177 (combinations) Write a program to find the number of ways to partition $a+b$ things into a things in the left part and b things in the right part. Include recursive time.

After trying the question, scroll down to the solution.

§ The number of ways to partition $a+b$ things into a things and b things is $(a+b)! / (a!\times b!)$ where ! is the factorial function. First without time.

$x{:=}\ (a+b)! / (a!\times b!) \ \Leftarrow$

  if $a=0$ **then** $x{:=}\ 1$

  else $a{:=}\ a-1.\ \ x{:=}\ (a+b)! / (a!\times b!).\ \ a{:=}\ a+1.\ \ x{:=}\ x\times(a+b)/a$ **fi**

The assignment $x{:=}\ (a+b)! / (a!\ b!)$ means $x' = (a+b)! / (a!\ b!)) \ \wedge\ a'=a \ \wedge\ b'=b$. On the right side it is a recursive call. Stating it as an assignment makes the proof easy: just use the substitution law and simplify. The proof is by cases. First case:

$a=0 \wedge (x{:=}\ 1) \ \Rightarrow\ (x{:=}\ (a+b)! / (a!\times b!))$ — definition of assignment

$=\ a=0 \wedge x'=1 \ \wedge\ a'=a \ \wedge\ b'=b \ \Rightarrow\ x' = (a+b)! / (a!\ b!)) \ \wedge\ a'=a \ \wedge\ b'=b$ — use $0!=1$

$=\ \top$

Second case, starting with the right side:

$a\ne0 \wedge (a{:=}\ a-1.\ \ x{:=}\ (a+b)! / (a!\times b!).\ \ a{:=}\ a+1.\ \ x{:=}\ x\times(a+b)/a)$ — assignment

$=\ a\ne0 \wedge (a{:=}\ a-1.\ \ x{:=}\ (a+b)! / (a!\times b!).\ \ a{:=}\ a+1.\ \ x'=x\times(a+b)/a \wedge a'=a \wedge b'=b)$ — substitution law 3 times

$=\ a\ne0 \ \wedge\ x' = (a-1+b)!/((a-1)!\times b!)\times(a+b)/a \ \wedge\ a'=a \ \wedge\ b'=b$ — simplify

$=\ a\ne0 \ \wedge\ x' = (a+b)! / (a!\times b!) \ \wedge\ a'=a \ \wedge\ b'=b$ — specialization

$\Rightarrow\ x{:=}\ (a+b)! / (a!\times b!)$

Now the time.

$t' = t+a \ \Leftarrow$ **if** $a=0$ **then** $x{:=}\ 1$

  else $a{:=}\ a-1.\ \ t{:=}\ t+1.\ \ t' = t+a.\ \ a{:=}\ a+1.\ \ x{:=}\ x\times(a+b)/a$ **fi**

Proof by cases. First case:

$a=0 \wedge (x{:=}\ 1) \ \Rightarrow\ t' = t+a$ — definition of assignment

$=\ a=0 \wedge x'=1 \ \wedge\ a'=a \ \wedge\ b'=b \ \wedge\ t'=t \ \Rightarrow\ t' = t+a$

$=\ \top$

Second case, starting with the right side:

$a\ne0 \wedge (a{:=}\ a-1.\ \ t{:=}\ t+1.\ \ t' = t+a.\ \ a{:=}\ a+1.\ \ x{:=}\ x\times(a+b)/a)$ — assignment

$=\ a\ne0 \wedge (a{:=}\ a-1.\ t{:=}\ t+1.\ t' = t+a.\ a{:=}\ a+1.\ x'=x\times(a+b)/a \wedge a'=a \wedge b'=b \wedge t'=t)$ — substitution law 3 times

$=\ a\ne0 \wedge (t' = t+a.\ \ x'=x\times(a+1+b)/(a+1) \wedge a'=a+1 \wedge b'=b \wedge t'=t)$ — sequential composition

$=\ a\ne0 \wedge (\exists x'', a'', b'', t''\cdot\ \ t''=t+a \wedge x'=x''\times(a''+1+b'')/(a''+1)$
$\wedge\ a'=a''+1 \wedge b'=b'' \wedge t'=t'')$ — one point 4 times

$=\ a\ne0 \ \wedge\ t' = t+a$ — specialization

$\Rightarrow\ t' = t+a$

When refining $x{:=}\ (a+b)! / (a!\times b!)$, there was no time variable. Adding the time variable, we cannot write this as an assignment, because that would mean $t'=t$. We can put the result and the timing together as

$x' = (a+b)! / (a!\ b!)) \ \wedge\ a'=a \ \wedge\ b'=b \ \wedge\ t'=t+a$

or as

$x{:=}\ (a+b)! / (a!\times b!).\ \ t{:=}\ \ t+a$

Here is a solution that is symmetric in a and b .

$x{:=}\ (a+b)! / (a!\times b!) \ \Leftarrow$

  if $a=0 \vee b=0$ **then** $x{:=}\ 1$

  else $a{:=}\ a-1.\ \ b{:=}\ b-1.\ \ x{:=}\ (a+b)! / (a!\times b!).$

   $a{:=}\ a+1.\ \ b{:=}\ b+1.\ \ x{:=}\ x/a/b\times(a+b-1)\times(a+b)$ **fi**

And its execution time is smaller: $a{\downarrow}b$.

Here is a solution with the same execution time and its recursion does not require a stack.

$x' \ = \ (a+b)! / (a!\times b!) \ \wedge\ t' \ = \ t + a{\downarrow}b \ \Leftarrow$

  $x{:=}\ 1.\ \ x' \ = \ x \times (a+b)! / (a!\times b!) \ \wedge\ t' \ = \ t + a{\downarrow}b$

$$x' = x \times (a+b)! / (a! \times b!) \wedge t' = t + a \downarrow b \Leftarrow$$
$$\textbf{if } a=0 \vee b=0 \textbf{ then } ok$$
$$\textbf{else } x := x/a/b \times (a+b-1) \times (a+b).\ a := a-1.\ b := b-1.\ t := t+1.$$
$$x' = x \times (a+b)! / (a! \times b!) \wedge t' = t + a \downarrow b \textbf{ fi}$$

Now, here is a **for**-loop solution. Define invariant

$$A\ k \ = \ x = (a+k)! / (a! \times k!)$$

Then

$$x' = (a+b)! / (a! \times b!) \Leftarrow x := 1.\ A\ 0 \Rightarrow A'b$$
$$A\ 0 \Rightarrow A'b \Leftarrow \textbf{for } k := 0;..b \textbf{ do } k: 0,..b \wedge A\ k \Rightarrow A'(k+1) \textbf{ od}$$
$$k: 0,..b \wedge A\ k \Rightarrow A'(k+1) \Leftarrow x := x \times (a+k+1)/(k+1)$$

with timing $t' = t+b$.

Finally, here are two functional solutions. Define

$$f = \langle a, b: nat \cdot (a+b)! / (a! \times b!) \rangle$$

Then

$$f\ a\ b = \textbf{if } a=0 \textbf{ then } 1 \textbf{ else } f\ (a-1)\ b \times (a+b) / a \textbf{ fi}$$

with execution time a . For execution time $a \downarrow b$

$$f\ a\ b = \textbf{if } a=0 \vee b=0 \textbf{ then } 1 \textbf{ else } f\ (a-1)\ (b-1) \times (a+b-1) \times (a+b) / a / b \textbf{ fi}$$