrow\; i=0 \land j=0 \land ok & \\
= & t' \le t+i+j+j^3 \;\Leftarrow\; i=0 \land j=0 \land i'=i \land j'=j \land t'=t & \text{context} \\
= & \top &
\end{array}
$$

middle case

$$
\begin{array}{lll}
& t' \le t+i+j+j^3 \;\Leftarrow\; i=0 \land j>0 \land (i:=j\times j.\; j:=j-1.\; t:=t+1.\; t' \le t+i+j+j^3) & \\
& & \text{substitution 3 times} \\
= & t' \le t+i+j+j^3 \;\Leftarrow\; i=0 \land j>0 \land t' \le t+1+j^2+j-1+(j-1)^3 & \text{portation} \\
= & i=0 \land j>0 \Rightarrow (t' \le t+1+j^2+j-1+(j-1)^3 \;\Rightarrow\; t' \le t+i+j+j^3) & \text{connection} \\
= & i=0 \land j>0 \Rightarrow (t+1+j^2+j-1+(j-1)^3 \le t+i+j+j^3) & \text{cube} \\
= & i=0 \land j>0 \Rightarrow (t+1+j^2+j-1+j^3-3\times j^2+3\times j-1 \le t+i+j+j^3) & \\
& & \text{cancel, and use } i=0 \\
= & i=0 \land j>0 \Rightarrow (2\times j^2-3\times j+1 \ge 0) & \text{arithmetic} \\
= & \top &
\end{array}
$$

last case

$$
\begin{array}{lll}
& t' \le t+i+j+j^3 \;\Leftarrow\; i\ne0 \land (i:=i-1.\; t:=t+1.\; t' \le t+i+j+j^3) & \text{substitution twice} \\
= & t' \le t+i+j+j^3 \;\Leftarrow\; i\ne0 \land t' \le t+i+j+j^3 & \text{specialization} \\
= & \top &
\end{array}
$$