156 Let P mean that the final values of natural variables a and b are the largest exponents of 2 and 3 respectively such that both powers divide evenly into the initial value of positive integer x .

(a) Define P formally.

(b) Define Q suitably and prove

$P \Leftarrow$ $a{:=}\,0.\ \ b{:=}\,0.\ \ Q$

$Q \Leftarrow$ **if** $x\!:\, 2{\times}nat$ **then** $x{:=}\, x/2.\ \ a{:=}\, a{+}1.\ \ Q$
else **if** $x\!:\, 3{\times}nat$ **then** $x{:=}\, x/3.\ \ b{:=}\, b{+}1.\ \ Q$
else *ok* **fi** **fi**

(c) Find an upper bound for the execution time of the program in part (b).

After trying the question, scroll down to the solution.

(a) Define P formally.

§ $P = x: 2^{a'}\times nat \land x: 3^{b'}\times nat \land \neg x: 2^{a'+1}\times nat \land \neg x: 3^{b'+1}\times nat$

(b) Define Q suitably and prove

$P \Leftarrow a:= 0.\ b:= 0.\ Q$

$Q \Leftarrow$ **if** $x: 2\times nat$ **then** $x:= x/2.\ a:= a+1.\ Q$

else if $x: 3\times nat$ **then** $x:= x/3.\ b:= b+1.\ Q$

else *ok* **fi fi**

§ $Q = x\times 2^a\times 3^b = x'\times 2^{a'}\times 3^{b'} \land \neg x': 2\times nat \land \neg x': 3\times nat$

$a:= 0.\ b:= 0.\ Q$

$= x = x'\times 2^{a'}\times 3^{b'} \land \neg x': 2\times nat \land \neg x': 3\times nat$

note that $x'\times 2^{a'}\times 3^{b'}: 2^{a'}\times nat$ so $x: 2^{a'}\times nat$.

Similarly $x: 3^{b'}\times nat$.

Also if $\neg x': 2\times nat$ then $\neg x'\times 2^{a'}\times 3^{b'}: 2^{a'+1}\times nat$

and so $\neg x: 2^{a'+1}\times nat$. Similarly $\neg x: 3^{b'+1}\times nat$.

$\Rightarrow P$

The Q refinement is proven by cases. First

$x: 2\times nat \land (x:= x/2.\ a:= a+1.\ Q)$

$= x: 2\times nat \land x/2\times 2^{a+1}\times 3^b = x'\times 2^{a'}\times 3^{b'} \land \neg x': 2\times nat \land \neg x': 3\times nat$

$= x: 2\times nat \land x\times 2^a\times 3^b = x'\times 2^{a'}\times 3^{b'} \land \neg x': 2\times nat \land \neg x': 3\times nat$

$\Rightarrow Q$

Second

$\neg x: 2\times nat \land x: 3\times nat \land (x:= x/3.\ b:= b+1.\ Q)$

$= \neg x: 2\times nat \land x: 3\times nat \land x/3\times 2^a\times 3^{b+1} = x'\times 2^{a'}\times 3^{b'} \land \neg x': 2\times nat \land \neg x': 3\times nat$

$= \neg x: 2\times nat \land x: 3\times nat \land x\times 2^a\times 3^b = x'\times 2^{a'}\times 3^{b'} \land \neg x': 2\times nat \land \neg x': 3\times nat$

$\Rightarrow Q$

Finally

$\neg x: 2\times nat \land \neg x: 3\times nat \land ok$

$= \neg x: 2\times nat \land \neg x: 3\times nat \land x'=x \land a'=a \land b'=b$

$\Rightarrow Q$

(c) Find an upper bound for the execution time of the program in part (b).

§ Replace P by $t' = t+a'+b'$ and replace Q by $t' = t+a'-a+b'-b$ and insert $t:= t+1$ before each of the two calls to Q . For a time bound in the initial values of variables, replace both P and Q by

$x\ge 1 \Rightarrow t' \le t + log\ x$

Proof of first refinement:

$a:= 0.\ b:= 0.\ x\ge 1 \Rightarrow t' \le t + log\ x$ — Substitution Law twice

$= x\ge 1 \Rightarrow t' \le t + log\ x$

The second refinement, first case, inserting time increment:

$x: 2\times nat \land (x:= x/2.\ a:= a+1.\ t:= t+1.\ x\ge 1 \Rightarrow t' \le t + log\ x)$ — Substitution Law

$= x: 2\times nat \land (x/2\ge 1 \Rightarrow t' \le t + 1 + log(x/2))$ — If x is even, then $x/2\ge 1 = x\ge 1$

$= x: 2\times nat \land (x\ge 1 \Rightarrow t' \le t + 1 + log(x/2))$ — law of logarithms and specialize

$\Rightarrow x\ge 1 \Rightarrow t' \le t + log\ x$

Second case, inserting time increment:

$\neg x: 2\times nat \land x: 3\times nat \land (x:= x/3.\ b:= b+1.\ t:= t+1.\ x\ge 1 \Rightarrow t' \le t + log\ x)$ — Subs Law

$= \neg x: 2\times nat \land x: 3\times nat \land (x/3\ge 1 \Rightarrow t' \le t + 1 + log(x/3))$ — If x is an odd multiple of 3 , then $x/3\ge 1 = x\ge 1$

$= \neg x: 2\times nat \land x: 3\times nat \land (x\ge 1 \Rightarrow t' \le t + 1 + log(x/3))$ — $log(x/3) \le log(x/2)$

$\Rightarrow \neg x: 2\times nat \land x: 3\times nat \land (x\ge 1 \Rightarrow t' \le t + 1 + log(x/2))$ — law of logarithms and specialize

$\Longrightarrow \quad x{\ge}1 \Rightarrow t' \le t + log\ x$

Final case:

$\neg\ x{:}\ 2{\times}nat \wedge \neg\ x{:}\ 3{\times}nat \wedge x'{=}x \wedge a'{=}a \wedge b'{=}b \wedge t'{=}t$ — If $x{\ge}1$ then $log\ x \ge 0$

$\Longrightarrow \quad x{\ge}1 \Rightarrow t' \le t + log\ x$